# Direct derivation of Minkowski space and special relativity from an invariant tensor statement and a condition of simultaneity for the null 4-vector. Solutions for time-like and space-like trajectories

**David Escors**

[1] Miguel Servet Foundation. Pamplona. Spain
Correspondence: descorsm@navarra.es
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**Abstract:** Derivation of special relativity classically relies on the use of two inertial reference frames separating at a constant velocity, combined with the constancy of the speed of light as a founding definition of simultaneity. Einstein´s original method for derivation provides the general framework in which special relativity is often taught. While this approach is intuitive for students, it is nevertheless confusing in some aspects and lacks clarity in certain steps. Hence, special relativity is very often questioned with paradoxes and apparent mathematical inconsistencies. In most instances, these paradoxes are set up under improper mathematical frameworks, or in systems which are not invariant under coordinate transformations. Here, Minskowski space and special relativity are fully derived from an invariant tensor statement valid in any reference frame, with the simultaneity condition imposed on the 4-dimensional null vector. This approach derives Minkowski space with a very limited number of steps in tensor space without the need of introducing separating inertial reference frames. From it, the main features of special relativity for bradyon and tachyon trajectories are derived. This solution is particularly adequate for students of special and general relativity, as it is simple, straightforward and mathematically consistent.

## 1. Introduction

General relativity (GR) arises from differential geometric principles through tensor equations, making this theory mathematically sound [1]. GR has been extensively tested at many levels, and it has been reinforced by new experimental data on gravitational waves and their velocity of propagation [2,3]. GR sets up a dynamical space-time structure in a pseudo-Riemannian manifold, which is shaped by mass-energy-momentum densities. Rather than focusing on the kinematics of objects in a gravitational field, GR stablishes local space-time geometries and shapes geodesics for particle trajectories. By the use of tensor equations, its overall mathematical structure is consistent and valid in any reference frame. Special relativity (SR) naturally arises from general relativity when the gravitational fields are removed, and therefore it is substantiated in GR for its mathematical structure. However, due to historical reasons, SR was developed before general relativity in a much simpler way. For its derivation, inertial reference frames separating with a constant velocity in the absence of gravitation were used to observe events [4]. The real novelty in SR was the definition of simultaneous events when linked by a path of light. Einstein explicitly postulated the constancy for the speed of light as a fundamental physical law, so it could comply with Maxwell equations, Lorentz transformations and known experimental data [5,6]. Following SR classical postulates, Lorentz transformations naturally came out, and the concept for a 4-dimentional space-time with a special metric invariant under Lorentz transformations was later generalized by Minkowski [7].

Einsteins´original derivation method is still taught because of its easy visualization. However, the derivation method requires a few steps that are not sufficiently justified. These steps make SR frequently questioned with paradoxes and apparent mathematical inconsistencies [8]. While most of the paradoxes disappear if SR is properly used within the framework of general relativity and tensor calculus, most relativity students miss this point. The main problematic steps in the derivation are two. First, the use of several inertial reference frames, and the need to choose one of them as "preferred" (static). The second one derives from the solution put forward by Einstein to the first problem, who imposed a symmetry condition with two opposite velocity terms in a founding equation [4]. The first term (c-v) is non-problematic, but the second term (c+v) implicitly leads to a breaking of the constancy of light. Although not inadequate *per se* to fulfill a symmetry condition, this step is sufficiently ambiguous to make it a target for discrediting the theory if taken out of the context of general relativity.

As SR is a special case of general relativity, it can be derived with a few steps by only defining the proper 4-dimensional space, without considering observable events. Here, Minkowski space and SR are derived from the definition of the differential invariant length in a vector space, and association of the 4-dimensional null vector to the definition of simultaneity. Then, particle trajectories for bradyons and tachyons are calculated within the SR framework.

## 2. Derivation of Minkowski space from an invariant tensor statement

A differential length vector $\boldsymbol{dS}$ within a vector space can be defined with components corresponding to differential changes in arbitrary coordinates.

(Equation 1)

$$\boldsymbol{ds} \equiv (\boldsymbol{dx^0}, \boldsymbol{dx^1}, \boldsymbol{dx^2}, \ldots\ldots \boldsymbol{dx^n})$$

A scalar can be constructed by contracting the differential length vector with its corresponding dual vector. This constitutes an invariant tensor statement for differential length squared in a tensor space valid in any reference frame. This contraction can be expressed in terms of the metric tensor.

(Equation 2)

$$\boldsymbol{dx^\alpha dx_\alpha = dx^\alpha dx^\rho g_{\rho\alpha} = (ds)^2}$$

The null vector represents no separation in the 4-dimensional vector space, and fulfils the following invariant tensor equation by equating the length squared (equation 2) to 0:

(Equation 3)

$$\boldsymbol{dx^\alpha dx^\rho g_{\rho\alpha} = 0}$$

To derive Minkowski space, orthogonal Euclidean spatial coordinates ($x$, $y$ and $z$; or $x^1$, $x^2$ and $x^3$) and a time coordinate ($t$, $x^0$) are used. The imposition of the condition of simultaneity (equation 3) can be re-written by separating temporal from spatial coordinates and applying orthogonality, as follows:

(Equation 4)

$$\boldsymbol{dx^0 dx^0 g_{00} + dx^m dx^m g_{mm} = 0}$$

A relationship between spatial and temporal components that fulfils the null vector can then be expressed:

(Equation 5)

$$\frac{dx^m dx^m g_{mm}}{(dx^0)^2} = -g_{00}$$

This condition for the null vector results in the temporal component of the metric to be of negative sign, which resembles the Minkowski space metric. The spatial components of the metric tensor are equal to 1 to fulfil orthogonality in flat space. Special relativity defines simultaneity as the coordinate path of light in space-time. This condition needs to be imposed on the null 4-vector, which would represent the length of two simultaneous points in 4-vector space. From equation (5) and imposing the definition of simultaneity, the following equivalences can be written:

(Equation 6)

$$\frac{dx^m dx^m g_{mm}}{(dx^0)^2} = \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(dx^0)^2} = c^2$$

The imposition of simultaneity could be in principle be met by any arbitrary expression, without the explicit statement of the constancy of the speed of light in vacuum. From equations (5) and (6), the temporal component of the metric tensor corresponds to

(Equation 7)

$$g_{00} = -c^2$$

This condition is valid in any reference frame as long as it fulfils the simultaneity condition. Then, we have the following invariant statement for the differential length, which corresponds to the metric of Minkowski space:

(Equations 8)

$$(ds)^2 = -c^2(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

or

$$(ds)^2 = -c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

**3. Derivation of time-like trajectories**

The derivation of founding equations for SR kinematics from the metric (equation 8) is straightforward using classical Lagrangian methodology, as this method satisfies geodesic equations. An Action can be constructed for time-like paths. This corresponds to the classical SR framework and requires the integration of the differential length for the action as follows:

(Equation 9)

$$\boldsymbol{A = k\int_a^b ds = k\int_a^b \sqrt{c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}}$$

The action includes an arbitrary constant "k", and can be rewritten in a classical form:

(Equation 10)

$$\boldsymbol{A = k\int_a^b \sqrt{1-\frac{v^2}{c^2}}\,dt}$$

Which includes the coordinate velocity in the path of stationary action. The Lagrangian function then corresponds to

(Equation 11)

$$\boldsymbol{L = k\sqrt{1-\frac{v^2}{c^2}} = \frac{kc}{\gamma}}$$

The relativistic gamma factor appears naturally in the Lagrangian function. By solving Euler-Lagrange equations, an expression for a 3-vector momentum is directly obtained.

(Equation 12)

$$\boldsymbol{P^m = \frac{\partial L}{\partial v^m} = \frac{-k\ v^m}{c^2\sqrt{1-\frac{v^2}{c^2}}}}$$

For this equation to be dimensionally consistent with momentum, the constant k needs to fulfil the following:

(Equation 13)

$$\boldsymbol{k \equiv -mc^2}$$

Leaving the special relativistic Lagrangian in its classical final form.

(Equation 14)

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

And a definition for proper velocity of the spatial components ($U^m$) which coincides with the classical definition.

(Equation 15)

$$P^m = m\frac{v^m}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv mU^m$$

The Hamiltonian for time-like trajectories can also be calculated following Lagrangian formalism which results in the classical expression for the mass-energy, as indicated:

(Equation 16)

$$H = P_m V^n - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In the framework of time-like trajectories, this solution is not defined for velocities greater than c, as they lead to imaginary solutions. These trajectories are characteristic of bradyons with infraluminal velocities.

**4. Derivation of space-like trajectories**

For space-like trajectories, there is a sign change in the temporal and spatial coordinates of the metric in equations (8) and (9), and these can be solved with the following action principle:

(Equation 17)

$$A = k\int_a^b \sqrt{-c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2} = -\,mc^2\int_a^b \sqrt{\frac{v^2}{c^2} - 1}\,dt$$

As the only difference is the change in the sign of the metric components, the constant k is equivalent for the calculated in time-like trajectories.

Euler-Lagrange equations can then be solved to obtain expressions for momentum.

(Equation 18)

$$\boldsymbol{P^m = \frac{\partial L}{\partial v^m} = \frac{-mv^m}{\sqrt{\frac{v^2}{c^2} - 1}}}$$

This solution is defined only when particles are moving with supraluminal velocities. Otherwise, imaginary solutions are obtained. Particles following a space-like trajectory have negative momentum, which can be associated to negative mass. If the solution of time-like trajectories is used instead, the mass of a tachyon would be imaginary. However, it has to be remarked that time-like trajectories are not defined for velocities higher than c. In space-like trajectories, the components of the spatial proper velocity take now this form:

(Equation 19)

$$\boldsymbol{U^m = \frac{v^m}{\sqrt{\frac{v^2}{c^2} - 1}}}$$

When the components of the coordinate velocity diverge to infinity, the components of the proper velocity will be asymptotic to 1. Similarly, the Hamiltonian can be calculated as follows:

(Equation 20)

$$\boldsymbol{H = \frac{-mv^2}{\sqrt{\frac{v^2}{c^2} - 1}} + mc^2\sqrt{\frac{v^2}{c^2} - 1} = \frac{-mc^2}{\sqrt{\frac{v^2}{c^2} - 1}}}$$

The energy of a tachyon in space-like trajectories is divergent to minus infinity with decreasing velocities as they approach to the speed of light. On the other hand, the energy goes asymptotically to 0 with increasing velocity, in agreement with the expected properties for tachyons.

**4. Discussion**

Special relativity is included within the framework of general relativity, which was formulated under principles of differential geometry through tensor equations. However, the classical derivation of SR starts from a purely "kinematic" approach by evaluating the observation of events from two independent inertial reference frames, and postulating simultaneity by light ray trajectories. Its classical mathematical derivation is appropriate but presents some weak points that are not sufficiently argued. The classical approach is still taught in relativity courses, and the change of different reference frames, relative velocities or the choice of particular coordinate systems make the theory to be prone to criticism.

However, as shown here SR can be easily and directly derived from a starting invariant tensor statement. Rather than using distinct inertial reference frames and observation of simultaneous events through light paths, here the Minkowski metric was derived through tensor principles. This invalidates any paradox that implies a change of reference frames, and makes observations of events from different reference frames mathematically consistent by tensor transformation rules.

Nevertheless, both the classical and formal derivation methods uncover a point in which special relativity can be critiqued upon: the definition of simultaneity. For the derivation of SR, here the speed of light has been identified with the null 4-vector in Minkowski space, to comply with the postulate of SR simultaneity. The definition of simultaneity by the speed of light is difficult to justify and derive from basic principles. However, all the experimental data supports this postulate, making relativity one of the most successful theories.

Once the Minkowski metric is derived, the kinematics of time-like and space-like trajectories can be straightforwardly obtained through Lagrangian mechanics. In this process, the speed of light appears as an event horizon separating bradyons and tachyons. Both particle energies diverge to infinity and minus infinity, respectively, when approaching the barrier of the speed of light. Additionally, while bradyons have standard positive energy, tachyons have negative energies, which could be similar to the crossing of the ergosphere horizon [9]. In any case, the SR equations for bradyons and tachyons are distinct and not interchangeable. Solutions derived for time-like trajectories are not defined for supraluminal velocities, and those for space-like trajectories are not defined for infraluminal velocities. Finally, the equations derived here are in agreement with published studies and support the concept for the mass of a tachyon to be non-imaginary [10].

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