Rotating Unbalance and Newton’s third Law of Motion

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Abstract. In this paper, we investigate the cause of vibration behind an isolated rotating unbalance. Such kind of system has a defined response and is described by the particular solution of a second-order inhomogeneous differential equation. However, the differential equation as also the response (particular solution) cannot justify why an isolated system should vibrate (or move in one direction) from the moment Newton’s third law of motion holds. The answer to this apparent contradiction comes from the study of the eccentric mass momentum transfer to the system. It reveals Newton’s third law does not hold for a rotating unbalance as also is a special case of a more general form, where the latter can be derived directly from the momentum conservation.

PACS numbers: 45.20.df

I. INTRODUCTION

In classical mechanics [1], any realization of the action-reaction principle presupposes two bodies exerting equal and opposite forces on each other. The same applies to an isolated system where its internal parts may interact with each other as also with the system itself. According to Newton’s third law [2, 3], an action force exerted upon a part (internal) of an isolated system, creates an opposite and of equal magnitude reaction force exerted upon the rest of the system. This has as result, the system does not acquire momentum through internal forces. However, classical physics and daily experience have taught us, an isolated rotating unbalance may vibrate, although the forces that act upon the system are internal. Most (or even all) of today’s literature and applications address the problem of vibration in terms of the system’s response by ignoring the actual mechanism that causes it. This work proposes the general form of Newton’s 3rd law for a rotating unbalance as also is a special case of a more general form, where the latter can be derived directly from the momentum conservation.

II. METHODS

The internal forces that act upon an isolated rotating unbalance system on the y-axis are

\[ (M - m)\frac{d^2 y}{dt^2} \]

+ \[ m\left(\frac{d^2 y}{dt^2} + y(r_e \sin(\omega t))\right) = -c\frac{dy}{dt} - ky, \]

\[ M\frac{d^2 y}{dt^2} - m\nu_e \omega^2 \sin(\omega t) = -c\frac{dy}{dt} - ky, \]  

(1)

(2)

\( M: \text{ system mass}, \ m: \text{ eccentric mass}, \ r_e: \text{ distance of } m \text{ from system’s center.} \)

Eq.(2) is essentially an inhomogeneous second-order differential equation, thus

\[ M\frac{d^2 y}{dt^2} + c\frac{dy}{dt} + ky = m\nu_e \omega^2 \sin(\omega t), \]

\[ \frac{d^2 y}{dt^2} + 2\zeta \omega_n \frac{dy}{dt} + \omega_n^2 y = m\nu_e \omega^2 \sin(\omega t). \]  

(3)

(4)

The particular solution is of the form:

\[ y = y_p \sin(\omega t - \phi), \]

\[ y_o = \frac{m\nu_e}{M} \frac{\omega^2}{\omega_n^2} \left(1 - \frac{\omega^2}{\omega_n^2}\right) + \left(2\zeta \frac{\omega}{\omega_n}\right)^2, \]

\[ \omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{c}{2\sqrt{km}}, \]

\[ \phi = \tan^{-1}\left(\frac{2\zeta \omega/\omega_0}{1 - (\omega/\omega_0)^2}\right), \]

\( k: \text{ stiffness of the spring,} \)

\( c: \text{ damping coefficient,} \)

\( \zeta: \text{ damping ratio,} \)

\( \omega: \text{ angular speed,} \)

\( \omega_n: \text{ natural angular speed.} \)

Assuming same stiffness and damping coefficient on the x-Axis, yields

\[ M\frac{d^2 x}{dt^2} + c\frac{dx}{dt} + kx = m\nu_e \omega \cos(\omega t), \]

\[ \frac{d^2 x}{dt^2} + 2\zeta \omega_n \frac{dx}{dt} + \omega_n^2 x = \frac{m\nu_e \omega^2}{M} \cos(\omega t), \]

\[ x = x_o \cos(\omega t - \phi), \]

\[ x_o = \frac{m\nu_e}{M} \frac{\omega^2}{\omega_n^2} \left(1 - \frac{\omega^2}{\omega_n^2}\right) + \left(2\zeta \frac{\omega}{\omega_n}\right)^2. \]  

(5)

(6)

(7)

(8)

(9)

(10)

(11)

(12)

We now add an identical eccentric mass placed symmetrically about the y-axis (see FIG. 1) that turns in the opposite direction (clockwise), however, synchronized with the motion of the first. Thus,
x-Axis:

\[ M \frac{d^2x}{dt^2} + \frac{dx}{dt} + kx = 0. \]  

The counter-rotating eccentric masses create a null net force along the x-Axis. This means no excitation net force results in no motion on the x-Axis:

\[ x = 0 \Rightarrow \frac{dx}{dt} = 0 \Rightarrow \frac{d^2x}{dt^2} = 0. \]  

y-Axis:

\[ M \frac{d^2y}{dt^2} + \frac{dy}{dt} + ky = 0. \]  

Consequently, from Eq.(13), Eq.(14) and Eq.(15) the system moves just along the y-axis. When the stiffness and damping coefficient is set to zero, Eq.(15) becomes

\[ M \frac{d^2y}{dt^2} = M \frac{dy}{dt} = 2mr_e \omega^2 \sin(\omega t). \]  

The particular solution of Eq.(15) is similar to Eq.(5), however, it also fails to justify, a) the internal momentum transfer from each eccentric mass to the rest of the system, b) Newton’s third law in the context of internal forces and c) the origin of the reaction force that causes the acceleration of the system. Consequently, the system should normally never move (or generally vibrate) using internal forces. The above observations lead to the conclusion that Newton’s 3rd law might be a special case of a more general form, as we will show below.

According to the conservation of momentum, the problem can be stated as the reduction of the tangential speed (see FIG. 2) of the eccentric masses or the reduction of the amplitude of the reactive centrifugal forces becomes the cause of the system’s linear acceleration. Stemming from Eq.(16) and the conservation of momentum, yields

\[ \pm \sum p_o = \pm \sum p = \pm p_0 \pm p_u = \pm p + p_t, \]  

\[ \pm Mu_o \pm 2mr_e \omega_o = \pm Mu \pm 2mr_e \omega, \]  

\[ \pm M(u - u_o) = \mp 2m(r_e \omega - r_e \omega_o), \]  

\[ \pm M(u - u_o) = \mp 2m(u_t - u_{e_t}), \]  

\[ \pm M \frac{\Delta \Delta t}{\Delta t} = \mp 2m \frac{\Delta u_t}{\Delta t}, \]  

\[ \pm M \frac{du}{dt} = \mp 2m \frac{du}{dt} = \mp 2ma_t \Rightarrow a_t \leq 0, \]  

\[ du: \text{ change in rectilinear speed of the system,} \]  

\[ da_t: \text{ change in tangential speed of the eccentric mass,} \]  

\[ a_t: \text{ tangential acceleration.} \]  

The negative sign in Eq.(22) indicates the direction of the force (reaction) is always from the center of the system, outwards. Moreover, due to momentum conservation, the change in tangential speed is in any case a deceleration \( (a_t \leq 0) \). Otherwise, it wouldn’t be possible to transfer momentum from the eccentric masses to the rest of the system. Alternatively, through the reduction of the reactive centrifugal forces (see FIG. 2), we have

\[ \pm M \frac{du}{dt} = \pm 2mr_e \omega_o^2 \mp 2mr_e \omega^2, \]  

\[ \pm M \frac{du}{dt} = \mp 2mr_e \left( \omega^2 - \omega_o^2 \right), \]  

\[ r_e \left( \omega^2 - \omega_o^2 \right) = \frac{2r_e a_o \phi}{\pi}, \]  

\[ -\pi/2 \leq \phi \leq \pi/2, \]
Eq. (28) describes clearly how the momentum is being transferred and this is achieved through the deceleration \(a_t < 0\) of the eccentric masses. Eq. (23) can be also written in terms of action (centripetal) and a reaction (reactive centrifugal) force as follow,

\[
+2mr_\omega^2 \pm M \frac{du}{dt} = +2mr_\omega^2, \quad F_A + F_R = +2mr_\omega^2. \tag{29}
\]

We usually state that an action upon a part of the system, creates an opposite and of equal magnitude reaction force upon the rest of the system. A slightly different formulation of this statement is, an action upon the center of mass of a part of the system, creates an opposite and of equal magnitude reaction force upon the center of mass of the system.

This enables us to use Eq. (30) as the general form of Newton’s third law. When the tangent to the eccentric masses motion follows a curvilinear trajectory \((F_A \perp u_t)\), the centripetal becomes effectively larger than the reactive centrifugal, thus

\[
0 \leq \omega \leq \omega_o \text{ and } F_A \perp u_t \Rightarrow F_A = +2mr_\omega^2, \tag{31}
\]

\[
F_R = \pm M \frac{du}{dt} \Rightarrow F_A + F_R = +2mr_\omega^2. \tag{32}
\]

A curvilinear trajectory presupposes the system’s center of mass to be not fixed on a surface. On the other hand when the system’s center of mass is fixed, the reactive centrifugal equals (turns into tension \((F_T)\)) to the centripetal forcing the eccentric masses to conduct a circular \((F_A \perp u_t)\) motion. Then, from Eq. (30) yields,

\[
\omega = 0 \text{ and } F_A \perp u_t \Rightarrow F_A = +2mr_\omega^2, \tag{33}
\]

\[
\frac{du}{dt} = 0 \Rightarrow F_R = F_T \Rightarrow F_A + F_R = 0. \tag{34}
\]

Finally, when the tangential velocity is in parallel \((F_A \parallel u_t)\) with the action force, the eccentric masses follow a rectilinear trajectory, hence

\[
\omega = \omega_o = 0 \text{ and } F_A \parallel u_t, \tag{35}
\]

\[
F_A = +2mr_\omega^2 - (+2mr_\omega^2) = 0, \tag{36}
\]

\[
F_R = \pm M \frac{du}{dt} \Rightarrow F_A + F_R = 0 \Rightarrow \pm M \frac{du}{dt} = 0. \tag{37}
\]

Deriving from Eq. (30), the energy conservation is expressed as follow

\[
U_A - W_R = U \Rightarrow W_R = U_K = -(U - U_A), \tag{38}
\]

\[
W_R = U_K = -(2m(r_\omega^2) - 2m(r_\omega^2)^2) \tag{39}
\]

As we see from Eq. (38) and Eq. (39), the work done by the reaction force is negative which means the system removes rotational energy from the eccentric masses and converts it to the system’s kinetic energy.

The general form of Newton’s 3rd law that applies to the interaction between two bodies or between a part of the system and the system as a whole, is written as follow

\[
0 \leq \omega \leq \omega_o \text{ and } F_A + F_R = +mr_\omega^2, \tag{40}
\]

\[
F_A \perp u_t \Rightarrow F_A + F_R = +mr_\omega^2, \tag{41}
\]

\[
F_A \perp u_t \text{ and } \omega = 0 \Rightarrow F_R = F_T \Rightarrow F_A + F_R = 0, \tag{42}
\]

\[
F_A \parallel u_t \text{ and } \omega = \omega_o = 0 \Rightarrow F_A + F_R = 0. \tag{43}
\]

The analysis of the system with the eccentric masses was based on the assumption of having ideal conditions where there is no energy conversion loss as also the system is isolated (in absence of external forces). Moreover, in order for all of these to work, the system must have onboard stored convertible energy, e.g. stored electric energy that powers the rotation of the eccentric masses.

### III. CONCLUSIONS

We showed that the differential equation and the response of an isolated rotating unbalance cannot justify the vibration of the system in the context of Newton’s 3rd law. The discovery of the general form of Newton’s third law provides an alternative interpretation about the cause behind the vibration of rotating unbalance systems in classical mechanics as also will encourage researchers to explore this subject in other branches of physics.

**Acknowledgments**

I thank my wife, who encouraged me to initiate and complete this work.