Galore of time independent invariants of one dimensional dissipative harmonic oscillator

Debasis Biswas
Chakdaha W. B – 741 222
India.
Email – biswasdebasis38@gmail.com

Abstract :

[In this work we find invariants of one dimensional dissipative harmonic oscillator from an elementary ansatz. It is shown that an elementary ansatz along with symmetry consideration yields new invariants of one dimensional dissipative harmonic oscillator.]

Key words : Invariant, dissipative harmonic oscillator, similarity variable

1. Introduction

Invariants or conservation laws are very important for investigation of mechanical systems. Generally knowledge of Lagrangian is essential for finding invariants of a dynamical system. Symmetry analysis is a very powerful tool to find invariants of a system. Among various symmetry approaches, Noether symmetry analysis [1] is well known for its elegance. It provides one to one correspondence between symmetry properties of Lagrangian and conservation laws. However in many cases without the knowledge of Lagrangian one can easily find conservation laws of the system. In this paper we shall find invariants of dissipative simple harmonic oscillator from an elementary ansatz.

2. Time independent invariants.

The differential equation for linearly damped harmonic oscillator is

$$\ddot{x} + \mu\dot{x} + \omega^2 x = 0 \qquad \ldots(2.1)$$

The Lagrangian of (2.1) is known as Caldirola-Kanai [2, 3] Lagrangian :

$$L = e^{\mu t}\left(\frac{\dot{x}^2}{2} - \frac{x^2}{2}\right) \qquad \ldots(2.2)$$

We shall pay no attention to the Lagrangian and assume an ansatz for invariant of (2.1) as

$$I = \dot{x}G = \text{Constant} \qquad \ldots(2.3)$$

$$\text{where } G = G(x, \dot{x}) \qquad \ldots(2.4)$$

Now

$$\dot{G} = \frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial \dot{x}}\ddot{x}$$

$$= \frac{\partial G}{\partial x}\dot{x} - (\mu\dot{x} + \omega^2 x)\frac{\partial G}{\partial \dot{x}} \;; \quad \text{using (2.1)} \qquad \ldots(2.5)$$

And from (2.3), taking derivative with respect to time

$$\ddot{x}G + \dot{x}\dot{G} = 0$$

$$\text{i,e., } \dot{x}\dot{G} - (\mu\dot{x} + \omega^2 x)G = 0 \;; \quad \text{using (2.1)} \qquad \ldots(2.6)$$

Using (2.5) one obtains from (2.6)

$$\dot{x}\left[\frac{\partial G}{\partial x}\dot{x} - (\mu\dot{x} + \omega^2 x)\frac{\partial G}{\partial \dot{x}}\right] - (\mu\dot{x} + \omega^2 x)G = 0$$

$$\text{i.e., } \frac{\partial G}{\partial x} - \frac{\partial G}{\partial \dot{x}}\left(\mu + \frac{\omega^2 x}{\dot{x}}\right) - \left(\frac{\mu}{\dot{x}} + \frac{\omega^2 x}{\dot{x}^2}\right)G = 0 \qquad \ldots(2.7)$$

To solve (2.7) for G, we seek a similarity variable ξ defined by

$$\xi = x^{\alpha}\dot{x}^{\beta} \quad ; \quad \alpha, \beta \text{ to be chosen later} \qquad \ldots(2.8)$$

$$\left.\begin{aligned} \text{Therefore } \frac{\partial}{\partial x} &\equiv \alpha x^{\beta-1}\dot{x}^{\beta}\frac{d}{d\xi} \\ \text{and } \frac{\partial}{\partial \dot{x}} &\equiv \beta\dot{x}^{\beta-1}x^{\alpha}\frac{d}{d\xi} \end{aligned}\right\} \qquad \ldots(2.9)$$

Using (2.9), equation (2.7) can be written as

$$\alpha x^{\alpha-1}\dot{x}^{\beta}\frac{dG}{d\xi} - \beta\dot{x}^{\beta-1}x^{\alpha}\frac{dG}{d\xi}\left(\mu + \frac{\omega^2 x}{\dot{x}}\right) - \left(\frac{\mu}{\dot{x}} + \frac{\omega^2 x}{\dot{x}^2}\right) G = 0$$

$$\text{i.e., } \frac{dG}{d\xi}\left[\alpha x^{\alpha-1}\dot{x}^{\beta} - \mu\beta\dot{x}^{\beta-1}x^{\alpha} - \beta\omega^2 x^{\alpha+1}\dot{x}^{\beta-2}\right] - \left(\frac{\mu}{\dot{x}} + \frac{\omega^2 x}{\dot{x}^2}\right) G = 0$$

$$\text{i.e., } \frac{dG}{d\xi}\left[1 - \frac{\mu\beta}{\alpha}\frac{x}{\dot{x}} - \frac{\beta\omega^2}{\alpha}\frac{x^2}{\dot{x}^2}\right] - \left[\frac{\mu}{\alpha}\frac{x^{1-\alpha}}{\dot{x}^{\beta+1}} + \omega^2\frac{x^{2-\alpha}}{\dot{x}^{2+\beta}}\right] G = 0 \qquad \ldots(2.10)$$

Case (i)

We now choose

$$\text{i)} \quad \alpha = 1, \quad \beta = -1 \qquad \ldots(2.11)$$

Hence from (2.8)

$$\xi = \frac{x}{\dot{x}} \qquad \ldots(2.12)$$

Using (2.11) and (2.12), equation (2.10) assumes a simplified form :

$$\frac{dG}{d\xi}\left[1 + \mu\xi + \omega^2\xi^2\right] = [\mu + \omega^2\xi]G$$

Therefore

$$\frac{dG}{G} = \frac{(\mu+\omega^2\xi)\,d\xi}{1+\mu\xi+\omega^2\xi^2}$$

Hence $\ln G = \mu\int \frac{1}{(1+\mu\xi+\omega^2\xi^2)}\,d\xi + \omega^2\int\frac{\xi}{(1+\mu\xi+\omega^2\xi^2)}\,d\xi$

$$\text{Therefore } \quad G = e^{\mu I_1 + \omega^2 I_2} \qquad \ldots(2.13)$$

$$\text{where } I_1 = \int\frac{d\xi}{1+\mu\xi+\omega^2\xi^2} \qquad \ldots(2.14)$$

$$\text{and } \quad I_2 = \int\frac{\xi\,d\xi}{1+\mu\xi+\omega^2\xi^2} \qquad \ldots(2.15)$$

Now Handbook of integrals [4] give

$$I_1 = \frac{1}{\sqrt{-\Delta}}\ln\frac{(\mu+2\omega^2\xi - \sqrt{-\Delta})}{(\mu+2\omega^2\xi + \sqrt{-\Delta})} \quad ; \quad \Delta = \xi\omega^2 - \mu^2 < 0$$

$$= \frac{1}{\sqrt{-\Delta}}\ln\frac{\left(\mu+2\omega^2\frac{x}{\dot{x}} - \sqrt{-\Delta}\right)}{\left(\mu+2\omega^2\frac{x}{\dot{x}} + \sqrt{-\Delta}\right)} \qquad \text{using (2.12)} \qquad \ldots(2.16)$$

$$= \frac{-2}{\mu+2\omega^2\xi} \quad ; \quad \Delta = 0$$

$$= \frac{-2}{\mu+2\omega^2\frac{x}{\dot{x}}} \qquad \text{using (2.12)} \qquad \ldots(2.17)$$

$$= \frac{2}{\sqrt{\Delta}}\tan^{-1}\frac{\mu+2\omega^2\xi}{\sqrt{\Delta}} \; ; \quad \Delta > 0$$

$$= \frac{2}{\sqrt{\Delta}}\tan^{-1}\frac{\mu+2\omega^2\;\frac{x}{\dot{x}}}{\sqrt{\Delta}} \qquad \text{using (2.12)} \qquad \ldots(2.18)$$

And $I_2 = \frac{1}{2\omega^2}\ln\left(1+\mu\xi+\omega^2\xi^2\right) - \frac{\mu}{2\omega^2} I_1$

$$= \frac{1}{2\omega^2}\ln\left(1+\mu\frac{x}{\dot{x}}+\omega^2\frac{x^2}{\dot{x}^2}\right) - \frac{\mu}{2\omega^2} I_1 \qquad \ldots(2.19)$$

Finally from (2.3), using (2.13) and (2.16), (2.17), (2.18) and (2.19) we get an invariant of (2.1) :

$$\dot{x}e^{\mu I_1+\omega^2 I_2} = \text{Constant} \qquad \ldots(2.20)$$

where I_1 and I_2 are given by (2.16), (2.17), (2.18) and (2.19).

Case ii)

To find another invariant of (2.1) we choose

$$\text{ii)} \quad \left.\begin{matrix}\alpha = -1 \\ \beta = 1\end{matrix}\right\} \qquad \ldots(2.21)$$

Then from (2.8)

$$\xi = \frac{\dot{x}}{x} \qquad \ldots(2.22)$$

And from (2.10)

$$\frac{dG}{d\xi}\left[1+\frac{\mu}{\xi}+\frac{\omega^2}{\xi^2}\right] - \left[\frac{\mu}{\xi^2}+\frac{\omega^2}{\xi^3}\right]G = 0$$

Hence $\frac{dG}{G} = \frac{\frac{\mu}{\xi^2}\quad d\xi}{\left[1+\frac{\mu}{\xi}+\frac{\omega^2}{\xi^2}\right]} + \frac{\frac{\omega^2}{\xi^3}\quad d\xi}{\left[1+\frac{\mu}{\xi}+\frac{\omega^2}{\xi^2}\right]} = \frac{\mu\, d\xi}{[\xi^2+\mu\xi+\omega^2]} + \frac{\omega^2\, d\xi}{\xi[\xi^2+\mu\xi+\omega^2]}$

Thus $\ln G = \mu\int\frac{d\xi}{\xi^2+\mu\xi+\omega^2} + \omega^2\int\frac{d\xi}{\xi\left(\xi^2+\mu\xi+\omega^2\right)}$

Therefore

$$G = e^{\mu I_3+\omega^2 I_4} \qquad \ldots(2.23)$$

Where $I_3 = \int\frac{d\xi}{\left(\xi^2+\mu\xi+\omega^2\right)}$ …(2.24)

$$I_4 = \int\frac{d\xi}{\xi\left(\xi^2+\mu\xi+\omega^2\right)} \qquad \ldots(2.25)$$

From table of integrals [4]

$$I_3 = -\frac{2}{\sqrt{-\Delta}}\tanh^{-1}\frac{\mu+2\omega^2\xi}{\sqrt{-\Delta}} \; ; \quad \Delta = 4\omega^2-\mu^2 < 0$$

$$= -\frac{2}{\sqrt{-\Delta}}\tanh^{-1}\frac{\mu+2\omega^2\;\frac{\dot{x}}{x}}{\sqrt{-\Delta}} \qquad \text{using (2.22)} \qquad \ldots(2.26)$$

$$= \frac{-2}{\mu+2\omega^2\xi} \; ; \quad \Delta = 0$$

$$= \frac{-2}{\mu + 2\omega^2 \frac{\dot{x}}{x}} \qquad \text{using (2.22)} \qquad \ldots(2.27)$$

$$I_3 = -\frac{2}{\sqrt{\Delta}} \tan^{-1} \frac{\mu + 2\omega^2 \xi}{\sqrt{\Delta}} \;; \qquad \Delta > 0$$

$$= -\frac{2}{\sqrt{\Delta}} \tan^{-1} \frac{\mu + 2\omega^2 \frac{\dot{x}}{x}}{\sqrt{\Delta}} \qquad \text{using (2.22)} \qquad \ldots(2.28)$$

and

$$I_4 = \frac{1}{2\omega^2} \ln \frac{\xi^2}{(\xi^2 + \mu\xi + \omega^2)} - \frac{\mu}{2\omega^2} I_3$$

$$= \frac{1}{2\omega^2} \ln \frac{\left(\frac{\dot{x}}{x}\right)^2}{\left(\frac{\dot{x}^2}{x^2} + \mu \frac{\dot{x}}{x} + \omega^2\right)} - \frac{\mu}{2\omega^2} I_3 \qquad \text{using (2.22)} \qquad .(2.29)$$

We thus get another invariant of (2.1) :
From (2.3), using (2.23) and (2.26), (2.27), (2.28) and (2.29)

$$\dot{x}\, G = \dot{x} e^{\mu I_3 + \omega^2 I_4} = \text{Constant} \qquad \ldots(2.30)$$

I_3 and I_4 are given by (2.26) to (2.29).

Likewise we can get many more invariants of (2.1) by assigning any arbitrary value of n to α and – n to β or in other words taking α = n and β = – n we can find galore of invariants of dissipative linear harmonic oscillator.

3. Conclusion and Comments.

Construction of invariants of a dynamical system is an important part of theoretical study. The above method of finding invariants has not been used so far. Time independent and time dependent invariants of dissipative harmonic oscillator have been worked out by various authors [4, 5, 6] using various methods. Using above method one can explicitly determine as many as invariants of dissipative harmonic oscillator as one wishes. However as equation (2.1) has two independent constants, only two of invariants calculated will be functionally independent.

References

1. E. Noether — Invariante Variations probleme Nach. Ges. Gottingen (1918)
2. P. Caldirda — Nuovo Cim. 18, 393 (1941)
3. E. kanai — Prog. Theo. Phys. 3, 440 (1949)

4.	I. S. Gradshteyn and I. M. Ryzhik	Table of Integrals, series and Products Eighth editioin, Elseveir
5.	N. A. Lemos	Am. J. Phys. 47, 857 (1979)
6.	A. Choudhuri et al	Pramana Vol 70. No.4 (2008)
7.	V. Tapia	ICTP preprint IC/96/161, (1996)