# A new proof of functional equation of Riemann Zeta function

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## Abstract :

[In this paper a new proof of functional equation of Riemann Zeta function is given using analytical expression of Riemann Xi function.]

**Keywords:** Functional equation, Riemann Xi function, Riemann Zeta function.

## 1. Introduction:

In a recent paper [1], analytical expression of Complex Riemann Xi function $\xi(s)$ was derived. The general expression for $\xi(s)$ is

$$\xi(s) = \xi(\sigma + it).$$

$$= F_2(l_1) + F_1(l_1)\,[\text{Cos } l_1 t \text{ Cos h } l_1\left(\sigma - \tfrac{1}{2}\right) + i \text{ Sin } l_1 t \text{ Sin h} l_1\left(\sigma - \tfrac{1}{2}\right)] \qquad \ldots(1.1)$$

$F_2(l_1)$ , $F_1(l_1)$ are all positive constants and can not be determined.

$l_1$ is a positive parameter.

Now using the identity

Cosx = Coshix and iSinx = Sinhix equation (1.1) can be written as.

$$\xi(s) = F_2(l_1) + F_1(l_1) \text{ Cosh } [il_1 t + l_1\left(\sigma - \tfrac{1}{2}\right)]$$

$$= F_2(l_1) + F_1(l_1) \text{ Cosh } [l_1\left(S - \tfrac{1}{2}\right)] \qquad \ldots(1.2)$$

Where $s = (\sigma + it)$.

In next section we will derive the functional equation of Riemann Zeta function using (1.2).

## 2. Proof of functional equation of Riemann Zeta function.

The functional equation of Riemann Zeta function [2] is, for all complex S

$$\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \qquad \ldots(2.1)$$

The Riemann Xi and Riemann zeta function are connected through the equation [3] :

$$\xi(s) = \frac{1}{2}\, s(s-1)\, \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) \qquad \ldots(2.2)$$

Therefore

$$\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) = \frac{2\xi(s)}{s(s-1)} \qquad \ldots(2.3)$$

Using (1.2) the equation (2.3) can be written as

$$\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) = \frac{2[F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\{l_1\left(S-\frac{1}{2}\right)\}]}{s(s-1)} \qquad \ldots(2.4)$$

In (2.4) replacing S by (1 – S) we find

$$\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \frac{2[F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\{l_1\left(1-S-\frac{1}{2}\right)\}]}{(1-s)(-s)}$$

$$= \frac{2[F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\{l_1\left(\frac{1}{2}-S\right)\}]}{s(s-1)}$$

$$= \frac{2[F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\{l_1\left(S-\frac{1}{2}\right)\}]}{s(s-1)} \qquad \ldots(2.5)$$

A comparison of (2.5) and (2.4) gives

$$\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) \qquad \ldots(2.6)$$

Equation (2.6) is nothing but the functional equation of Riemann Zeta function given by (2.1). Thus the proof is established.

## 3. Conclusion:

The above proof is new and depends on the general analytical expression of Riemann Xi function ξ(s) derived earlier [1].

## References:


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