Quaternion Space-Time and Matter

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In this work, we use the key assumption that division algebras play a key role in description and prediction of natural phenomena. Consequently, we use division algebra of quaternions to describe the four-dimensional space-time intervals. Then, we demonstrate that the quaternion space-time together with the finite speed of signal propagation allow for a simple, intuitive understanding of the space-time interval transformation during arbitrary motion between a signal source and observer. We derive a quaternion form of the Lorentz time dilation and suggest that it's real scalar norm is the traditional form of the Lorentz transformation, representing experimental measurements of the space-time interval. Thus, the new quaternion theory is inseparable from the experimental process. We determine that the space-time interval in the observer reference frame is given by a conjugate quaternion expression, which is essential for a proper definition of quaternion gradient operator. Then, we apply the quaternion gradient to an arbitrary quaternion potential function, which leads to the unified expressions of force fields. The second quaternion differentiation results in the unified Maxwell equations. Finally, we apply the resulting unified formalism to electromagnetic and gravitational interactions and show that the new expressions are similar to the traditional equations, with the novel terms related to scalar fields and velocity dependent components. Furthermore, we obtain two types of force fields and four types of matter density expressions, which require further theoretical and experimental study. Therefore, the new mathematical framework based on quaternion algebra and quaternion calculus may serve as the foundation for a unified theory of space-time and matter, leading to a useful enhancement of the traditional theories of special and general relativity.

I. INTRODUCTION

In this work, we are driven by natural curiosity rather than practical utility and hope to achieve a more intuitive understanding of physical theory. The goal is to develop a conceptual basis for a unified theory of space-time and matter, which naturally requires a unified and robust mathematical foundation. The key assumption of this work states that division algebras are valid and essential for description and prediction of natural phenomena. In other words, we assume that mathematical properties of division algebras form the foundation of physical laws. This is conceptually different from the traditional physical research method, where physical experience is described by a variety of convenient mathematical methods, regardless of their universal validity.

We begin by proposing the division algebra of real quaternions [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], as an alternative to the traditional mathematical formalism of the relativity theory [11], [12], [13], [14]. Our goal is not to reproduce the conceptual results of relativity, but to achieve an alternative interpretation of the Lorentz time transformation [15], [16].

We define the four-dimensional quaternion space-time interval and proceed by deducing physical interpretation and experimental methods that follow directly from the mathematical properties of quaternions. Note that in our approach, experimental results are inseparable from theoretical predictions and form a single coherent framework of mathematical laws and physical experience.

An interesting method of quaternion differentiation was developed by Jack [17], [18]. Then, Dunning-Davies and Norman considered using a similar method for the gravitational field [19]. Application of the electromagnetic analogy to gravitation was previously extensively studied [20], [21], [22], [23], [24], [25], [26].

We switch from quaternion algebra to quaternion calculus by introducing a proper form of the quaternion gradient operator. We apply this differentiation procedure to a general quaternion potential in order to derive the unified forms of the force fields and Maxwell equations. The new components in electromagnetic and gravitational interactions may lead to exciting new discoveries of physical phenomena.

Therefore, using mathematical tools of quaternion algebra that were widely available at the end of the nineteenth century, we derive a novel unified theory of space-time and matter that may enhance the theories of special and general relativity.

II. QUATERNION SPACE-TIME

Curiously, there are only three associative division algebras in existence: real numbers, complex numbers, and quaternions, as proven by the Frobenius division algebra theorem [27].

Historically, Rodrigues [1] introduced quaternion multiplication while searching for a method to describe ro-

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tation of three-dimensional solids. His discovery can be considered the precursor to quaternion division algebra, which was formally introduced and extensively studied by Hamilton [2], [3], who discovered quaternions while searching for a well-defined mathematical division in three-dimensional space. Hamilton spent over twenty years developing quaternion theory and was quoted saying: "Time is said to have only one dimension, and space to have three dimensions. The mathematical quaternion partakes of both these elements" [4]. Thus, in Hamilton's view, time is a real scalar and space is a three-dimensional imaginary vector, which seems like a brilliant insight predating the discovery of modern four-dimensional space-time.

The key advantages of quaternion algebra over other mathematical methods are: a positive Euclidean norm, description of both rotation and propagation in three dimensions, and most importantly, a well-defined division leading to the quaternion differentiation and calculus.

Since the algebra of real quaternions is the only four-dimensional division algebra, we introduce the four-dimensional quaternion manifold,

$$\boldsymbol{\tau}^4 = (\,\hat{\tau}_0\,,\vec{\tau}_1,\vec{\tau}_2,\vec{\tau}_3) = (\,\hat{\imath}_0\tau_0\,,\vec{\imath}_1\tau_1,\vec{\imath}_2\tau_2,\vec{\imath}_3\tau_3)\,, \qquad (1)$$

which we identify with time [10] in order to facilitate an intuitive physical interpretation. Here, $\hat{\imath}_0$, is a unity directional scalar, $\vec{\imath}_1,\vec{\imath}_2,\vec{\imath}_3$, are purely imaginary unit vectors, and $\tau_0,\tau_1,\tau_2,\tau_3 \in \mathbb{R}$, are real numbers. The relationships between the Euclidean quaternion units, $\hat{\imath}_0,\vec{\imath}_1,\vec{\imath}_2,\vec{\imath}_3$, are essential for the present theory and are defined using complex multiplication [2] as,

$$\begin{cases} \hat{\imath}_0\,\hat{\imath}_0 = \hat{\imath}_0 = 1\,, \\ \vec{\imath}_1\,\vec{\imath}_1 = \vec{\imath}_2\,\vec{\imath}_2 = \vec{\imath}_3\,\vec{\imath}_3 = \vec{\imath}_1\,\vec{\imath}_2\,\vec{\imath}_3 = -\hat{\imath}_0 = -1\,, \\ \vec{\imath}_1\,\vec{\imath}_2 = \vec{\imath}_3, \quad \vec{\imath}_2\,\vec{\imath}_3 = \vec{\imath}_1, \quad \vec{\imath}_3\,\vec{\imath}_1 = \vec{\imath}_2\,, \\ \vec{\imath}_2\,\vec{\imath}_1 = -\vec{\imath}_3, \quad \vec{\imath}_3\,\vec{\imath}_2 = -\vec{\imath}_1, \quad \vec{\imath}_1\,\vec{\imath}_3 = -\vec{\imath}_2\,. \end{cases} \qquad (2)$$

In the current work, we use a scalar coefficient of proportionality between space and time, c, which is equal to the maximum saturation velocity in isotropic materials [10], for example, the speed of light in vacuum. This allows us to express four-dimensional space-time in terms of the four-dimensional quaternion time expression,

$$\boldsymbol{t} = \left(\,\hat{\imath}_0 t_0\,,\vec{\imath}_1\frac{x_1}{c},\vec{\imath}_2\frac{x_2}{c},\vec{\imath}_3\frac{x_3}{c}\,\right), \qquad (3)$$

where, $x_1,x_2,x_3 \in \mathbb{R}$, are real Cartesian space coordinates. Interestingly, this may be reminiscent of the modern approach to distance measurement with the Global Positioning System, using electromagnetic wave time-of-flight with the help of at least four satellites [28].

Thus using (3), we established quaternion time as a coordinate point location in the four-dimensional space-

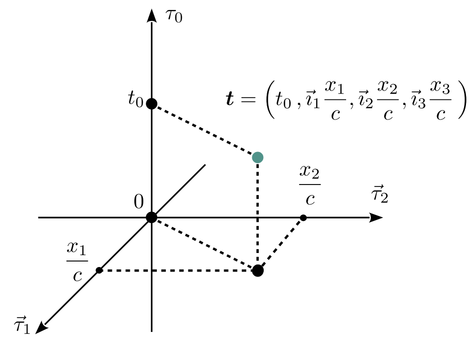

FIG. 1. A three-dimensional representation of the quaternion time-point.

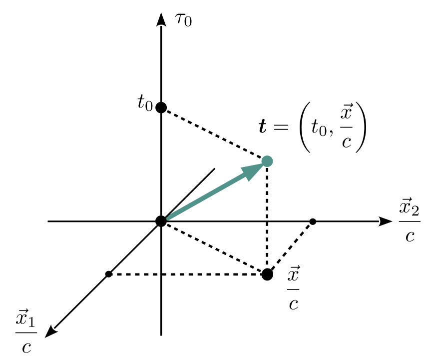

FIG. 2. A three-dimensional representation of the quaternion time interval.

time coordinate system, which we can express in the following simplified notation,

$$\boldsymbol{t} = (t_0,\,\vec{t}_v\,) = \left(t_0,\,\frac{\vec{x}}{c}\right)\,, \qquad (4)$$

where $\vec{x}$ is a pure imaginary space vector, as originally introduced by Hamilton [2]

$$\vec{x} = (\,\vec{\imath}_1x_1,\vec{\imath}_2x_2,\vec{\imath}_3x_3\,)\,, \qquad (5)$$

and, t_0 is a real scalar time,

$$t_0 = \hat{\imath}_0 t_0 = t_0\,. \qquad (6)$$

The space-time coordinate point (4) is defined relative to the quaternion zero-point,

$$\mathbf{0} = \left(\,0\,,\vec{0}\,\right) = (\,\hat{\imath}_0 0\,,\vec{\imath}_1 0,\vec{\imath}_2 0,\vec{\imath}_3 0\,)\,. \qquad (7)$$

Consequently, a quaternion space-time coordinate point is represented by a four-dimensional quaternion interval starting at the zero-point and ending at the coordinate point defined by (4).

In Fig. 1 and Fig. 2, we show diagrams of a space-time point and a quaternion interval respectively using a three-dimensional representation, with a real scalar dimension, τ_0, and two imaginary vector dimensions, $\vec{\tau}_1 = \vec{x}_1/c$ and $\vec{\tau}_2 = \vec{x}_2/c$. For simplicity of visualization, we neglect the third vector dimension, $\vec{\tau}_3 = \vec{x}_3/c$.

As usual, we define the norm or length of the quaternion space-time interval,

$$t = |\boldsymbol{t}| = \sqrt{\boldsymbol{t}\,\bar{\boldsymbol{t}}} = \sqrt{\bar{\boldsymbol{t}}\,\boldsymbol{t}}\,, \tag{8}$$

where we use the conjugate quaternion time definition,

$$\bar{\boldsymbol{t}} = \left(t_0\,,\, -\vec{t}_v\ \right) = \left(t_0,\, -\frac{\vec{x}}{c} \right)\,. \tag{9}$$

Therefore, we defined the quaternion space-time coordinate system and populated it with quaternion intervals.

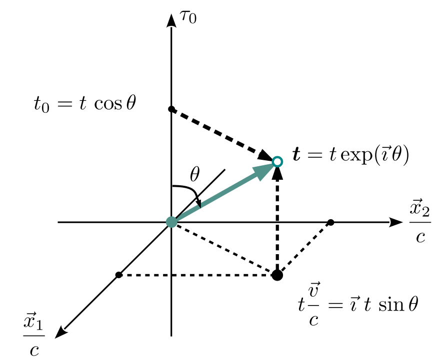

FIG. 3a. Polar quaternion form of the Lorentz time interval dilation.

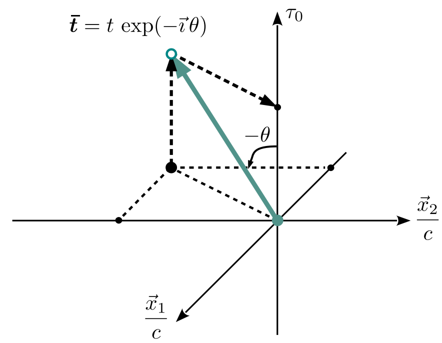

FIG. 3b. Conjugate polar form of the Lorentz time interval dilation.

III. POLAR REPRESENTATION OF QUATERNION SPACE-TIME INTERVALS

Let us assume that the quaternion space-time interval signifies a transition in space-time from the quaternion zero-point to a space location, $\vec{x}$.

To describe this motion, we introduce a vector velocity,

$$\vec{v} = \frac{\vec{x}}{t}\,, \tag{10}$$

where, $\vec{x}$, is a space vector and, $t = |\boldsymbol{t}|$, is the absolute value of the space-time interval given by (8). Note that previously we defined a full quaternion form of velocity [10].

Then, we write the quaternion space-time interval in terms of its norm and vector velocity,

$$\begin{cases} \boldsymbol{t} = (t_0,\, \vec{t}_v\,) = \left(t_0\,, \dfrac{\vec{v}}{c} t \right)\,, \\[2ex] \bar{\boldsymbol{t}} = (t_0,\, -\vec{t}_v\,) = \left(t_0\,, -\dfrac{\vec{v}}{c} t \right), \end{cases} \tag{11}$$

where we note that the quaternion space-time interval consists of two components: a scalar time, t_0, and a vector, $\vec{t}_v = (\vec{v}/c)\,t$, which appears as a high-speed correction due to motion.

Next, we introduce a purely imaginary unit-vector,

$$\vec{\imath} = \frac{\vec{x}}{x} = \frac{\vec{v}}{v}\,, \tag{12}$$

which signifies the direction of motion. Finally from (11) and (12), we express the quaternion space-time interval in polar form as in [29],

$$\begin{cases} \boldsymbol{t} = t\left(\dfrac{t_0}{t}, \dfrac{\vec{v}}{c} \right) = t\,\left(\cos\theta\,,\, \vec{\imath}\sin\theta\right) = t\,\exp\left(\vec{\imath}\theta\right)\,, \\[2ex] \bar{\boldsymbol{t}} = t\left(\dfrac{t_0}{t}\,, -\dfrac{\vec{v}}{c} \right) = t\left(\cos\theta, -\vec{\imath}\sin\theta\right) = t\,\exp\left(-\vec{\imath}\theta\right), \end{cases} \tag{13}$$

where the angle, θ, is a function of the velocity, $\vec{v}$, and is defined as,

$$\begin{cases} \cos\theta = \dfrac{t_0}{t} = \sqrt{1 - \dfrac{v^2}{c^2}}\,, \\[2ex] \sin\theta = \dfrac{v}{c}. \end{cases} \tag{14}$$

We see from (10), (12), and (14) that the maximum velocity is,

$$\vec{v} = \vec{c} = \vec{\imath}c\,, \tag{15}$$

where, c, is the absolute value of the maximum speed in isotropic materials [10]. Then, we obtain the full polar form of the space-time interval from (13) and (14),

$$\begin{cases} \boldsymbol{t} = \dfrac{t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}\ \exp\left(\vec{\imath}\theta\right)\,, \\[4ex] \bar{\boldsymbol{t}} = \dfrac{t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}\ \exp\left(-\vec{\imath}\theta\right)\,, \end{cases} \tag{16}$$

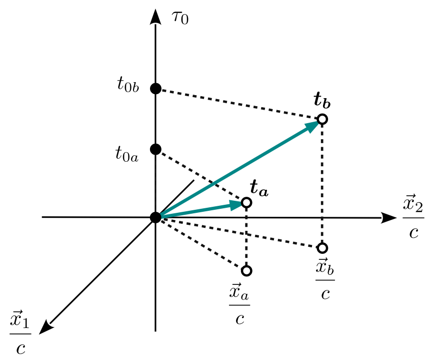

FIG. 4. A three-dimensional representation of two arbitrary time points.

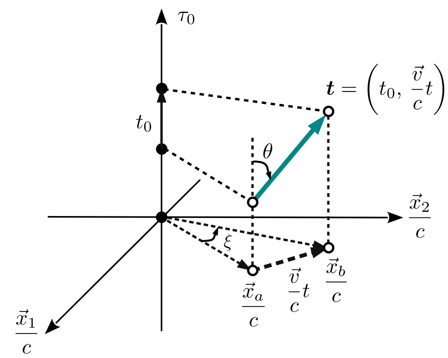

FIG. 5a. A simplified three-dimensional representation of the arbitrary quaternion space-time interval.

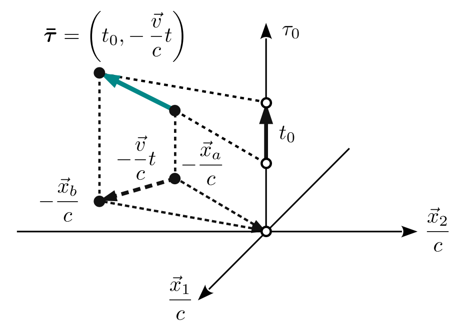

FIG. 5b. A simplified three-dimensional representation of the conjugate quaternion space-time interval.

which we interpret as a quaternion form of the Lorentz time dilation.

Now, we can easily determine the scalar norm of the quaternion time interval from (16),

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \,, \tag{17}$$

which we immediately recognize as the traditional scalar form of the Lorentz time dilation.

In Fig. 3a and Fig. 3b, we demonstrate diagrams of a quaternion space-time interval and its conjugate.

IV. SPACE-TIME TRANSFORMATIONS FOR ARBITRARY INTERVALS

Next, let us define an arbitrary quaternion space-time interval as the difference between any two quaternion space-time points, $\boldsymbol{t_a}$ and $\boldsymbol{t_b}$, as can be seen in Fig. 4,

$$\begin{cases} \boldsymbol{t_a} = \left(t_{0a}\,, \dfrac{\vec{x}_a}{c}\right), \\[2ex] \boldsymbol{t_b} = \left(t_{0b}, \dfrac{\vec{x}_b}{c}\right). \end{cases} \tag{18}$$

Let us calculate the space-time interval,

$$\boldsymbol{t} = \boldsymbol{t_b} - \boldsymbol{t_a} = \left(t_0, \frac{\vec{x}}{c}\right) = \left(t_0, \frac{\vec{v}}{c}\,t\right), \tag{19}$$

where we define the velocity with arbitrary direction as, $\vec{v} = \vec{x}/t$. Similarly, for the conjugate quaternion,

$$\bar{\boldsymbol{t}} = \bar{\boldsymbol{t}}_{\boldsymbol{b}} - \bar{\boldsymbol{t}}_{\boldsymbol{a}} = \left(t_0, -\frac{\vec{x}}{c}\right) = \left(t_0, -\frac{\vec{v}}{c}\,t\right). \tag{20}$$

In Fig. 5a and Fig. 5b, we demonstrate an arbitrary quaternion space-time interval and its conjugate. Note that in Fig. 5a the angle, ξ, represents rotation in the three-dimensional space, while the angle, θ, represents velocity of propagation based on the same expressions for the quaternion space-time interval as in (14) and (16).

We can now express the final space-time point in terms of the initial point and the quaternion space-time interval from (16), (19), and (20), which is a quaternion form of the space-time transformation for arbitrary motion,

$$\begin{cases} \boldsymbol{t}_b = \boldsymbol{t}_a + \boldsymbol{t} = \boldsymbol{t}_a + \dfrac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \exp\left(\vec{\imath}\theta\right)\,, \\[3ex] \bar{\boldsymbol{t}}_b = \bar{\boldsymbol{t}}_a + \bar{\boldsymbol{t}} = \bar{\boldsymbol{t}}_a + \dfrac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \exp\left(-\vec{\imath}\theta\right)\,. \end{cases} \tag{21}$$

These transformations for space-time are similar to expressions of the special theory of relativity, however because of the quaternion origin, they are truly four-dimensional and should be suitable for representation of arbitrary motion.

Let us now consider a stationary case where, $v = 0$, as in Fig. 6a and Fig. 6b. We calculate the space-time

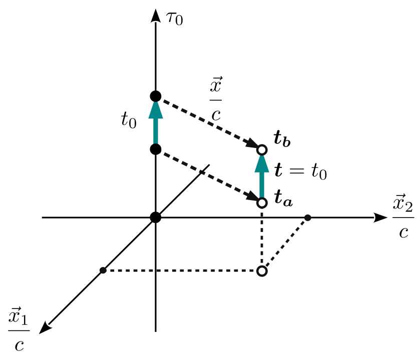

FIG. 6a. A three-dimensional representation of the stationary space-time interval.

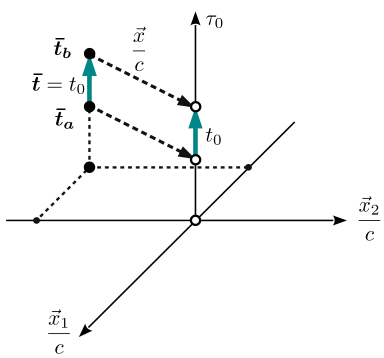

FIG. 6b. The stationary conjugate space-time interval.

interval using (21),

$$\boldsymbol{t} = \boldsymbol{t_b} - \boldsymbol{t_a} = t_0 = t. \tag{22}$$

Similarly, for the conjugate interval, we obtain,

$$\bar{\boldsymbol{t}} = \bar{\boldsymbol{t}}_{\boldsymbol{b}} - \bar{\boldsymbol{t}}_{\boldsymbol{a}} = t_0 = t\,. \tag{23}$$

Therefore in the stationary case, the absolute value of the space-time interval is equal to the stationary value, t_0.

We realize that the quaternion space-time interval, $\boldsymbol{t}$, depends on both the absolute value of speed, v, and the vector direction of motion, $\vec{\imath} = \vec{v}/v$. On the other hand, the absolute value of the space-time interval, t, is a function of the absolute speed only. Consequently using quaternion algebra, we can express any arbitrary propagation including the stationary case.

V. PHYSICAL INTERPRETATION OF QUATERNION SPACE-TIME INTERVALS

Next, we will elaborate on the physical meaning of quaternion space-time intervals including the general transformation defined by (21).

Let us assume the existence of time sources such as clocks, and signal detectors, such as observers with measuring instruments.

Let us consider a stationary clock located on a train platform, which we designate as a signal source. First, we perform an experiment in the source reference frame of the stationary clock, where the location of the clock is the zero of space, $\vec{x} = 0$. Next, let us consider an observer with a video camera on a train passing the platform at midnight, when the time on the platform clock is zero. We assume that the train is moving along a straight track with a constant vector velocity, $\vec{v}$. The observer synchronizes the camera timer with the platform clock at midnight and then starts filming the time on the platform clock while simultaneously observing the time-stamp of the camera.

After synchronization, the starting time for both the platform clock and the observer camera timer is zero. The observer stops filming when the time, t_0, is observed on the platform clock. What would be the time-stamp on the observer's camera at the end of the recording? Due to the finite speed of light propagation, we expect that the recorded time of the platform clock will appear delayed relative to the time-stamp of the observer's camera. Also, we expect that the delay should be a function of the train speed relative to the speed of light as the light signal from the clock is chasing the observer on the moving train.

Let us express mathematically a simple model of the thought experiment. Assume that the last photon left the clock at the end of the time interval, t_0, in the source reference frame. Then, the photon traveled along the vector, $\vec{x}$, with velocity, c, arriving at the observer location at the end of the space-time interval, $\boldsymbol{t}$. Thus, we summarize using quaternions,

$$\boldsymbol{t} = \left(t_0\,, \vec{0}\right) + \left(0\,, \frac{\vec{x}}{c}\right) = \left(t_0\,, \frac{\vec{x}}{c}\right) = \left(t_0\,, \frac{\vec{v}}{c}\,t\right)\,. \tag{24}$$

Note that the vector correction, $\vec{x}/c = (\vec{v}/c)\,t$, can be considered a vector representation of the photon time-of-flight. Thus, we can think of the quaternion space-time interval as the a stationary time interval and a time-of-flight vector correction at a distance, justifying our choice of the quaternion time to describe the four-dimensional space-time. Therefore, we represent the space-time interval in the source reference frame by a quaternion (17),

$$\boldsymbol{t} = \left(t_0\,, \frac{\vec{v}}{c}t\right) = \frac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}}\,\exp\left(\vec{\imath}\theta\right)\,, \tag{25}$$

as presented in Fig. 3a in a polar quaternion form.

Also, let us suggest that the measured time interval on the camera time-stamp is a real scalar value, equal to the quaternion norm of the interval given by (17), which is the Lorentz time interval dilation generally accepted as a verified experimental result.

Next, let us consider the same experiment in the observer's reference frame. Clearly, we expect to obtain the same experimental result even though the platform is now moving away from the observer with a constant velocity $-\vec{v}$. The starting time of the measurement is the clock synchronization equal to zero, as in the source reference frame. However, the end time-point is now given by the conjugate quaternion due to space inversion when switching from the source to the observer reference frame,

$$\bar{\boldsymbol{t}} = \left(t_0 \,, -\frac{\vec{v}}{c}\, t \right) = \frac{t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} \, \exp\left(-\vec{\imath}\theta\right) \,, \tag{26}$$

as shown in Fig. 3b.

We can repeat the experiment with the clock located some distance away from the train trajectory, thus leading to a more general time interval presented in Fig. 5a and Fig. 5b and calculated by (21). The measured space-time interval duration in the observer reference frame is again given by (17), due to the symmetry of the quaternion norm relative to conjugation.

Furthermore, the solution can be applied to stationary clocks and observers as in Fig. 6a and Fig. 6b, where we note that the measured time interval is equal to the stationary time, t_0 from (22) and (23). This is due to photon time-of-flight canceling out at the beginning and end of the measured interval. As expected, the measured time duration remains the same despite the conjugate form of the space-time interval in the observer reference frame. In general, our physical interpretation resembles the relativistic Doppler effect approach of [14], however using quaternion mathematical formalism.

Therefore, the quaternion expressions (25) and (26) describe the space-time transformations, expressed by a linear vector correction due to motion, $(\vec{v}/c)\, t$. The regular quaternion, $\boldsymbol{t}$, describes the space-time interval transformation in the source reference frame, while the quaternion conjugate, $\bar{\boldsymbol{t}}$, describes the interval in the observer reference frame with a negative sign of the vector correction, $-(\vec{v}/c)t$. For a stationary case, the measured time interval is unchanged and equal to the source value, t_0.

The unidirectional progression of time appears to be one of the most fundamental physical properties of nature. Yet the direction of time is not reflected in most physical expressions. Eddington popularized the concept of the arrow-of-time based on thermodynamic considerations [30]. Recently, a quaternion arrow-of-time was suggested [9].

It is a common physical convention that the positive time direction is from the past and into the future. Therefore, we assume that the time direction of the zero-point interval, $t_0 = \hat{\imath}_0 t_0$, is indicated by the directional scalar $\hat{\imath}_0$. Because of the finite speed of light propagation, any signal leaving a signal source propagates into the future, while any signal received by an observer originated from the past. Consequently, the direction of signal propagation in the three-dimensional space is indicated by the sign of the three-dimensional vector component, $\pm(\vec{v}/c)t$, which we identified as the vector representation of the photon time-of-flight. The direction of the quaternion space-time interval signifies the arrow-of-time, which is positive in the direction from start-to-finish.

Remarkably, both quaternion expressions for the time interval transformation (25) and (26) carry the embedded values of the measured time interval at the signal source, t_0, and at the remote observer, t. The experimentally measured intervals are connected together by the scalar Lorentz time dilation (17), which is dependent only on the absolute velocity, v, relative to the maximum saturation velocity, c, regardless of the direction of motion. Since we did not make any assumptions, about the type of motion inertial or otherwise, we expect the the new quaternion space-time interval transformation (25), (26), and (17) will have universal validity for any arbitrary motion.

Therefore, our physical interpretation of quaternion space-time intervals includes concepts of experimental measurement at-a-distance and arrow-of-time, as integral parts of the quaternion space-time theory.

VI. INVERSE TIME INTERVAL - QUATERNION FREQUENCY, ENERGY, AND MASS

Let us consider a periodic signal source such as a clock on a kitchen wall or a distant quasar. Assume that such a signal source produces a wave signal with a period described by the time interval, t_0. Then, assume that the periodic signal in the observer reference frame can be described by a wave with a period, t. Let us introduce the definition of quaternion frequency, $\boldsymbol{\omega}$, by using the definition of the quaternion multiplicative inverse,

$$\begin{cases} 2\pi\, \boldsymbol{t}^{\,\boldsymbol{-1}} = \dfrac{2\pi\bar{\boldsymbol{t}}}{t^2} = \bar{\boldsymbol{\omega}}, \\ \\ 2\pi\, \bar{\boldsymbol{t}}^{\,\boldsymbol{-1}} = \dfrac{2\pi\boldsymbol{t}}{t^2} = \boldsymbol{\omega}, \end{cases} \tag{27}$$

Note that the above definition is possible due to well defined quaternion division. Also, note that in the source reference frame, where the space-time interval is represented by a regular quaternion, $\boldsymbol{t}$, the frequency assumes the conjugate form, $\bar{\boldsymbol{\omega}}$. On the other hand, in the observer reference frame, the space-time interval is given by a complex-conjugate quaternion, $\bar{\boldsymbol{t}}$, resulting in the normal quaternion frequency, $\boldsymbol{\omega}$, due to mathematical inversion during quaternion division.

Since we are primarily interested in physical results in the observer reference frame, we assume that the quaternion space-time interval is given by the conjugate quaternion, $\bar{\boldsymbol{t}}$, while the frequency is given by the regular quaternion, $\boldsymbol{\omega}$. Then, we calculate frequency in the observer reference frame using (25) and (27),

$$\boldsymbol{\omega} = \omega\left(\frac{t_0}{t}\,,\,\frac{\vec{v}}{c}\right) = \left(\omega_0\,,\,\frac{\vec{v}}{c}\omega\right), \tag{28}$$

where we define the absolute value of frequency,

$$\omega = |\boldsymbol{\omega}| = |\bar{\boldsymbol{\omega}}| = \frac{2\pi}{t}\,, \tag{29}$$

and the zero-point frequency from (17) and (28),

$$\omega_0 = \omega\frac{t_0}{t} = \omega\sqrt{1-\frac{v^2}{c^2}}\,. \tag{30}$$

We see from (28) that quaternion frequency can be described as static zero-point frequency, ω_0, and a linear vector correction, $(\vec{v}/c)\,\omega$, due to motion.

Let us assume that we can express energy from frequency in the observer reference frame using (28),

$$\boldsymbol{\epsilon} = \hbar\,\boldsymbol{\omega} = \left(\epsilon_0\,,\,\frac{\vec{v}}{c}\,\epsilon\right), \tag{31}$$

where, $\hbar$, is the reduced Planck constant and, $(\vec{v}/c)\,\epsilon$, is the linear vector correction to the rest energy, ϵ_0.

Next, we apply a similar procedure to express quaternion mass from energy using (31),

$$\boldsymbol{m} = \frac{\boldsymbol{\epsilon}}{c^2} = \frac{\hbar\,\boldsymbol{\omega}}{c^2} = \left(m_0\,,\,\frac{\vec{v}}{c}\,m\right) = \left(m_0\,,\,\frac{\vec{p}}{c}\right), \tag{32}$$

where we define the momentum as, $\vec{p} = m\vec{v}$, while, $(\vec{v}/c)m = \vec{p}/c$, becomes a vector correction to the stationary rest-mass, m_0.

This leads to the traditional energy-momentum relation from (31) and (32),

$$\epsilon^2 = m_0^2c^4 + p^2c^2. \tag{33}$$

For small velocities, $v \ll c$, and $\vec{p} \sim 0$, we obtain the scalar rest mass using (30) and (32),

$$\boldsymbol{m} \simeq m_0 = \frac{\hbar\omega}{c^2}\sqrt{1-\frac{v^2}{c^2}}\,, \tag{34}$$

which is a scalar quantity clearly showing the wave origin of the rest-mass concept, possibly as a standing wave.

For large velocities, $v \simeq c$, we derive an approximation,

$$\begin{cases} m_0 = m\sqrt{1-\dfrac{v^2}{c^2}} \simeq 0\,, \\[2ex] \dfrac{\vec{v}}{c} \simeq \dfrac{\vec{c}}{c} \simeq \vec{\imath}. \end{cases} \tag{35}$$

This brings us to an approximation for particles with a zero rest-mass, such as photons or electrons in graphene [31],

$$\boldsymbol{m} \simeq (0\,,\,\vec{\imath}m) = \left(0\,,\,\vec{\imath}\,\frac{\hbar\,\omega}{c^2}\right), \tag{36}$$

which is a pure vector quantity, showing that movement with the maximum saturation velocity, c, results in zero rest-mass and that the vector mass correction gets a life of its own.

Therefore, it appears that all physical quantities under observation in the configuration space, such as frequency, energy, and mass are essentially the same quaternion quantity, differentiated only by a constant scalar coefficient. Each one consists of a static component measured at rest and a positive vector correction, proportional to, $\vec{v}/c$, which accounts for motion. This is different from the space-time interval in the observer reference frame, which has a negative vector correction proportional to, $-\vec{v}/c$. As usual, the measured value of any quaternion quantity is given by a real scalar norm, similar to the Lorentz transformation.

VII. THE FIRST QUATERNION DERIVATIVE AND THE UNIFIED FORCE FIELDS

Let us assume that any physical interaction can be described by a general interaction potential, which we define by analogy with other quaternion quantities in the configuration space,

$$\boldsymbol{\phi} = (\phi_0\,,\,\vec{\phi}_v\,) = \left(\phi_0\,,\,\frac{\vec{v}}{c}\phi\right), \tag{37}$$

where, ϕ_0, is the scalar static potential at the signal source, while, $\vec{\phi}_v = (\vec{v}/c)\phi$, is a linear vector potential correction due to motion.

Next, we take advantage of quaternion division in order to deduce proper quaternion gradient operators in the source and observer reference frames. Using the definition of the quaternion multiplicative inverse, as in (27),

$$\begin{cases} \boldsymbol{t}^{-1} = \dfrac{\bar{\boldsymbol{t}}}{t^2}, \\[2ex] \bar{\boldsymbol{t}}^{-1} = \dfrac{\boldsymbol{t}}{t^2}, \end{cases} \tag{38}$$

we can determine the proper sign of the vector component in the quaternion gradient operators, $\boldsymbol{\nabla}$ and $\bar{\boldsymbol{\nabla}}$. As before (27), the correct form of the quaternion gradient operator assumes the conjugate form, $\bar{\boldsymbol{\nabla}}$, in the source reference frame, where the space-time interval is given by a regular quaternion, $\boldsymbol{t}$. On the other hand, in the observer reference frame, where the space-time interval is represented by the conjugate quaternion, $\bar{\boldsymbol{t}}$, the expression for the gradient operator has the regular form, $\boldsymbol{\nabla}$, due to space inversion during division in (38). Therefore, we deduce,

$$\begin{cases} \bar{\boldsymbol{\nabla}} = \dfrac{1}{c}\,\dfrac{d}{d\boldsymbol{t}} = \left(\dfrac{\partial}{c\partial t_0}, -\vec{\imath}_1\dfrac{\partial}{\partial x_1}, -\vec{\imath}_2\dfrac{\partial}{\partial x_2}, -\vec{\imath}_3\dfrac{\partial}{\partial x_3}\right), \\ \boldsymbol{\nabla} = \dfrac{1}{c}\,\dfrac{d}{d\bar{\boldsymbol{t}}} = \left(\dfrac{\partial}{c\partial t_0}, +\vec{\imath}_1\dfrac{\partial}{\partial x_1}, +\vec{\imath}_2\dfrac{\partial}{\partial x_2}, +\vec{\imath}_3\dfrac{\partial}{\partial x_3}\right). \end{cases} \tag{39}$$

We write the resulting four-dimensional gradients in the simplified vector notation as,

$$\begin{cases} \bar{\boldsymbol{\nabla}} = \left(\dfrac{\partial}{c\partial t_0}\,, -\vec{\nabla}\right) = \left(\nabla_0\,, -\vec{\nabla}\right), \\ \boldsymbol{\nabla} = \left(\dfrac{\partial}{c\partial t_0}\,,\ \vec{\nabla}\right) = \left(\nabla_0\,,\ \vec{\nabla}\right). \end{cases} \tag{40}$$

Since we are primarily interested in the reference frame of the measuring apparatus, which is the observer reference frame, we will use the conjugate quaternion time interval, $\bar{\boldsymbol{t}}$ and the regular form of the gradient operator, $\boldsymbol{\nabla}$.

Let us suggest that the negative quaternion gradient of the potential function, $\boldsymbol{\phi}$, can be interpreted as the unified quaternion force field, similar to electromagnetic interaction. Then, we apply the definitions of quaternion multiplication for any two quaternions, $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\begin{cases} \boldsymbol{a}\,\boldsymbol{b} = \left(a_0 b_0 - \vec{a}\cdot\vec{b},\ a_0\vec{b} + b_0\vec{a} + \vec{a}\times\vec{b}\right), \\ \boldsymbol{b}\,\boldsymbol{a} = \left(a_0 b_0 - \vec{a}\cdot\vec{b},\ a_0\vec{b} + b_0\vec{a} - \vec{a}\times\vec{b}\right), \end{cases} \tag{41}$$

and obtain two force fields as the gradients of the general potential function, $\boldsymbol{\phi}$, with a customary minus sign as in electromagnetic interaction,

$$\begin{cases} \boldsymbol{\mathcal{F}}^{+} = -\boldsymbol{\phi}\,\boldsymbol{\nabla}, \\ \boldsymbol{\mathcal{F}}^{-} = -\boldsymbol{\nabla}\boldsymbol{\phi}, \end{cases} \tag{42}$$

where we use a mathematical convention that, $\boldsymbol{\phi}\,\boldsymbol{\nabla}$, is a quaternion gradient operator multiplying the function, $\boldsymbol{\phi}$, on the right side, while, $\boldsymbol{\nabla}\boldsymbol{\phi}$, is the gradient operator multiplying, $\boldsymbol{\phi}$, on the left, as introduced by Jack [17].

The two derivative expressions in (42), and consequently the two types of forces, are due to the non-commutative nature of quaternion multiplication (41).

Then using (37), (41), and (42), we obtain quaternion expressions for the total unified force fields,

$$\begin{cases} \boldsymbol{\mathcal{F}}^{+} = \left(-\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v,\ -\dfrac{\partial\vec{\phi}_v}{c\partial t_0} - \vec{\nabla}\phi_0 + \vec{\nabla}\times\vec{\phi}_v\right), \\ \boldsymbol{\mathcal{F}}^{-} = \left(-\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v,\ -\dfrac{\partial\vec{\phi}_v}{c\partial t_0} - \vec{\nabla}\phi_0 - \vec{\nabla}\times\vec{\phi}_v\right). \end{cases} \tag{43}$$

Note that the two expressions differ by the sign of the vector multiplication term, as expected from (41). We suggest that (42) and (43) represent the unified force fields for two different types of matter particles, similar to the Lorentz force field for electromagnetic interaction.

Let us look for the single-valued functions that determine the field components, by using commutator and anti-commutator relations [17],

$$\begin{cases} \boldsymbol{\mathcal{F}_a} = \dfrac{1}{2}\left(\boldsymbol{\mathcal{F}}^{+} + \boldsymbol{\mathcal{F}}^{-}\right), \\ \boldsymbol{\mathcal{F}_c} = \dfrac{1}{2}\left(\boldsymbol{\mathcal{F}}^{+} - \boldsymbol{\mathcal{F}}^{-}\right). \end{cases} \tag{44}$$

This brings us to two types of unified quaternion field components derived from (43) and (44),

$$\begin{cases} \boldsymbol{\mathcal{F}_a} = \left(\mathcal{F}_0\,,\ \vec{\mathcal{F}}_a\right) = \left(-\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v\,,\ -\dfrac{\partial\vec{\phi}_v}{c\partial t_0} - \vec{\nabla}\phi_0\right), \\ \boldsymbol{\mathcal{F}_c} = \vec{\mathcal{F}}_c = \vec{\nabla}\times\vec{\phi}_v\,. \end{cases} \tag{45}$$

We see that the first field component, $\boldsymbol{\mathcal{F}_a}$, is a full quaternion, with both a scalar component, $\mathcal{F}_0$, and a driving vector component, $\vec{\mathcal{F}}_a$, while the second field component, $\boldsymbol{\mathcal{F}_c} = \vec{\mathcal{F}}_c$, is a pure vector torsion field.

Using the velocity dependent vector potential, $\phi_v = (\vec{v}/c)\phi$, we can express the scalar and vector field components in the velocity dependent form,

$$\begin{cases} \mathcal{F}_0 = -\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v = -\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\left(\phi\,\dfrac{\vec{v}}{c}\right), \\ \vec{\mathcal{F}}_a = -\vec{\nabla}\phi_0 - \dfrac{\partial\vec{\phi}_v}{c\partial t_0} = -\vec{\nabla}\phi_0 - \dfrac{\partial(\phi\,\vec{v})}{c^2\partial t_0}, \\ \vec{\mathcal{F}}_c = \vec{\nabla}\times\vec{\phi}_v = \vec{\nabla}\times\left(\phi\,\dfrac{\vec{v}}{c}\right). \end{cases} \tag{46}$$

Note that the torsion field component, $\vec{\mathcal{F}}_c$, has only velocity dependent terms and therefore does not exist in the absence of motion. The existence of the velocity dependent scalar and vector field components is a unique novel feature of this theory.

We re-write the unified force fields in terms of the field components from (44),

$$\begin{cases} \boldsymbol{\mathcal{F}}^{+} = \boldsymbol{\mathcal{F}_a} + \boldsymbol{\mathcal{F}_c} \\ \boldsymbol{\mathcal{F}}^{-} = \boldsymbol{\mathcal{F}_a} - \boldsymbol{\mathcal{F}_c}\,, \end{cases} \tag{47}$$

where the scalar and vector field components can be substituted from (45) and (46).

Thus, we obtained unified field equations for an arbitrary physical interaction defined by a quaternion potential function, $\boldsymbol{\phi}$. These fields combine into two expressions (47) for the unified force fields, $\boldsymbol{\mathcal{F}}^+$ and $\boldsymbol{\mathcal{F}}^-$, describing two different kinds of matter. We believe that the novel scalar and vector field components (46), which are velocity dependent, are some of the the most remarkable results of the unified quaternion field theory.

VIII. THE SECOND DERIVATIVE AND THE UNIFIED MAXWELL EQUATIONS

Next, let us introduce the unified density of matter for continuous media, which we write in the observer frame as a quaternion,

$$\boldsymbol{\rho} = \left(\rho_0,\, \frac{\vec{v}}{c}\,\rho\right) = \left(\rho_0,\, \frac{\vec{j}}{c}\right), \tag{48}$$

where, ρ_0, is the static matter density and, $(\vec{v}/c)\rho = \vec{j}/c$, is the vector current density due to motion.

Let us assume that the second derivative of the potential function, $\boldsymbol{\phi}$, can be interpreted similar to Poisson's equation, as the unified density of matter, $\boldsymbol{\rho}$.

Using the density of matter expression (48), we apply the left and right quaternion differentiation to the two kinds of the unified force fields, $\boldsymbol{\mathcal{F}}^+$ and $\boldsymbol{\mathcal{F}}^-$. We obtain by applying (42),

$$\begin{cases} \boldsymbol{\rho}^+ = \boldsymbol{\mathcal{F}}^+\boldsymbol{\nabla} = (-\boldsymbol{\phi}\boldsymbol{\nabla})\boldsymbol{\nabla} = -\boldsymbol{\phi}\boldsymbol{\nabla}\boldsymbol{\nabla}, \\ \boldsymbol{\rho}^- = \boldsymbol{\nabla}\boldsymbol{\mathcal{F}}^- = \boldsymbol{\nabla}(-\boldsymbol{\nabla}\boldsymbol{\phi}) = -\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\phi}, \\ \boldsymbol{\rho}^\pm = \boldsymbol{\nabla}\boldsymbol{\mathcal{F}}^+ = \boldsymbol{\nabla}(-\boldsymbol{\phi}\boldsymbol{\nabla}) = -\boldsymbol{\nabla}\boldsymbol{\phi}\boldsymbol{\nabla}, \\ \boldsymbol{\rho}^\mp = \boldsymbol{\mathcal{F}}^-\boldsymbol{\nabla} = (-\boldsymbol{\nabla}\boldsymbol{\phi})\boldsymbol{\nabla} = -\boldsymbol{\nabla}\boldsymbol{\phi}\boldsymbol{\nabla}. \end{cases} \tag{49}$$

Thus, we see four types of matter density due to four second derivatives, resulting from the lack of commutativity in quaternion differentiation. Note that the last two expressions in (49) appear identical because quaternion multiplication is distributive.

We can write matter density in terms of the quaternion field components using (47),

$$\begin{cases} \boldsymbol{\rho}^+ = \boldsymbol{\mathcal{F}}^+\boldsymbol{\nabla} = \boldsymbol{\mathcal{F}_a}\boldsymbol{\nabla} + \boldsymbol{\mathcal{F}_c}\boldsymbol{\nabla}, \\ \boldsymbol{\rho}^- = \boldsymbol{\nabla}\boldsymbol{\mathcal{F}}^- = \boldsymbol{\nabla}\boldsymbol{\mathcal{F}_a} - \boldsymbol{\nabla}\boldsymbol{\mathcal{F}_c}, \\ \boldsymbol{\rho}^\pm = \boldsymbol{\nabla}\boldsymbol{\mathcal{F}}^+ = \boldsymbol{\nabla}\boldsymbol{\mathcal{F}_a} + \boldsymbol{\nabla}\boldsymbol{\mathcal{F}_c}, \\ \boldsymbol{\rho}^\mp = \boldsymbol{\mathcal{F}}^-\boldsymbol{\nabla} = \boldsymbol{\mathcal{F}_a}\boldsymbol{\nabla} - \boldsymbol{\mathcal{F}_c}\boldsymbol{\nabla}. \end{cases} \tag{50}$$

First, we calculate the left and right derivatives of the unified quaternion field, $\boldsymbol{\mathcal{F}_a}$, using quaternion multiplication (41) as before,

$$\begin{cases} \boldsymbol{\nabla}\,\boldsymbol{\mathcal{F}_a} = \left(\nabla_0\mathcal{F}_0 - \vec{\nabla}\cdot\vec{\mathcal{F}}_a\ ,\ \nabla_0\vec{\mathcal{F}}_a + \vec{\nabla}\mathcal{F}_0 + \vec{\nabla}\times\vec{\mathcal{F}}_a\right), \\ \boldsymbol{\mathcal{F}_a}\,\boldsymbol{\nabla} = \left(\nabla_0\mathcal{F}_0 - \vec{\nabla}\cdot\vec{\mathcal{F}}_a\ ,\ \nabla_0\vec{\mathcal{F}}_a + \vec{\nabla}\mathcal{F}_0 - \vec{\nabla}\times\vec{\mathcal{F}}_a\right). \end{cases} \tag{51}$$

Similarly, we obtain for the unified torsion field, $\boldsymbol{\mathcal{F}_c}$, using (41),

$$\begin{cases} \boldsymbol{\nabla}\,\boldsymbol{\mathcal{F}_c} = \left(\vec{\nabla}\cdot\vec{\mathcal{F}}_c\ ,\ \nabla_0\vec{\mathcal{F}}_c + \vec{\nabla}\times\vec{\mathcal{F}}_c\right), \\ \boldsymbol{\mathcal{F}_c}\,\boldsymbol{\nabla} = \left(\vec{\nabla}\cdot\vec{\mathcal{F}}_c\ ,\ \nabla_0\vec{\mathcal{F}}_c - \vec{\nabla}\times\vec{\mathcal{F}}_c\right). \end{cases} \tag{52}$$

Since the divergence of the curl is zero,

$$\vec{\nabla}\cdot\vec{\mathcal{F}}_c = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{\phi}_v\right) = 0, \tag{53}$$

we derive the final result for the unified torsion field,

$$\begin{cases} \boldsymbol{\nabla}\,\boldsymbol{\mathcal{F}_c} = \left(0\ ,\ \nabla_0\vec{\mathcal{F}}_c + \vec{\nabla}\times\vec{\mathcal{F}}_c\right), \\ \boldsymbol{\mathcal{F}_c}\,\boldsymbol{\nabla} = \left(0\ ,\ \nabla_0\vec{\mathcal{F}}_c - \vec{\nabla}\times\vec{\mathcal{F}}_c\right). \end{cases} \tag{54}$$

Also, from the last two expressions of (49) and (50), we derive an equality,

$$\frac{\partial\vec{\mathcal{F}}_c}{c\partial t_0} + \vec{\nabla}\times\vec{\mathcal{F}}_a = 0. \tag{55}$$

Writing quaternion density (50) in terms of scalar and vector components, (51) and (54), we obtain a complete set of quaternion equations for the four distinct types of unified density of matter,

$$\begin{cases} \boldsymbol{\rho}^+ = \left(-\dfrac{\partial\mathcal{F}_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\mathcal{F}}_a, -\dfrac{\partial\vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 - \vec{\nabla}\times\vec{\mathcal{F}}_c + 2\dfrac{\partial\vec{\mathcal{F}}_c}{c\partial t_0}\right), \\ \boldsymbol{\rho}^- = \left(-\dfrac{\partial\mathcal{F}_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\mathcal{F}}_a, -\dfrac{\partial\vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 - \vec{\nabla}\times\vec{\mathcal{F}}_c - 2\dfrac{\partial\vec{\mathcal{F}}_c}{c\partial t_0}\right), \\ \boldsymbol{\rho}^\pm = \left(-\dfrac{\partial\mathcal{F}_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\mathcal{F}}_a, -\dfrac{\partial\vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 + \vec{\nabla}\times\vec{\mathcal{F}}_c\right), \\ \boldsymbol{\rho}^\mp = \left(-\dfrac{\partial\mathcal{F}_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\mathcal{F}}_a, -\dfrac{\partial\vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 + \vec{\nabla}\times\vec{\mathcal{F}}_c\right). \end{cases} \tag{56}$$

Note that the four matter density components result from the two types of differentiation of the two force field expressions (43).

Using the matter density expressions (48) and (56), together with supplementary conditions (53) and (55), we derive the complete set of unified Maxwell equations,

$$\begin{cases} \rho_0 = -\dfrac{\partial \mathcal{F}_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\mathcal{F}}_a, \\ \vec{j}^{\,+} = -\dfrac{\partial \vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 - \vec{\nabla}\times\vec{\mathcal{F}}_c + 2\dfrac{\partial \vec{\mathcal{F}}_c}{c\partial t_0}, \\ \vec{j}^{\,-} = -\dfrac{\partial \vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 - \vec{\nabla}\times\vec{\mathcal{F}}_c - 2\dfrac{\partial \vec{\mathcal{F}}_c}{c\partial t_0}, \\ \vec{j}^{\,\pm} = -\dfrac{\partial \vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 + \vec{\nabla}\times\vec{\mathcal{F}}_c, \\ \vec{j}^{\,\mp} = -\dfrac{\partial \vec{\mathcal{F}}_a}{c\partial t_0} - \vec{\nabla}\mathcal{F}_0 + \vec{\nabla}\times\vec{\mathcal{F}}_c, \\ \dfrac{\partial \vec{\mathcal{F}}_c}{c\partial t_0} + \vec{\nabla}\times\vec{\mathcal{F}}_a = 0, \\ \vec{\nabla}\cdot\vec{\mathcal{F}}_c = 0. \end{cases} \tag{57}$$

Thus, we derived a new set of unified Maxwell equations, where, ρ_0, represents the static density of matter and the four different currents densities result from the four different second derivatives of the quaternion potential. Importantly, the unified force fields (43) and the unified Maxwell equations (57) were derived by differentiation directly from the quaternion interaction potential, thus avoiding incompatibility between potentials, fields, and densities of matter. Furthermore, the unified quaternion theory is fundamentally four-dimensional and preserves the intrinsic validity of the scalar Lorentz time transformation. This seems to imply that no gauge fixing is needed or possible, as it would upset the self-consistency of the theory.

IX. QUATERNION ELECTROMAGNETIC FIELDS AND MAXWELL EQUATIONS

Let us consider electromagnetic interaction expressed by a quaternion potential, $\boldsymbol{\phi}$, written in the observer reference frame,

$$\frac{\boldsymbol{\phi}}{\varepsilon} = (\phi_0\,,\,\vec{\phi}_v\,) = \left(\phi_0\,,\,\frac{\vec{v}}{c_\varepsilon}\phi\right). \tag{58}$$

Here, ε, is a coefficient of proportionality accounting for material properties, similar to relative electrical permittivity. Also, c_ε, represents maximum saturation velocity, or the speed of light, in a particular material.

Next, by inspection we introduce the electric and magnetic fields using (44), (46),

$$\begin{cases} \dfrac{\boldsymbol{\mathcal{F}_a}}{\varepsilon} = \boldsymbol{\mathcal{E}} = \left(\mathcal{E}_0\,,\,\vec{\mathcal{E}}\right), \\ \dfrac{\boldsymbol{\mathcal{F}_c}}{\varepsilon} = \boldsymbol{\mathcal{B}} = \left(0,\,\vec{\mathcal{B}}\right), \end{cases} \tag{59}$$

where we identify the scalar electric field, $\mathcal{E}_0$, the driving electric field, $\vec{\mathcal{E}}$, and the torsion magnetic field, $\vec{\mathcal{B}}$, which we obtain from the corresponding unified field components (45).

Next, we derive the full expressions for the electromagnetic fields from (45) and (59) while using the velocity dependent definition of the vector potential, $\vec{\phi}_v = (\vec{v}/c_\varepsilon)\phi$,

$$\begin{cases} \mathcal{E}_0 = -\dfrac{\partial\phi_0}{c_\varepsilon\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v = -\dfrac{\partial\phi_0}{c_\varepsilon\partial t_0} + \vec{\nabla}\cdot\left(\phi\,\dfrac{\vec{v}}{c_\varepsilon}\right), \\ \vec{\mathcal{E}} = -\vec{\nabla}\phi_0 - \dfrac{\partial\vec{\phi}_v}{c_\varepsilon\partial t_0} = -\vec{\nabla}\phi_0 - \dfrac{\partial(\phi\,\vec{v})}{c_\varepsilon^2\partial t_0}, \\ \vec{\mathcal{B}} = \vec{\nabla}\times\vec{\phi}_v = \vec{\nabla}\times\left(\phi\,\dfrac{\vec{v}}{c_\varepsilon}\right). \end{cases} \tag{60}$$

These are novel expressions for the electromagnetic fields showing dependence on both the electrostatic potential and the velocity-dependent vector potential. We clearly see that the scalar and vector electric fields have both stationary as well as velocity dependent components, while the magnetic vector field is purely velocity dependent.

Furthermore, the magnetic field in (60) originates from the high-speed correction to the potential function, despite the fact that we experience effects of magnetism in everyday life even at low speeds, pointing to a relatively high impact of motion on magnetic interaction.

Note that the scalar electric field was previously derived by Jack [17] and discussed by Dunning-Davies and Norman [19]. Jack suggested [17], [18] that the scalar electric field may be related to thermal electricity. The scalar electric field may explain the work on scalar waves by Tesla [35], which were recently discussed and demonstrated experimentally by Meyl [36].

For the quasi-stationary case, $\vec{v}/c \sim 0$, the electromagnetic fields can be approximated as,

$$\begin{cases} \mathcal{E}_0 \simeq -\dfrac{\partial\phi_0}{c_\varepsilon\partial t_0}, \\ \vec{\mathcal{E}} \simeq -\vec{\nabla}\phi_0, \\ \vec{\mathcal{B}} \simeq \vec{0}. \end{cases} \tag{61}$$

Remarkably, the scalar electric field is still present for high-frequency variations of electrostatic potential. For the static case, when $\partial\phi_0/(c\partial t_0) \sim 0$, we obtain the classical electrostatic field expressions,

$$\begin{cases} \vec{\mathcal{E}} \simeq -\vec{\nabla}\phi_0 \,, \\ \\ \vec{\mathcal{B}} \simeq \vec{0} \,. \end{cases} \tag{62}$$

Using the unified force fields (47), we obtain the electromagnetic force fields, which can be considered a new quaternion form of the Lorentz force field equations,

$$\begin{cases} \dfrac{\boldsymbol{\mathcal{F}}^{+}}{\varepsilon} = \left(\mathcal{E}_0 \,,\; \vec{\mathcal{E}} + \vec{\mathcal{B}}\right) \,, \\ \\ \dfrac{\boldsymbol{\mathcal{F}}^{-}}{\varepsilon} = \left(\mathcal{E}_0 \,,\; \vec{\mathcal{E}} - \vec{\mathcal{B}}\right) \,, \end{cases} \tag{63}$$

where the values of the scalar electric field, $\mathcal{E}_0$, vector electric field, $\vec{\mathcal{E}}$, and vector magnetic field, $\vec{\mathcal{B}}$, are given by (60). Note that the magnetic field switches direction in the two equations, which predicts the existence of positive and negative electric charges, as expected from the Hall effect [16], [32]. Also, note from (60) that the magnetic field includes a curl of the vector velocity and consequently does not require additional multiplication by velocity as appear in the traditional Lorentz force expression. Therefore, we expect the traditional Lorentz field to be an approximation of the quaternion force field equations (63), which requires further verification.

Next, using the unified Maxwell equations (57), with the supplementary electromagnetic conditions (53) and (55), we obtain the new form of the electromagnetic Maxwell equations,

$$\begin{cases} \dfrac{\rho_0}{\varepsilon} = \vec{\nabla}\cdot\vec{\mathcal{E}} - \dfrac{\partial\mathcal{E}_0}{c_\varepsilon\partial t_0}, \\ \\ \dfrac{\vec{j}^{\,+}}{\varepsilon} = -\dfrac{\partial\vec{\mathcal{E}}}{c_\varepsilon\partial t_0} - \vec{\nabla}\mathcal{E}_0 - \vec{\nabla}\times\vec{\mathcal{B}} + 2\dfrac{\partial\vec{\mathcal{B}}}{c_\varepsilon\partial t_0}, \\ \\ \dfrac{\vec{j}^{\,-}}{\varepsilon} = -\dfrac{\partial\vec{\mathcal{E}}}{c_\varepsilon\partial t_0} - \vec{\nabla}\mathcal{E}_0 - \vec{\nabla}\times\vec{\mathcal{B}} - 2\dfrac{\partial\vec{\mathcal{B}}}{c_\varepsilon\partial t_0}, \\ \\ \dfrac{\vec{j}^{\,\pm}}{\varepsilon} = -\dfrac{\partial\vec{\mathcal{E}}}{c_\varepsilon\partial t_0} - \vec{\nabla}\mathcal{E}_0 + \vec{\nabla}\times\vec{\mathcal{B}}, \\ \\ \dfrac{\vec{j}^{\,\mp}}{\varepsilon} = -\dfrac{\partial\vec{\mathcal{E}}}{c_\varepsilon\partial t_0} - \vec{\nabla}\mathcal{E}_0 + \vec{\nabla}\times\vec{\mathcal{B}}, \\ \\ \vec{\nabla}\times\vec{\mathcal{E}} + \dfrac{\partial\vec{\mathcal{B}}}{c_\varepsilon\,\partial t_0} = 0 \,, \\ \\ \vec{\nabla}\cdot\vec{\mathcal{B}} = 0 \,. \end{cases} \tag{64}$$

The impact of the magnetic field on the current density in the new Maxwell equations (64) is more complicated than predicted by the simplified theory of the Hall effect [32]. Specifically, two of the current density equations in (64) include a time-varying term in the direction of the magnetic field, $\partial\vec{\mathcal{B}}/(c_\varepsilon\partial t_0)$, which may explain the anomalous Hall effect [37].

The new current density equations include a gradient of a scalar function, $\vec{\nabla}\mathcal{E}_0$, which may correspond to the main current component in solid-state materials, which is calculated as a gradient of the quasi-Fermi potentials [38]. Also, the four separate kinds of current density may correspond to the four different types of charges in solid-state: electrons, holes, positive and negative ions [38].

Therefore, we derived novel expressions for the electromagnetic Lorentz fields and Maxwell equations directly from the quaternion unified field theory. The novel components of the resulting equations may predict new phenomena, requiring further theoretical and experimental investigation.

X. QUATERNION GRAVITATIONAL FIELDS AND GRAVITATIONAL MAXWELL EQUATIONS

Next, let us assume that the gravitational fields and gravitational matter densities can be obtained from a quaternion potential function (37) by differentiation. However, we need to multiply the potential by the gravitational constant, $-G$, to account for the difference of gravitational interaction strength relative to electromagnetic interaction. Also, the minus sign signifies gravitational attraction of two bodies with the same mass. This leads to the following quaternion form of the gravitational potential,

$$-G\boldsymbol{\phi} = \left(\phi_0 \,,\, \vec{\phi}_v\,\right) = \left(\phi_0 \,,\, \frac{\vec{v}}{c_g}\phi\right), \tag{65}$$

where c_g is the speed of gravitational interaction in a particular material. Let us assume that the gravitational fields can be obtained via multiplication of the unified fields (42) by the gravitational constant, $-G$, resulting in two expressions for the gravitational field components,

$$\begin{cases} -G\,\boldsymbol{\mathcal{F}_a} = \boldsymbol{\varGamma} = \left(\varGamma_0 \,,\; \vec{\varGamma}\right) \,, \\ \\ -G\,\boldsymbol{\mathcal{F}_c} = \boldsymbol{\varOmega} = \left(0,\; \vec{\varOmega}\right) \,. \end{cases} \tag{66}$$

The components of the gravitational field include the novel scalar field, $\varGamma_0$, in addition to two vector fields, $\vec{\varGamma}$, and, $\vec{\varOmega}$, which are the gravitational driving field and the gravitational torsion field respectively. We calculate the gravitational field components from the unified field expressions (46) as,

$$
\begin{cases}
\mathit{\Gamma}_0 = -\dfrac{\partial\phi_0}{c_g\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v = -\dfrac{\partial\phi_0}{c_g\partial t_0} + \vec{\nabla}\cdot\left(\phi\,\dfrac{\vec{v}}{c_g}\right)\,, \\[2ex]
\vec{\mathit{\Gamma}} = -\vec{\nabla}\phi_0 - \dfrac{\partial\vec{\phi}_v}{c_g\partial t_0} = -\vec{\nabla}\phi_0 - \dfrac{\partial(\phi\,\vec{v})}{c_g^2\partial t_0}\,, \\[2ex]
\vec{\mathit{\Omega}} = \vec{\nabla}\times\vec{\phi}_v = \vec{\nabla}\times\left(\phi\,\dfrac{\vec{v}}{c_g}\right)\,.
\end{cases}
\tag{67}
$$

Based on (60) and (67), we realize that the scalar and vector components of the electromagnetic and gravitational fields are essentially the same, while the total quaternion fields, (59) and (66), differ by the relative strength shown by the electrical permittivity, ε, and the gravitational constant, $-G$.

Assuming small variations of the gravitational potential with time, $\partial\phi_0/\partial t_0 \sim 0$, and $\partial\vec{\phi}/\partial t_0 \sim 0$, we obtain an approximate form of the gravitational field,

$$
\begin{cases}
\mathit{\Gamma}_0 \simeq= \vec{\nabla}\cdot\left(\phi\,\dfrac{\vec{v}}{c_g}\right)\,, \\[2ex]
\vec{\mathit{\Gamma}} \simeq -\vec{\nabla}\phi_0, \\[2ex]
\vec{\mathit{\Omega}} \simeq \vec{\nabla}\times\left(\phi\,\dfrac{\vec{v}}{c_g}\right)\,,
\end{cases}
\tag{68}
$$

which is a new form of the gravitational field expressions for slow varying fields. The scalar gravitational field is still present, as well as the torsion gravitational vector field. For the stationary case, when, $\vec{v} \sim 0$, the field expressions reduce to,

$$
\begin{cases}
-G\boldsymbol{\mathcal{F}}^{+} = -G\boldsymbol{\mathcal{F}}^{-} = \vec{\mathit{\Gamma}} \simeq -\vec{\nabla}\phi_0\,, \\[2ex]
\vec{\mathit{\Omega}} \simeq \vec{0}\,,
\end{cases}
\tag{69}
$$

which is the gravitational static field, while the scalar field and the vector torsion field are absent.

Using the unified force field expressions (47) and (66) we derive new quaternion expressions for the gravitational force fields similar to electromagnetic interaction,

$$
\begin{cases}
-G\,\boldsymbol{\mathcal{F}}^{+} = \left(\mathit{\Gamma}_0\,,\ \vec{\mathit{\Gamma}} + \vec{\mathit{\Omega}}\right)\,, \\[2ex]
-G\,\boldsymbol{\mathcal{F}}^{-} = \left(\mathit{\Gamma}_0\,,\ \vec{\mathit{\Gamma}} - \vec{\mathit{\Omega}}\right)\,,
\end{cases}
\tag{70}
$$

where the values of the scalar gravitational field, $\mathit{\Gamma}_0$, gravitational driving field, $\vec{\mathit{\Gamma}}$, and gravitational torsion field, $\vec{\mathit{\Omega}}$ are given by (67). The two gravitational force fields, $-G\,\boldsymbol{\mathcal{F}}^{+}$ and $-G\,\boldsymbol{\mathcal{F}}^{-}$, differ by the direction of the torsion field, $\vec{\mathit{\Omega}}$, similar to the magnetic field in electromagnetic field equations. Therefore, we interpret the two field expressions as the two types of gravitational matter with the opposite mass sign.

Again, we assume that for calculation of the gravitational matter density, $-G\boldsymbol{\rho}$, replaces unified matter density $\boldsymbol{\rho}$. Then, from (57) and (67), we derive a new form of the gravitational Maxwell equations, with the supplementary gravitational conditions (53) and (55),

$$
\begin{cases}
-G\rho_0 = \vec{\nabla}\cdot\vec{\mathit{\Gamma}} - \dfrac{\partial\mathit{\Gamma}_0}{c_g\partial t_0}, \\[2ex]
-G\vec{j}^{\,+} = -\dfrac{\partial\vec{\mathit{\Gamma}}}{c_g\partial t_0} - \vec{\nabla}\mathit{\Gamma}_0 - \vec{\nabla}\times\vec{\mathit{\Omega}} + 2\dfrac{\partial\vec{\mathit{\Omega}}}{c_g\,\partial t_0}, \\[2ex]
-G\vec{j}^{\,-} = -\dfrac{\partial\vec{\mathit{\Gamma}}}{c_g\partial t_0} - \vec{\nabla}\mathit{\Gamma}_0, -\vec{\nabla}\times\vec{\mathit{\Omega}} - 2\dfrac{\partial\vec{\mathit{\Omega}}}{c_g\,\partial t_0}, \\[2ex]
-G\vec{j}^{\,\pm} = -\dfrac{\partial\vec{\mathit{\Gamma}}}{c_g\partial t_0} - \vec{\nabla}\mathit{\Gamma}_0 + \vec{\nabla}\times\vec{\mathit{\Omega}}, \\[2ex]
-G\vec{j}^{\,\mp} = -\dfrac{\partial\vec{\mathit{\Gamma}}}{c_g\partial t_0} - \vec{\nabla}\mathit{\Gamma}_0 + \vec{\nabla}\times\vec{\mathit{\Omega}}, \\[2ex]
\vec{\nabla}\times\vec{\mathit{\Gamma}} + \dfrac{\partial\vec{\mathit{\Omega}}}{c_g\,\partial t_0} = 0, \\[2ex]
\vec{\nabla}\cdot\vec{\mathit{\Omega}} = 0.
\end{cases}
\tag{71}
$$

Thus, we derived a new form of the gravitational force fields and Maxwell equations directly from the unified quaternion theory. In addition, we obtained the novel scalar gravitational field, $\mathit{\Gamma}_0$, as well as the new velocity dependent gravitational torsion field, $\vec{\mathit{\Omega}}$. The four gravitational current densities differ by the impact of the torsion field, $\vec{\mathit{\Omega}}$, which is negligible under low speed conditions. The new gravitational components are absent in the general theory of relativity [39] and may predict exciting new physical phenomena.

XI. CONCLUSIONS

Assuming the fundamental importance of division algebras, we introduced the division algebra of quaternions as a framework for description of four-dimensional space-time intervals. We derived the quaternion form of the Lorentz time dilation and presented an intuitive physical interpretation of the space-time transformations in the source and observer reference frames. We showed that the resulting physical interpretation is inseparable from experimental measurements. Then, we used quaternion algebra to develop quaternion calculus by choosing the correct quaternion form of the differential operators. We applied the new differentiation to the generalized potential function and suggested that the results can be interpreted as the two unified force fields. We repeated the differentiation, assuming that the second derivative of the potential function can be interpreted as the unified matter density. This resulted in the four unified matter density equations and a unified quaternion form of Maxwell equations. Then, we applied the unified fields and unified Maxwell equations to electromagnetic and gravitational interactions. Notably, the expressions for electromagnetic and gravitational interactions are similar, moreover, they were derived from the same quaternion potential function, pointing to a possible procedure for unification of electromagnetism and gravitation. The novel components appearing in expressions for the force fields and Maxwell equations originated from scalar fields and velocity dependent potentials, and require further theoretical and experimental investigation. Consequently, the proposed quaternion framework may be suitable as the foundation for a unified theory of space-time and matter, while enhancing the traditional theories of special and general relativity.

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