Proof of i) Balazard-Saias-Yor and ii) Sondow-Dumitrescu criteria for validity of Riemann Hypothesis

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Abstract

[In this paper two proofs of Riemann Hypothesis equivalent will be given. One equivalent is due to Balazard-Saias-Yor and another equivalent is due to Sondow-Dumitrescu.]

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1. Introduction

In a recent paper [1] it was shown that Riemann Xi function can have explicit analytical expression containing two arbitrary positive constants and an unknown positive parameter. This general expression of ξ(s) was utilized to prove Riemann Hypothesis. Two different proofs were given in that paper. In this paper two more proofs of Riemann Hypothesis equivalent will be given. First we will consider Balazard-Saias-Yor equivalent.

2. Balazard-Saias-Yor equivalent and its proof

Balazard-Saias-Yor showed [2] that validity of Riemann Hypothesis is equivalent to

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+4t^2)} \log\left|\zeta\left(\frac{1}{2}+ it\right)\right| = 0 \qquad \ldots(2.1)$$

It is known [3] that Riemann Xi function ξ(s) and Riemann Zeta function ζ(s) are connected through relation

$$\xi(s) = \frac{1}{2} s(s-1)\, \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right) \zeta(s) \qquad \ldots(2.2)$$

Therefore

$$\zeta(s) = \frac{2\, \pi^{\frac{s}{2}}\, \xi(s)}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \qquad ..(2.3)$$

Then from (2.3)

$$\zeta(\tfrac{1}{2} + it) = \frac{2\,\pi^{\frac{1}{4}+\frac{it}{2}}\,\xi(\frac{1}{2}+it)}{\left(\frac{1}{2}+it\right)\left(-\frac{1}{2}+it\right)\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} \qquad \ldots(2.4)$$

It was shown in [1] that

$$\xi(s) = \xi(\sigma + it) = F_2(l_1) + F_1(l_1)\,[\mathrm{Cos}\, l_1 t\ \mathrm{Cos\,h}\, l_1\left(\sigma - \tfrac{1}{2}\right) + i\ \mathrm{Sin}\, l_1 t\ \mathrm{Sin\,h} l_1\left(\sigma - \tfrac{1}{2}\right)] \qquad \ldots(2.5)$$

where $F_2(l_1)$ and $F_1(l_1)$ both being real and positive and l_1 is an unknown positive parameter

Now from (2.5)

$$\xi(\tfrac{1}{2} + it) = F_2(l_1) + F_1(l_1)\,\mathrm{Cos}\, l_1 t \qquad \ldots(2.6)$$

Therefore from (2.4) and (2.6)

$$\zeta(\tfrac{1}{2} + it) = \frac{2\,\pi^{\frac{1}{4}+\frac{it}{2}}\,[F_2(l_1) + F_1(l_1)\ \mathrm{Cos}\, l_1 t]}{\left(\frac{1}{2}+it\right)\left(-\frac{1}{2}+it\right)\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}$$

$$= \frac{2\,\pi^{\frac{1}{4}}\,\pi^{\frac{it}{2}}\,[F_2(l_1) + F_1(l_1)\ \mathrm{Cos}\, l_1 t]}{-\left(\frac{1}{4}+t^2\right)\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} \qquad \ldots(2.7)$$

Therefore

$$\left|\zeta(\tfrac{1}{2} + it)\right| = \frac{|2|\,\left|\pi^{\frac{1}{4}}\right|\left|\pi^{\frac{it}{2}}\right|\,|F_2(l_1) + F_1(l_1)\ \mathrm{Cos}\, l_1 t|}{\left|-\left(\frac{1}{4}+t^2\right)\right|\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|}$$

$$= \frac{2\,\pi^{\frac{1}{4}}\,.1.|F_2(l_1) + F_1(l_1)\ \mathrm{Cos}\, l_1 t|}{\left(\frac{1}{4}+t^2\right)\,\Gamma\left(\frac{1}{4}\right)\,\sqrt{\prod_{k=0}^{\infty}\frac{1}{1+\frac{\left(\frac{t}{2}\right)^2}{\left(\frac{1}{4}+K\right)^2}}}} \qquad \ldots(2.8)$$

Therefore from (2.1) using (2.8)

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+4t^2)} \log \left[\frac{2\,\pi^{\frac{1}{4}}.1.|F_2(l_1) + F_1(l_1)\, \mathrm{Cos}\, l_1 t|}{\left(\frac{1}{4}+t^2\right)\Gamma\left(\frac{1}{4}\right)\sqrt{\prod_{k=0}^{\infty}\frac{1}{1+\frac{\left(\frac{t}{2}\right)^2}{\left(\frac{1}{4}+K\right)^2}}}} \right] dt \qquad \ldots(2.9)$$

To evaluate the integral I in (2.9) we will use mean value theorem for definite integrals [4].

Equation (2.9) can be written as

$$I = \int_{-N}^{N} \frac{1}{(1+4t^2)} \log \left[\frac{2\,\pi^{\frac{1}{4}}.1.|F_2(l_1) + F_1(l_1)\, \mathrm{Cos}\, l_1 t|}{\left(\frac{1}{4}+t^2\right)\Gamma\left(\frac{1}{4}\right)\sqrt{\prod_{k=0}^{\infty}\frac{1}{1+\frac{\left(\frac{t}{2}\right)^2}{\left(\frac{1}{4}+K\right)^2}}}} \right] dt \qquad \ldots(2.10)$$

$N \to \infty$

Now the factor $\frac{1}{(1+4t^2)}$ is continually positive and decreasing in [-N, N]. And $\log \left[\frac{2\,\pi^{\frac{1}{4}}.1.|F_2(l_1) + F_1(l_1)\, \mathrm{Cos}\, l_1 t|}{\left(\frac{1}{4}+t^2\right)\Gamma\left(\frac{1}{4}\right)\sqrt{\prod_{k=0}^{\infty}\frac{1}{1+\frac{\left(\frac{t}{2}\right)^2}{\left(\frac{1}{4}+K\right)^2}}}} \right]$ is continuous in [-N, N] and > 0. Hence using mean value theorem for integrals [4] we can write from (2.10)

$$I = \frac{1}{1+4(-N)^2} \int_{-N}^{C} \log \left[\frac{2\,\pi^{\frac{1}{4}}.1.|F_2(l_1) + F_1(l_1)\, \mathrm{Cos}\, l_1 t|}{\left(\frac{1}{4}+t^2\right)\Gamma\left(\frac{1}{4}\right)\sqrt{\prod_{k=0}^{\infty}\frac{1}{1+\frac{\left(\frac{t}{2}\right)^2}{\left(\frac{1}{4}+K\right)^2}}}} \right] dt$$

$N \to \infty$

where $-N < C < N$

Therefore $$I = \frac{1}{1+4(N)^2}\int_{-N}^{C} \log\left[\frac{2\,\pi^{\frac{1}{4}}.1.|F_2(l_1)+F_1(l_1)\,\mathrm{Cos}\,l_1 t|}{\left(\frac{1}{4}+t^2\right)\Gamma\left(\frac{1}{4}\right)\sqrt{\prod_{k=0}^{\infty}\frac{1}{1+\frac{\left(\frac{t}{2}\right)^2}{\left(\frac{1}{4}+K\right)^2}}}}\right] dt$$

$N \to \infty$

$$= 0\ , \quad \text{as } \frac{1}{1+4(N)^2} \to 0 \text{ as } N \to \infty \qquad \ldots(2.11)$$

Therefore from (2.1) and (2.11) $I = \int_{-\infty}^{\infty} \frac{1}{(1+4t^2)} \log\left|\zeta\left(\frac{1}{2}+ it\right)\right| = 0$

This completes the proof of Riemann Hypothesis.

3. Sondow-Dumitrescu equivalent and its proof

This Riemann Hypothesis equivalent is known as Sondow-Dumitrescu criteria [5]. It states that Riemann Hypothesis is true if and only if for each fixed value of t, $|\xi(\sigma + it)|$ is strictly increasing for $\frac{1}{2} < \sigma < \infty$. Riemann Hypothesis is also true if and only if $|\xi(\sigma + it)|$ decreasing for $-\infty < \sigma < \frac{1}{2}$ for a fixed t. The proof of these are as follows

From (2.5)

$$\xi(s) = \xi(\sigma + it) = F_2(l_1) + F_1(l_1)\left[\mathrm{Cos}\, l_1 t\ \mathrm{Cos\,h}\, l_1\left(\sigma - \tfrac{1}{2}\right) + i\ \mathrm{Sin}\, l_1 t\ \mathrm{Sin\,h} l_1\left(\sigma - \tfrac{1}{2}\right)\right]$$

$$= \left\{F_2(l_1) + F_1(l_1)\ \mathrm{Cos}\, l_1 t\ \mathrm{Cos\,h}\, l_1\left(\sigma - \tfrac{1}{2}\right)\right\} + i\left\{F_1(l_1)\ \mathrm{Sin}\, l_1 t\ \mathrm{Sin\,h} l_1\left(\sigma - \tfrac{1}{2}\right)\right\} \quad \ldots(3.1)$$

where $F_2(l_1)$ and $F_1(l_1)$ both being real and positive and l_1 is an unknown positive parameter

Therefore

$$|\xi(\sigma + it)| = \sqrt{\left\{F_2(l_1) + F_1(l_1)\mathrm{Cos}\, l_1 t\ \mathrm{Cos\,h}\, l_1\left(\sigma - \tfrac{1}{2}\right)\right\}^2 + \left\{F_1(l_1)\mathrm{Sin}\, l_1 t\ \mathrm{Sin\,h} l_1\left(\sigma - \tfrac{1}{2}\right)\right\}^2} \quad \ldots(3.2)$$

Now, l_1, $F_2(l_1)$ and $F_1(l_1)$ are all positive [1]. And $\mathrm{Cos\,h}\, l_1\left(\sigma - \frac{1}{2}\right)$ and $\mathrm{Sinh} l_1\left(\sigma - \frac{1}{2}\right)$ are gradually increasing for $\frac{1}{2} < \sigma < \infty$. Hence $|\xi(\sigma + it)|$ is increasing for fixed value of t for $\frac{1}{2} < \sigma < \infty$. This clearly proves Sondow-Dumitrescu criteria for validity of Riemann Hypothesis.

And for $-\infty < \sigma < \frac{1}{2}$ the R.H.S of (3.2) is obviously strictly decreasing for a fixed value of t. Thus $|\xi(\sigma + it)|$ is strictly decreasing fixed value of t. This also confirms Riemann Hypothesis.

4. Conclusion

As both Balazard-Saias-Yor and Sondow-Dumitrescu criteria follows from above analysis it turns out that Riemann Hypothesis is true.

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