Evaluating the Alignment of the Polarized radio waves from 27 QSOs in a region near the NGP

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Abstract

The sample of 27 quasars with polarized radio emissions located in a region near the North Galactic Pole is shown to have highly aligned polarization directions. Furthermore, by extending their polarization directions around the Celestial Sphere, the convergence of their polarization directions is close to the sources. Thus, parallax forces the position angles to vary with locations of individual sources. The QSOs are taken from the JVAS1450 subset of the JVAS/CLASS 8.4-GHz surveys. The alignment is analyzed by the Hub Test. Fewer than about 70,000 randomly directed such samples would be as well aligned, a $4\sigma$ result. Some underlying calculations are presented in a Mathematica-coded Appendix. Access to a .nb notebook is provided in the references.

Keywords: Polarized Radio Sources; Alignment; Quasi-stellar objects

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0. Preface

The pdf version of this notebook is available online from the viXra archive, try https://vixra.org/abs/2105.0091
To find the ready-to-run notebook follow the link in Ref. 1. The notebooks in this series were created using Wolfram Mathematica, Version Number: 12.1, Ref. 2.

Note(s):

(1) Random numbers should be reliable. Thus, numerical quantities in the pdf version in Sec. 6 Uncertainty and Sec. 7 Probability and Significance should differ from the live ready-to-run version in Ref. 1. Different sets of random runs and uncertainty runs, for a sufficiently large number of runs, should provide numerical values that differ only slightly.

(2) To shorten the document, some Mathematica Notebook Cells have been hidden. The first cell is hidden before the Title and Abstract cell. It contains a list of notebook files that are potentially useful for me to know. Starting with Fig. 8 in Part II the Appendix, most of the cells that produce the captions for the figures are hidden in cells.

To open a cell for viewing in the live ready-to-run version from Ref.1: Highlight the little nub that shows a cell exists there, Find Cell: Cell Properties: Check “Open” to see one of the hidden cells.
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1. Introduction to Part I

To have regions of the sky containing QSOs with aligned polarizations, or some other way correlated, is certainly remarkable. Large scale alignments are found for both optical and radio quasi stellar objects (QSOs), Refs. 3-8. In some studies, the tests that determine significant alignment compare the polarization direction of the electromagnetic radiation from one of the QSOs with one or more of its neighbors. An example of the potential value of such research is the finding of correlations between polarization directions and the local large scale structure, Refs. 6,7.

Polarization by way of interaction with local interstellar grains in the Milky Way Galaxy has been discounted, Ref. 5. The typical QSO polarization levels are too strong, a percent to several percent, for the cause to be local to the Milky Way. Taking a step further out, with the Virgo Supercluster in the same general direction as these QSOs, some mechanism related to the supercluster may be able to explain the alignment. Perhaps there are intergalactic magnetic fields, Ref. 9. Or, as mentioned above, the polarizations could exist when the radio waves are emitted. In any case, the alignment is intriguing.

The Hub Test does not compare polarizations directly with each other, but indirectly, by finding points of convergence of the great circle geodesics obtained by extending polarization directions around the Celestial Sphere. Places where the geodesics are most dense are called “hubs” much as International Travel Hubs are places where the paths of passenger jets converge. The Hub Test is especially useful, compared to direct-comparison tests, when the convergence is strong near the sources. In that case, as is true for these 27 QSOs, there is parallax which masks the alignment for direct-comparison tests. Some other studies, Refs. 10,11, employ the Hub Test that is used here.

All tests, direct or indirect, serve to add to the information defining the behavior of QSOs and informing other topics of interest, such as Large Scale Structure, intergalactic magnetic fields, and the properties of these objects.

2. Sample selection and the Hub Test

The sample of 27 QSOs in this report are taken from JVAS1450, Ref. 12,13, a catalog of 1450 QSOs that was kindly communicated to me by one of the authors of Ref. 12. Details of the dataset can be found in Ref. 12. As explained there, the JVAS1450 catalog builds on data from the earlier large JVAS/CLASS 8.4-GHz catalog, Ref. 14.

To find candidate samples in the JVAS1450 to study, a survey was conducted. The QSO sources were binned, assigned to 5° radius circular regions centered on the grid points of a 2° mesh. A minimum of seven sources was enforced. The regions were sorted by the significance of their alignments according to the Hub Test. Another report, Ref. 10, evaluates a clump of 13 QSOs, Clump 2 in Fig. 1.

In this report we investigate Clump 1, which consists of the QSOs inhabiting the overlap of fourteen significantly aligned regions near the Vernal Equinox and the North Galactic Pole. The sample occupies a roughly 11° radius patch of sky centered on (RA,dec) = (178°,10°). The alignment of these 27 QSOs is evaluated with the Hub Test.
Figure 1. Survey of 1450 polarized radio QSOs. (Equatorial Coordinates, centered at \((\alpha, \delta) = (180^\circ, 0^\circ)\), East to the right.) The QSOs were grouped into 5° radius regions centered on grid points. Those regions having at least 7 QSOs are plotted as gray dots. Just 35 regions showed very significant alignment, i.e. \(S \leq 0.01 = 10^{-2}\), or, equivalently, \(-\log_{10} S \geq 2.0\), and these are shaded in color. Clump 1 has 14 regions containing 27 QSOs and is selected for analysis here. Clump 2 has 3 regions containing 13 QSOs and is analyzed in Ref. 10.

The Hub Test is discussed more fully in Ref. 15. The basic idea is analogous to a well-known guide to find Polaris, the North Star. Assume one can find the stars Merak and Dubhe which are two stars in the constellation Ursa Major. Then the direction from Merak to Dubhe aligns with the direction from Merak to Polaris. While Fig. 2 is not drawn for this case, with the labelling of Fig. 2, let the source \(S\) be the star Merak, take the direction from Merak to Dubhe to be the direction of polarization \(\vec{v}_\psi\), and let Polaris be the point \(H\). Then the alignment of the Merak-to-Dubhe direction \(\vec{v}_\psi\) with the direction toward Polaris, the point \(H\), illustrates the concept of alignment in the Hub Test. The alignment angle \(\eta\) for Merak-Dubhe and Merak- Polaris would be about \(\eta = 3.47^\circ\) and the blue great circle would almost coincide with the purple great circle.

Figure 2: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source \(S\). The linear polarization direction \(\vec{v}_\psi\) lies in the tangent plane and determines the purple great circle on the sphere. A point \(H\) on the sphere together with the point \(S\) determine a second great circle, the blue circle drawn on the sphere. Clearly, \(H\) and \(S\) must be distinct in order to determine a great circle. The angle \(\eta\) measures the alignment of the polarization direction \(\psi\) with the point \(H\).

In Fig. 2, the “alignment angle” \(\eta\) is the acute angle \(\eta\) between two great circles at \(S\), \(0^\circ \leq \eta \leq 90^\circ\). The alignment angle \(\eta\)
measures how well the polarization direction $\hat{\nu}$ matches the direction $\hat{\nu}_H$ toward the point $H$. Perfect alignment occurs when $\eta = 0^\circ$ and the two great circles overlap. Perpendicular great circles, $\eta = 90^\circ$, indicates maximum “avoidance” of the polarization direction $\hat{\nu}$ with the point $H$ on the sphere. The halfway value, $\eta = 45^\circ$, favors neither alignment nor avoidance.

With $N$ sources $S_i$, $i = 1, ..., N$, there are $N$ alignment angles $\eta_{iH}$ at each point $H$. One can calculate an average alignment angle $\bar{\eta}$ at $H$,

$$\bar{\eta}(H) = \frac{1}{N} \sum_{i=1}^{N} \eta_{iH}, \quad (1)$$

where

$$\cos(\eta_{iH}) = |\hat{\nu}_i \cdot \hat{\nu}_H|. \quad (2)$$

Each angle $\eta_{iH}$ is taken to be the acute angle solving (2). Then the average alignment angle $\bar{\eta}(H)$ at the point $H$ must also be acute.

The alignment angle $\eta(H)$ is a function of position $H$ on the sphere. It is symmetric across diameters, $\eta(H) = \eta(-H)$, because great circles are symmetric across diameters. The function $\eta(H)$ measures convergence and divergence of the great circles determined by the polarization directions. For random polarization directions, the average $\eta(H)$ is most likely near $45^\circ$, since each alignment angle $\eta_{iH}$ is acute, $0^\circ \leq \eta_{iH} \leq 90^\circ$, and random polarization directions should not favor any one value. Points $H$ where the alignment angle $\eta(H)$ is smaller than $45^\circ$, the great circles tend to converge; where $\eta(H)$ is larger than $45^\circ$, the great circles can be said to diverge.

In this article and notebook, we often use “min” to label the smallest alignment angle $\eta_{\text{min}}$ and the associated points on the sphere, the “hubs” $H_{\text{min}}$ and $-H_{\text{min}}$. Thus “min” is associated with convergence of the polarization directions. For divergence, the hubs $H_{\text{max}}$ and $-H_{\text{max}}$ locate places where the polarization directions avoid, as indicated by the largest alignment angle $\eta_{\text{max}}$. Thus, we very often label an avoidance related quantity with “max”.

3. The alignment of the polarization directions for the 27 QSOs

For the 27 sources considered in this report, the alignment angle function $\eta(H)$, Eq. 1, makes the following contour map. The global and local maps are computed in the Mathematica program below in Part II, Secs. 5b,c.

![Figure 3: The alignment angle function $\eta(H)$, Eq. 1, mapped on the Celestial Sphere (Aitoff plot, centered on $(\alpha, \delta) = (180^\circ, 0)$, East to the right). The QSOs are shaded green. To guide the eye, two Great Circles are plotted in gray, one through the sources’ center point and the avoidance hubs $H_{\text{max}}$ and $-H_{\text{max}}$ while the other Great Circle runs through the sources’ enter and the alignment hubs $H_{\text{min}}$ and $-H_{\text{min}}$. The circles cross at an angle of $88.1^\circ \pm 3.7^\circ$. The smallest alignment angle, $\eta_{\text{min}} = 21.64^\circ \pm 0.86^\circ$, is located at the hubs $H_{\text{min}}$ and $-H_{\text{min}}$, where the polarization directions converge best. One alignment hub $H_{\text{max}}$ is located very close to the QSOs.](image-url)
Figure 4: The region near the QSOs. The QSOs are located at the green dots. The short black lines through the QSOs indicate the polarization directions. Measuring polarization directions $\psi$ clockwise from North, one sees that the angles $\psi$ range from about $\psi = 150^\circ$ for many of the northern-most QSOs to a little more than $90^\circ$ or so for the more southerly QSOs. The QSOs display parallax: almost all are in the general direction of the alignment hub $H_{\text{min}}$ at $(\alpha, \delta) = (189^\circ, -1^\circ)$, but the directions depend on where the sources are located. A couple, say 2 or 3 of the 27 QSOs, have polarization directions that do not point toward $H_{\text{min}}$, but somewhat perpendicular to the direction favored by the others.

4. Experimental uncertainty

A fundamental characteristic of measurements such as polarization is uncertainty. Some measured quantities, such as the location of the sources, are so accurately measured that their uncertainty, while not zero, is considered negligible. The maps above were drawn based on the “best” polarization directions reported in the JVAS1450 catalog. The catalog also reports uncertainties in the polarization directions. In Part II Sec. 6, below, the uncertainties are carried through the calculations yielding the uncertainties in the results.

The uncertainties reported with the observed polarization directions are assumed to make normal distributions, i.e. Gaussians that integrate to unity. For example, one of the QSOs, the sixth one, has a measured polarization position angle of $\psi_{\text{obs}} \pm \sigma = 146.3^\circ \pm 2.9^\circ$. We take this to mean that the probability that the actual value of $\psi$ was not $\psi_{\text{obs}} = 146.3^\circ$, but some other value $\psi_1$, is given by the Gaussian

$$P(\psi_1) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\psi_1 - \psi_{\text{obs}}}{\sigma} \right)^2 \right].$$

The Mathematica software has a special command, “RandomVariate”, that produces random values of $\psi_1$ with respect to the probability distribution in Eq. (3). Thus, an “uncertainty run” begins by selecting a set of polarization directions for the 27 QSOs conforming to the uncertainty distributions like the one in Eq. (3). The alignment angle function $\eta(H)$ in Eq. (1) is evaluated to find the smallest alignment angle $\eta_{\text{min}}$. As expected, the small changes to the observed polarization directions make small changes to the resulting angle $\eta_{\text{min}}$. By repeating the process many times, one obtains a distribution of values for the smallest alignment angle $\eta_{\text{min}}$.

The many uncertainty run values for the smallest alignment angle $\eta_{\text{min}}$ produce a distribution of the smallest alignment angle $\eta_{\text{min}}$, as well as the locations of alignment hubs. These distributions have corresponding mean values and distribution widths. See Fig. 5. The distribution of the uncertainty run values for the smallest alignment angle $\eta_{\text{min}}$ in Fig. 5 can be summarized by $\eta_{\text{min}} = 0.377 \pm 0.015$ radians = $21.64^\circ \pm 0.86^\circ$. This disagrees a little with the observed value, $\eta_{\text{min}} = 21.09^\circ$, i.e. the value found using the recorded polarization directions $\psi_{\text{obs}}$, the “best” values of $\psi$. But all is well, since $21.09^\circ$ is in the range, $\eta_{\text{min}} = 21.64^\circ \pm 0.86^\circ$, of most likely values determined by experimental uncertainty.
Figure 5: Histogram of the smallest alignment angle $\eta_{\text{min}}$ for $R = 10,000$ uncertainty runs. The height $\Delta R$ is the number of uncertainty runs with a value of $\eta_{\text{min}}$ in the ‘bin’, the range covered by each bar. This Gaussian distribution peaks at a mean value of $\eta_{\text{min}}$ of 0.3777 radians = 21.64° and has a half-width of $\sigma = 0.0150 = 0.86°$ where the distribution is down from the peak by a fraction $e^{-1/2} = 0.607 = 60.7\%$. One writes the result as $\eta_{\text{min}} = 0.3777 \pm 0.0150 \text{ radians} = 21.64° \pm 0.86°$.

Besides the uncertainty in the smallest alignment angle $\eta_{\text{min}}$, the uncertainty runs yield uncertainty ranges for other quantities such as the largest avoidance angle $\eta_{\text{max}}$. Each uncertainty run has its own set of alignment and avoidance hubs, $H_{\text{min}}$ and $H_{\text{max}}$, respectively. A plot of the polarization directions with their uncertainties and the locations of many of the uncertainty run hubs is displayed in Fig. 6.

Figure 6: The QSOs as green dots plotted with the experimental uncertainties in polarization directions. The most likely locations of the nearest alignment hub $H_{\text{min}}$ are enclosed in the orange oval. In the following section, we find that the avoidance of the hubs $H_{\text{max}}$ and $-H_{\text{max}}$ is very significant and the alignment of the polarization directions with the hubs $H_{\text{min}}$ and $-H_{\text{min}}$ is also very significant.
5. Significance

Finally, we need to determine the significance of the alignment found for the polarization directions of these 27 QSOs. ‘Significance’ means how likely it is that randomly directed polarization vectors would give the same or better alignments than the observed polarization directions give.

To determine significance, we repeatedly find the smallest alignment angle function $\eta(H)$ many times, but with random values of $\psi$ chosen for the 27 QSOs. The only experimental data used in this process are the locations of the 27 QSO sources. The goal is to see what fraction of random runs yield a value with a lower $\eta_{\text{min}}$ than the value $\eta_{\text{min}} = 21.09^\circ$ obtained with the observed data.

For this study, we created 10,000 random runs. By sorting those 10,000 runs by the value of $\eta_{\text{min}}$, smaller $\eta_{\text{min}}$ before larger $\eta_{\text{min}}$, one can find how many of those 10,000 runs gives a smaller alignment angle $\eta_{\text{min}}$ than the observed value of $\eta_{\text{min}}$, i.e. $\eta_{\text{min}} = 21.09^\circ$ using the recorded polarization directions $\psi_{\text{obs}}$ from the catalog. For this batch of 10,000 random runs, none of the 10,000 runs is smaller than 21.09°, with 22.04° being the closest random value of $\eta_{\text{min}}$. But the smallest random run value, 22.04°, is quite close to the observed value 21.09°, so one can roughly estimate the significance of the observed $\eta_{\text{min}} = 21.09^\circ$ is about one in 10,000 or 0.0001, probably less. Clearly, we would need many more sets of 10,000 random runs for such considerations to produce a reliable value of significance.

Rather than expending a large amount of computer time generating more random runs, we follow conventional practice and make assumptions so we can get a significance from the set of already-completed 10,000 random runs. We start by finding a function that fits the distribution of the 10,000 $\eta_{\text{min}}$, which is the number of $\eta_{\text{min}}$ since there is one smallest alignment angle $\eta_{\text{min}}$ per random run. See Part II the Appendix Sec. 7 for details. Having found a function that fits the distribution, we assume that the function accurately describes the distribution down along the “tail” of the function where our well-aligned QSOs have their $\eta_{\text{min}}$.

A histogram of the resulting smallest alignment angles $\eta_{\text{min}}$ from 10,000 runs is displayed in Fig. 7. Look closely at the distribution in Fig. 7. The right side, the side toward $\eta_{\text{min}} \rightarrow \pi/4 \sim 0.79$, has a steeper slope than the left side, the side toward $\eta_{\text{min}} \rightarrow 0$. Thus, the low $\eta_{\text{min}}$ side is favored; probability is pushed from the right side to the left side. A simple, symmetrical Gaussian would not fit the data well. The fitting curve shown combines a Gaussian with a unit step-function, that is unity to the left, and zero to the right, of the peak. Since the 27 QSOs have an alignment angle $\eta_{\text{min}}$ that is about 0.38 radians, it occurs down the tail of the curve on the side where the step-function is unity and the curve is a Gaussian.

It is important for the application here to notice that the step-function is unity along the tail of the distribution on the left, the $\eta_{\text{min}} \rightarrow 0$, side. The well-aligned sample of 27 QSOs has a smallest alignment angle around $\eta_{\text{min}} = 0.38$ radians, which is down the tail a bit, see the blue arrow in Fig. 7. The net effect of the steep right side of the distribution is to raise the probability of the observed $\eta_{\text{min}}$ by about 20%. Since random runs are thereby more likely in the region of the observed result, that makes the observed result somewhat less significant than if the distribution were symmetric and Gaussian.
Figure 7. The distribution of the 10,000 values of the smallest alignment angle $\eta_{\text{min}}$ from $R = 10,000$ random runs. The height $\Delta R$ is the number of runs with $\eta_{\text{min}}$ in the designated range of each bin. The fraction $\Delta R/R$ represents the likelihood that a random run result $\eta_{\text{min}}$ is in the bin. Thus the histogram approximates the shape of the probability distribution, aside from a normalizing scale factor. The observed polarization directions determine a value of $\eta_{\text{min}}$ at the blue arrow that is just a little bit lower than the lowest populated bin.

To find the significance of the observed smallest alignment angle $\eta_{\text{min}} = 21.09^\circ$, we integrate the probability distribution to find the likelihood that a random run would produce a smaller value. The significance is found to be $0.44$ to $4.5 \times 10^{-5}$ or about one in 22 to 230 thousand random runs would be better aligned than is observed for these QSOs. The range here is based on the distribution of random runs in Fig. 7.

Experimental uncertainty of the polarization directions yields $\eta_{\text{min}} = 21.64^\circ \pm 0.86^\circ$, as found in Sec. 4 above and Part II Appendix Sec. 6b below. The corresponding range in significances is $0.72 \times 10^{-5}$ to $5.25 \times 10^{-5}$, or about one in 19 to 139 thousand would give a smaller alignment angle than the observed polarization directions provide. The range here is based on the distribution of uncertainty runs in Fig. 5. The alignment of the polarization directions with the hub $H_{\text{min}}$ is, therefore, very significant.

6. Conclusions

The polarization directions of these 27 QSOs are well-aligned with a point on the Celestial Sphere, the hub $H_{\text{min}}$, that is very close to the sample, see Figs. 4 and 6. Finding a correlation among polarization directions that display parallax is a property that distinguishes the Hub Test from other tests. Thus, the 27 QSOs offer a satisfying illustration of the Hub Test.

It is unlikely that the alignment is a consequence of selection bias. These 27 QSOs, Clump 1 in Fig. 1, are not alone; a sample of 13 QSOs, Clump 2, has been evaluated by the Hub Test. Clump 1 is better aligned than one in more than 20,000 random runs, similar to the significance of the alignment for Clump 2. Since the survey of $5^\circ$-radius regions, Fig. 1, involves 1863 regions, the number of regions considered is not close to the number 20,000 taken twice. It seems that the alignments are not due to selection bias.

By the survey in Fig. 1, one sees that very significantly aligned regions are rare with QSOs. This is unlike polarized starlight sources in the Milky Way which has a large proportion of $5^\circ$ regions well aligned, with $-\log_{10}(S)$ often well over 9, when surveyed as in Fig. 1. See Ref. 11. Another difference is the percent polarization. While typical QSOs have percent polarizations of 1% or more, starlight is usually less, a few tenths of a percent. Thus, QSOs, in general, have a higher percent polarization, but with lower significances of the alignments, than is typical with starlight. It may be worthwhile to search the sky near the very significant regions, the color dots in Fig. 1, or near the hubs in Fig. 3, for objects that may have a polarizing effect on the radio waves from the QSOs.

While the article, Ref. 6, relating alignments to Large Scale Structure constrains the QSOs to have like-redshifts, one might argue that the alignment found in this article is due to a subset of the 27 QSOs with more-or-less equal redshifts. One might try the 15 or so QSOs in the sample with redshifts between 1.0 and 1.5. Then the alignment would speak to Large Scale Structures, as in Ref. 6.

Astronomical data is being acquired at fantastic rates. Investigations of new data would be intriguing. However, the main motivation for this study is to illustrate an application of the Hub Test, an application involving parallax which makes it special. Interpreting the results is deemed beyond the scope of the study. One hopes the results are of interest and potentially useful.
7. References

1. Shurtleff, R., the ready-to-run Mathematica version of this notebook is available at the following URL:
   https://www.dropbox.com/s/rpamuesvg0zu17w/20211030ReplaceClump1PaperFirst.nb?dl=0
13. The JVAS1450 catalog was emailed to me with kind thanks from V. Pelgrims.
17. The NASA/IPAC Extragalactic Database (NED), https://ned.ipac.caltech.edu/
Part II  Computer Program

1. Introduction to Part II

The following computer program, a Mathematica notebook, performs the calculations made to evaluate the alignment of the sources in the sample under consideration. The setup is similar to that in Refs. 10 and 11.

Since Mathematica encodes the instructions, it is inconvenient to try to run the computer program from the pdf version of this work. A viable .nb version that runs on Mathematica is available by following the link in Ref. 1.

2. Coordinates, utility functions, derivation of basic formula

2a. Coordinates, utility functions

Consider the “Celestial Sphere”, a sphere with unit radius in 3 dimensional Euclidean space. See Figs. 1, 2, 3 in the article, Part 1 above. The sphere is also called the “sphere” or sometimes “the sky”. Picture the dome of a planetarium viewed from the outside. The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates (x, y, z). The direction of the positive z-axis is due “North”. Equatorial longitude is the Right Ascension $\alpha$ and latitude is the declination $\delta$.

Definitions:

- homeDirectory directory containing the notebook and data files
- Utilities:
  - $er$, $eN$, $eE$ unit vectors in a 3D Cartesian coordinate system
  - $(\alpha, \delta)$ equatorial coordinates longitude and latitude
  - $er(\alpha, \delta)$ radial unit vectors from Origin
  - $eN(\alpha, \delta)$ local North at a point on the Celestial Sphere
  - $eE(\alpha, \delta)$ local East at a point on the Celestial Sphere
  - $\alpha$-FROMr($er$) $\alpha$ determined by a radial unit vector $er$
  - $\delta$-FROMr($er$) $\delta$ determined by a radial unit vector $er$

Aitoff Plot Functions:

- $\alpha$HA($\alpha, \delta$), $xH(\alpha, \delta)$, $yH(\alpha, \delta)$, where $xH$ is centered on $\alpha$ = 0 and $\alpha$ increases from left-to-right, with $\alpha$ = -180° on the left and +180° on the right
- $xH180(\alpha, \delta)$, $yH180(\alpha, \delta)$, where $xH$ is centered on $\alpha$ = 180° and $\alpha$ increases from left-to-right, with $\alpha$ = 0° on the left and 360° on the right

mean the arithmetic average of a set of numbers, $\frac{1}{N} \sum_{i=1}^{N} n_i$

stanDev the standard deviation. Given a set of $N$ numbers $n_i$ with mean value $m$, the standard deviation is $\sqrt{\frac{1}{N} \sum_{i=1}^{N} (n_i - m)^2}$, the square root of the average of the squares of the differences of the numbers with the mean. Note that we divide by $N$ to get the average of the deviations squared.

Derivation of $\eta_H$:

denoSquared1 magnitude of $r_H - (r_H, r_S) r_S$ part of the formula for $v_{Ht}$, see Fig. 2
In[1]:= \[\text{Print["The computer time expended so far is ", \text{TimeUsed[]}, " seconds."\}]
\text{Print["The date and time that this statement was evaluated: ", \text{Now}\}]

The computer time expended so far is 0.594 seconds.

The date and time that this statement was evaluated: Sat 6 Nov 2021 08:24:53 GMT–4.

In[2]:= \text{homeDirectory = "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJ_PCEJPetc\\SendViXra\\20200715AlignmentMethod\\20210505AlignmentMethod\v4\\20210515Clump1QSOsNearNGP"; (*The notebook file and data files for this notebook are put in this directory. *)

In[3]:= \{\text{Check er.er = 1, er.eN = 0, er.eE = 0, eNeN = 1, erXeE = enN, eEXen = er, enXer = ee: }, True\}

Out[3]= \{\text{Check er.er = 1, er.eN = 0, er.eE = 0, eNeN = 1, erXeE = enN, eEXen = er, enXer = ee: }, True\}

Get \((\alpha, \delta)\) in radians from a radial vector \(r\):

In[4]:= \{\text{aFROMr}[r_] := N[\text{ArcTan}[\text{Abs}[r[[2]]/r[[1]]]]] \}; (r[[2]] \geq 0 \&\& r[[1]] > 0)
\text{aFROMr}[r_] := N[\pi - \text{ArcTan}[\text{Abs}[r[[2]]/r[[1]]]]] \}; (r[[2]] \geq 0 \&\& r[[1]] < 0)
\text{aFROMr}[r_] := N[\pi + \text{ArcTan}[\text{Abs}[r[[2]]/r[[1]]]]] \}; (r[[2]] < 0 \&\& r[[1]] > 0)
\text{aFROMr}[r_] := \pi/2. \}; (r[[2]] \geq 0 \&\& r[[1]] = 0)
\text{aFROMr}[r_] := 3/\pi . 2. \}; (r[[2]] < 0 \&\& r[[1]] = 0)
\text{aFROMr}[r_] := N[\text{ArcTan}[r[[3]]/(\sqrt{(r[[1]]^2 + r[[2]]^2)]})] \}; (\sqrt{(r[[1]]^2 + r[[2]]^2)]}) > 0)
\text{dFROMr}[r_] := \text{Sign[r[[3]]] \(\pi/2. \}; (\sqrt{(r[[1]]^2 + r[[2]]^2)]}) = 0\}

The following Aitoff Plot formulas can be found in Wikipedia, Ref. 16.

For these formulas the angles \(\alpha\) and \(\delta\) should be in degrees.

They give an Aitoff Plot that is centered on \(0^0,0^0\).

The quantity "\(\text{H}\)" is the RA coordinate of a point H on the Celestial Sphere. Thus, we use "\(\alpha\text{H}\)" for Aitoff function.

In[6]:= \text{aHA}[\alpha_, \delta_] := \text{ArcCos}[\text{Cos[}((2. \pi) / 360. \) \text{Cos[}((2. \pi) / 360. \) \alpha / 2.)]]
\text{xH}[\alpha_, \delta_] := \text{2. Cos[}((2. \pi) / 360. \) \delta \text{Sin[}((2. \pi) / 360. \) \alpha / 2.)]/\text{Sinc[aHA[\alpha, \delta]]]
\text{yH}[\alpha_, \delta_] := \text{Sin[}((2. \pi) / 360. \) \delta /\text{Sinc[aHA[\alpha, \delta]]]"}
Using the following functions produces an Aitoff Plot that is centered on \((180^\circ, 0^\circ)\)

\[
\begin{align*}
\text{In}[19] &= \\
\text{xH180} &= \\
xH180[\alpha, \delta_] := \\
xH180[\alpha, \delta_] = \left(2 \cos \left(\frac{(2 \cdot \pi)}{360.} \cdot \delta\right) \sin \left(\frac{(2 \cdot \pi)}{360.} \cdot (\alpha - 180.) / 2.\right) / \text{Sinc}\left[\text{aHA}\left[(\alpha - 180.), \delta\right]\right]\right) \\
yH180 &= \\
yH180[\alpha, \delta_] := \text{yH180}[\alpha, \delta_] = \sin \left(\frac{(2 \cdot \pi)}{360.} \cdot \delta\right) / \text{Sinc}\left[\text{aHA}\left[(\alpha - 180.), \delta\right]\right]
\end{align*}
\]

\[
\begin{align*}
\text{In}[21] &= \text{mean}[\text{data}_-] := \\
\text{mean}[\text{data}_-] &= \left(\frac{1}{\text{Length}[\text{data}]}\right) \text{Sum}[\text{data}[[i4]], \{i4, \text{Length}[\text{data}]\}]; \\
\text{(* arithmetic average *)} \\
\text{stdDev}[\text{data}_-] := \\
\text{stdDev}[\text{data}_-] &= \left(\left(\frac{1}{\text{Length}[\text{data}]}\right) \text{Sum}[\left(\text{data}[[i5]] - \text{mean}[\text{data}]\right)^2, \{i5, \text{Length}[\text{data}]\}\right)^{1/2} \\
\text{(*standard deviation*)}
\end{align*}
\]

2b. Derivation of a formula for the alignment angle \(\eta_{ih}\) given the position \(r_S\) of the \(i\)th source, the location \(r_H\) of point \(H\), and the polarization direction \(\psi\) for the \(i\)th source

From Fig 2b, we see that \(\cos \psi = \vec{v_H} \cdot \vec{r_H}\), Eq. 2.

\[
\vec{v_H} = \frac{\frac{dH}{d}\left(\frac{dH}{d}\right) \vec{r_H}}{\left(\frac{dH}{d} - \left(\frac{dH}{d}\right) \vec{r_H}\right) \vec{r_S}} : \text{unit vector in the 2D tangent plane at S, in the direction of H from S, vH.rS = 0, where}
\]

\(\text{er}[\alpha H, \delta H].\text{er}[\alpha S, \delta S] = \vec{r_H} \cdot \vec{r_S}\) is the inner product of the radial unit vectors \(\vec{r_H}\) and \(\vec{r_S}\) to point \(H\) and source \(S\)

Since \(\psi\) is also perpendicular to \(\vec{r_S}\), it follows that \(\psi \cdot \vec{r_S} = 0\), and we have \(\frac{dH}{d} - \left(\frac{dH}{d}\right) \vec{r_H}\) as the part of \(\vec{v_H}\) that contributes to the dot product \(\cos \psi = \vec{v_H} \cdot \vec{r_H}\). Therefore, define

\[
\vec{v_H}_{\perp \vec{r_S}} = \frac{dH}{d} \left(\frac{dH}{d} - \left(\frac{dH}{d}\right) \vec{r_H}\right) \vec{r_S}
\]

Simplify the denominator,

\[
\text{In}[23] = \text{denoSquared1} = \text{FullSimplify}[\text{er}[\alpha H, \delta H] - (\text{er}[\alpha H, \delta H] \cdot \text{er}[\alpha S, \delta S]) \cdot \text{er}[\alpha S, \delta S]]; \\
\text{denoSquared1} = \left\{\text{er}[\alpha H, \delta H] - (\text{er}[\alpha H, \delta H] \cdot \text{er}[\alpha S, \delta S]) \cdot \text{er}[\alpha S, \delta S]\right\} \\
\text{d} = \text{denoSquared1} = \text{FullSimplify}\left[\text{denoSquared1} - \left(1 - (\text{er}[\alpha H, \delta H] \cdot \text{er}[\alpha S, \delta S])^2\right)^2\right] \text{(*check that*)}
\]

\[
\text{Out}[24] = 0
\]

Write the formula for the vector \(\vec{v_H}_{\perp \vec{r_S}}\), with a denominator of \(\left(1 - (\text{er}[\vec{r_H}])^2\right)^{1/2}\):

\[
\text{In}[25] = \text{vH}_{\perp \vec{r_S}} = \text{er}[\alpha H, \delta H] \left(\left\{\frac{dH}{d} - (\text{er}[\alpha H, \delta H] \cdot \text{er}[\alpha S, \delta S])\right\}^2\right)^{1/2}
\]
Thus, \( \alpha \delta H = S * \approx \) since \( (*) \delta H \) with Indeterminate, \( S \), \( H \), \( (*) \delta \psi \) is the acute angle between \( v \) and \( v_H \) in the 2D tangent plane at \( S \). By Eq. 2,

\[
\alpha \delta H, H, \delta, S \left[ \delta, S, e^\psi \right] + \left[ \right]_\delta \psi \]

The alignment angle \( \eta \) is the acute angle between \( v_H \) and \( v \) in the 2D tangent plane at \( S \). By Eq. 2,

\[
\eta \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] := \text{ArcCos} \left[ \frac{\text{Abs} \left[ \psi \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] . vH \right] }{vH \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] } \right] \]

FullSimplify \[ \eta \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] \]

Out[29]=

\[
\text{ArcCos} \left[ \frac{\text{Cos} \left[ \delta S \right] \text{Cos} \left[ \psi \right] \text{Sin} \left[ \delta H \right] + \text{Cos} \left[ \delta H \right] \left( - \text{Cos} \left[ \alpha H - \alpha S \right] \text{Cos} \left[ \psi \right] \text{Sin} \left[ \delta S \right] + \text{Sin} \left[ \alpha H - \alpha S \right] \text{Sin} \left[ \psi \right] \right) }{\sqrt{\left( 1 - \left( \text{Cos} \left[ \alpha H - \alpha S \right] \text{Cos} \left[ \delta H \right] \text{Cos} \left[ \delta S \right] + \text{Sin} \left[ \delta H \right] \text{Sin} \left[ \delta S \right] \right)^2 }^2 }} \right] \]

In[30]= (*The following function is well-behaved everywhere except where \( \pm H \) coincides with \( \pm S \).*)

\[
\eta H\text{withIndeterminate} \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] := \text{ArcCos} \left[ \frac{\left( \text{Cos} \left[ \delta S \right] \text{Cos} \left[ \psi \right] \text{Sin} \left[ \delta H \right] + \text{Cos} \left[ \delta H \right] \left( - \text{Cos} \left[ \alpha H - \alpha S \right] \text{Cos} \left[ \psi \right] \text{Sin} \left[ \delta S \right] + \text{Sin} \left[ \alpha H - \alpha S \right] \text{Sin} \left[ \psi \right] \right) }{\sqrt{\left( 1 - \left( \text{Cos} \left[ \alpha H - \alpha S \right] \text{Cos} \left[ \delta H \right] \text{Cos} \left[ \delta S \right] + \text{Sin} \left[ \delta H \right] \text{Sin} \left[ \delta S \right] \right)^2 }^2 }} \right] \]

In[31]= (*Since \( \eta \) is an acute angle, let us take the halfway value, \( \eta = \pi/4 \) in the neighborhood where \( H = S \).* )

\[
\eta H \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] := \eta H\text{withIndeterminate} \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] ; \left( 1 - \left( \text{er} \left[ \alpha H, \delta H \right] . \text{er} \left[ \alpha S, \delta S \right] \right)^2 \right) \geq 0.000001 \]

\[
\eta H \left[ \alpha S, \delta S, \alpha H, \delta H, \psi \right] := \pi / 4. ; \left( 1 - \left( \text{er} \left[ \alpha H, \delta H \right] . \text{er} \left[ \alpha S, \delta S \right] \right)^2 \right) < 0.000001 \]

Print[

"Thus \( \eta H = \pi/4 \) wherever \( \pm H \) is 'close' to \( \pm S \), with 'close' meaning within an angle of ",

\[ \text{ArcSin}[0.000001^{1/2}] \], " radians, or ", \[ \text{ArcSin}[0.000001^{1/2}] \left( \frac{360}{2 \pi} \right) \] "°."

Thus \( \eta H = \pi/4 \) wherever \( \pm H \) is 'close' to \( \pm S \), with 'close' meaning within an angle of 0.001 radians, or 0.0572958°.

3. Polarization and Position Data

3a. Source Data

The JVAS1450 catalog incorporates data from the large JVAS/CLASS 8.4 Ghz catalog Jackson 2007, Refs. 12,13,14. The
JVAS1450 catalog sources were filtered from Jackson 2007 sources by identification as QSOs. Filters: for percent polarization, \( p > 0.6\% \), for the largest fractional uncertainty in percent polarization, \( \sigma_p/p < 0.6\% \), and for uncertainty in the polarization position angle \( \sigma_\psi < 16^\circ \).

We consider Quasi-Stellar Objects, QSOs. From the data in JVAS1450, \( 5^\circ \) radius regions are constructed, one centered at each
of the 10518 grid points of a \( 2^\circ \times 2^\circ \) mesh. The 1450 QSOs were assigned to the regions based on location and we calculated the
significance of the alignment of the polarization directions for the sources in each region.

The three such QSO regions selected for this notebook satisfied many requirements: (i) have 7 or more sources in order to use
the significance formulas in Sec. 4 accurately, (ii) have longitude RA \( 160^\circ \leq \alpha \leq 180^\circ \), (iii) have latitude dec \( 40^\circ \leq \delta \leq 55^\circ \), (iv)
whose QSOs are very significantly aligned, \( S \leq 10^{-2} \). There are 3 regions satisfying (i) - (iv) containing a total of 27 sources. See Fig.
1.

Definitions:

data00 the catalog data, JVAS1450
secondClumpQsoIDInData001450 - record numbers in the catalog of the QSOs in the sample
nSrc number of sources
\( \alpha_{\text{Src}} \) right ascension of the sources, longitude (radians)
\( \delta_{\text{Src}} \) declination of the sources, latitude (radians)
\( \psi_{\text{Src}} \) PPA, polarization position angle of the sources: clockwise from North with East to the right.
\( \sigma_{\psi_{\text{Src}}} \) uncertainty in PPA
percentPol percentage of linear polarization of the sources
redshift redshift, no uncertainty reported
rSrc unit vectors from the Origin to Sources on Celestial Sphere
eNSrc Local North at each Source
eESrc Local East at each Source
\( \eta_{\text{BarAtHwithAny}} \psi \) alignment angle function \( \eta(H) \), Eqn. 1, obtained using the location of the sources
sourceCenter unit radial vector to the arithmetic center of the sources
\( \alpha_{\text{SourceCenter}} \) Right Ascension at the sourceCenter
\( \delta_{\text{SourceCenter}} \) Declination at the sourceCenter
angleSourceToCenter angle from each Source to the sourceCenter
\( \rho_{\text{RgnRadius}} \) angle to the furthest QSO from the sourceCenter
\( \rho_{\text{RMS}} \) root-mean-square angular distance to the sources from the sourceCenter

Alternate names:
A position search of the NASA/IPAC Extragalactic Database (NED)*, Ref. 17, returned the following names of 27 QSOs whose
position is coincident with those reported in the JVAS1450 catalog:
\{QSO #, ID from NED\} =
{1, [HB89] 1111-149}, {2, WISEA J111609.96+082922.1}, {3, [HB89] 1116-128},
{4, [HB89] 1119-183}, {5, WISE J112736.52+055532.0}, {6, WISEA J112907.69-164322.6},
{7, WISE J113036.99-105401.2}, {8, WISEA J113613.49+144819.7},
{9, WISEA J114120.70+100524.3}, {10, WISEA J114207.75+154754.0},
{11, [HB89] 1142-052}, {12, WISEA J115225.91+073357.5}, {13, [HB89] 1150-095},
{14, [HB89] 1151-102}, {15, [HB89] 1155-169}, {16, WISE J115910.42+030211.0},
{17, WISEA J115923.73+015223.8}, {18, WISEA J120301.01+063441.1},
{19, PKS 1200-045}, {20, WISEA J120518.70+052748.4}, {21, WISEA J120712.62+121145.8},
{22, WISEA J121459.93-082922.5}, {23, LBQS 1213-0922}, {24, LBQS 1215-1121},
{25, WISEA J121827.99-061659.0}, {26, [HB89] 1219-044}, {27, WISEA J122354.62+065002.7}

Note that there is a disagreement in the redshift values for object 10. “WISEA J114207.75+154754.0”, JVAS: z = 0.299 and NED: z = -0.000435. The other redshifts were nearly the same in both NED and JVAS1450.

These identifications are FYI, for your information. No data from the NED search is used in this notebook.

*The NASA/IPAC Extragalactic Database (NED) is funded by the National Aeronautics and Space Administration and operated by the California Institute of Technology.*

```
In[34]:= (*Recorded here for personal use. The QSO data needed is copied below. *)
firstClumpQsosIDinData001450 = {659, 660, 663, 667, 674, 680, 682, 690, 695, 696, 698,
707, 712, 714, 718, 720, 721, 727, 728, 731, 734, 744, 746, 751, 752, 762, 764};

In[35]:= (*right ascension in radians*)
αSrc = 10^-6.
{2940786, 2950332, 2962501, 2977947, 3000259, 3006888, 3013383, 3037854, 3060196,
3063615, 3077693, 3108571, 3111962, 3114578, 3111037, 3137907, 3138954, 3154756,
3156278, 3164771, 3173054, 3207036, 3209928, 3222030, 3222168, 3239225, 3245921};

In[36]:= nSrc = Length[αSrc]
Out[36]= 27

In[37]:= (*declination in radians*)
δSrc = 10^-6.
{256694, 148170, 219533, 315742, 103421, 291870, 190246, 258405,
176105, 275734, 85942, 132052, 161164, 173344, 290596, 52995, 32695, 114811,
73978, 95356, 212862, 148171, 158862, 193466, 109659, 73672, 119278};

In[38]:= (* position angle in radians*)
ψSrc = 10^-6.
{1788962, 1120501, 2185152, 2724459, 2022837, 2553417, 2045526, 2857104, 1733112,
2485349, 1877974, 2331760, 2460809, 2277655, 1937315, 1106539, 1799434, 2961824,
2586578, 2912955, 1925098, 2605541, 2188643, 2352704, 2827433, 1527163, 2905973};

In[39]:= histψData = Histogram[ψSrc 360. / 2. π], {12}, PlotLabel -> "PPA ψ, number ΔR per bin",
AxesLabel -> {"ψ", "ΔR"}, PlotRange -> {{0, 200}, Automatic}];
Figure 8: Distribution of position angles for the 27 polarization directions in the sample. Note the wide distribution over a hundred degrees or so, $\psi = 60^\circ$ to $\psi = 160^\circ$.

In[42]:= (*uncertainty in $\psi$ in radians*)
$\sigma_{\psi}\text{Src} = 10^{-6}$.
{4242, 252, 2254, 99, 106992, 51458, 112351, 26729, 137622, 18357, 10877, 271821, 37352, 134004, 48856, 98592, 277921, 7249, 5633, 5724, 66923, 35001, 138200, 114372, 105062, 7815, 7653};

In[43]:= (* % polarization*)
percentPol = $10^{-6}$.
{2386846, 4130478, 2023713, 1658885, 1784232, 1979194, 2210679, 6381769, 5954787, 2903853, 3866300, 3070517, 1080690, 1854161, 492130, 2652914, 1021777, 3754306, 1874058, 3174907, 604797, 653203, 5457402, 271821, 37352, 134004, 48856, 98592, 277921, 7249, 5633, 5724, 66923, 35001, 138200, 114372, 105062, 7815, 7653};

In[44]:= (*uncertainty in % polarization*)
$\sigma_{\text{percentPol}} = 10^{-6}$.
{20249, 2078, 9121, 328, 381771, 203679, 341137, 1638906, 106607, 84105, 1669146, 80727, 496898, 48084, 523076, 5679057, 54428, 21111, 36344, 80945, 45723, 1508313, 140783, 3405959, 14090, 50611};

In[45]:= (*Redshift*)
redshift = $10^{-6}$.
{867400, 486000, 2125700, 1040000, 2217000, 1996700, 1323900, 603700, 1051400, 299000, 1343600, 876100, 695900, 895000, 1061200, 1009800, 2440000, 2189000, 1226000, 1300000, 890500, 2359000, 2721600, 1404000, 2078200, 966000, 1189000};

In[46]:= redshiftFromNED = N[10^{-6} (865791, 486000, 2125285, 1040957, 575, 1996476, 1323300, 603397, 1049919, -435, 1338667, 876665, 695517, 893539, 1060245, 1008851, 2440000, 2180078, 1224294, 1297708, 891817, 2366153, 2720127, 1402712, 2073279, 966360, 1189000)]
Out[46]= {0.865791, 0.486, 2.12529, 1.04096, 0.000575, 1.99648, 1.3223, 0.603397, 1.04992, -0.000435, 1.33867, 0.876665, 0.695517, 0.893539, 1.06025, 1.008851, 2.44, 2.18008, 1.22429, 1.29771, 0.891817, 2.366153, 2.720127, 1.402712, 2.073279, 0.966360, 1.189}

In[47]:= rSrc = Table[er[ αSrc[[i]], δSrc[[i]] ], {i, nSrc}];(*calculated from Input.*)
enSrc = Table[eN[ αSrc[[i]], δSrc[[i]] ], {i, nSrc}];(*calculated from Input.*)
eESrc = Table[eE[ αSrc[[i]], δSrc[[i]] ], {i, nSrc}];(*calculated from Input.*)
ηBarAtHwithAnyΨ[αH_, δH_, ψ_] :=
1/nSrc Sum[ηH[αSrc[[i]], δSrc[[i]], αH, δH, ψ[[i]], {i, nSrc}]
(*ηBarAtHwithAnyΨ[3.5,0.6,ψSrc]*) (* An example with a selected αH and δH and with the observed polarization directions for ψ*)

sourceCenter0 = 1/nSrc Sum[rSrc[[i]], {i, nSrc}];
sourceCenter = sourceCenter0 - sourceCenter0.sourceCenter0 1/2;
(*unit radial vector to the arithmetic average center of the sources.*) αSourceCenter = αFROMr[sourceCenter]; δSourceCenter = δFROMr[sourceCenter]; angleSourceToCenter = Table[ArcCos[rSrc[[i]].sourceCenter], {i, nSrc}]; ρRgnRadius = Sort[angleSourceToCenter][[-1]]; (*Furthest source from center*)
ρRMS = 1/nSrc Sum[angleSourceToCenter[i]², {i, nSrc}] 1/2;
3b. Section Summary

In[58]:= Print["There are ", nSrc, " sources in the sample."
Print["Check that the Sample obeys the data cuts:"
Print["Check that the smallest % polarization p in the sample is 0.5% or more. Smallest: ", Sort[percentPol][[1]], ", % ."
Print["Check that the largest fractional uncertainty in % polarization, σp/p, is less than 0.6 . Largest: ", Sort[σpercentPol/percentPol][[-1]], ", ."
Print["Check that the largest PPA ψ uncertainty σψ is less than 16°. Largest: ", Sort[σψSrc][[-1]] (360.)/(2. π), ", ° ."]
There are 27 sources in the sample.
Check that the Sample obeys the data cuts:
Check that the smallest % polarization p in the sample is 0.5% or more. Smallest: 0.49213% .
Check that the largest fractional uncertainty in % polarization, σp/p, is less than 0.6 . Largest: 0.555802 .
Check that the largest PPA ψ uncertainty σψ is less than 16°. Largest: 15.9237° .

In[63]:= αδForSrc = ListPlot[
Table[{{αSrc[[j]], δSrc[[j]]} (360.)/(2. π), {j, nSrc}],
PlotRange -> {{0, 360}, {-90, 90}},
Ticks -> {Table[{i, i}, {i, 0, 360, 60}], Table[{j, j}, {j, -90, 90, 30}]},
PlotLabel -> "Sources", AxesLabel -> {"α, degrees", "δ, degrees"}, PlotStyle -> Green];
4. Grid

While we have a formula \( \eta(H) \) for the alignment angle at a point \( H \) on the Celestial Sphere, there are occasions when it is better not to use it and, instead, construct a discrete table of values. To locate the values \( \eta(H) \) at a finite number of points \( H \) on the sphere, we create a grid, or ‘mesh’.

When building the grid, we avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle \( d\theta \).

We grid one hemisphere. Symmetry across diameters gives the other hemisphere. The grid is conveniently built centered at the North pole and then moved so that it is centered on the sample of sources. For detailed work near the sources a 30° finely spaced grid cap is produced to supplement the more coarsely spaced grid. The fine and coarse grids are offset so that no grid points are common to the two grids.

4a. Construct the grid

Definitions:

- \( \text{gridSpacing} \), \( \text{coarseGridSpacing} \) - fine, coarse grid separation in degrees between grid points on and between constant latitude circles
- \( \text{fineCapRadius} \) - radius of the fine grid cap in radians
- \( d\theta_1, d\theta_2 \) - fine, coarse grid spacing in radians
- \( i^dN, a^i, j^j, \delta^j \) - dummy indices
- \( \alpha^H, \delta^H \) - \( \alpha \) and \( \delta \) of the grid points \( H_j \)
- \( \text{fineGrid}, \text{coarseGrid}, \text{gridN}, \text{grid} \) - tables of data associated with grid points, record descriptions below
- \( \text{rotzToSample} \) - rotation matrix from North pole to sourceCenter
- \( \text{lpgrid} \) - plot of the radial unit vectors to the grid points
- \( n\text{Grid} \) - number of grid points
- \( \alpha\text{Grid} \) - longitudes at the grid points (\( -\pi \leq \alpha \leq +\pi \))
- \( \delta\text{Grid} \) - latitudes at the grid points (\( -\pi/2 \leq \alpha \leq \pi/2 \))
\[ \text{rGrid} \quad \text{radial unit vectors from origin to grid points, in 3D Cartesian coordinates} \]

\begin{verbatim}
In[66]:= gridSpacing = 0.6(*degrees*);
    fineCapRadius = 0.5;

In[68]:= (*KEEP this cell - DO NOT DELETE*)
    (*The Northern Grid "gridN".*)
    \[ \delta = \frac{2 \pi}{360} \text{ gridSpacing} \text{ (Convert gridSpacing to radians*)}; \]
    fineGrid = {};
    idN = 1;
    \[ \text{For}\[\delta j = 0., \delta j < \frac{\text{fineCapRadius}}{\delta \text{do1}}, \delta j ++, \text{deltaH} = \frac{\pi}{2} - \delta j \text{ do1} - \frac{\text{do1}}{2.1/2}; \]
    (*\text{Print}["(\delta j,\text{deltaH}) = \{\},\{\delta j,\text{deltaH}\}];\*)
    \[ \text{For}\[\text{ai} = 0., \text{ai} < \text{Ceiling}\left[\frac{2 \pi}{\text{deltaH}}(\text{Cos}[\text{deltaH}] + 0.01)\right], \text{ai} ++, \text{apointH} = \text{ai} \text{ do1}/(\text{Cos}[\text{apointH}] + 0.01); \]
    (*\text{Print}["\{\text{ai},\text{apointH}\} = \{\},\{\text{ai},\text{apointH}\}\];\*)
    \[ \text{AppendTo}\[\text{fineGrid}, \{\text{idN}, \text{ai}, \delta j, \text{apointH}, \text{deltaH}, \text{er}[\text{apointH}, \text{deltaH}]\}\]; \]
    \[ \text{idN} = \text{idN} + 1 \]
]\]
  \(\text{Length}[\text{fineGrid}];\)
  \(\text{lpFine} = \text{ListPointPlot3D}[\text{Table}[\text{fineGrid}[[i, 6]], \{i, 1, \text{Length}[\text{fineGrid}], 10\}], \text{PlotRange} \rightarrow \{(\text{-1.2, 1.2}), (\text{-1.2, 1.2}), (\text{-1.2, 1.2})\}, \text{AxesLabel} \rightarrow \{"x", "y", "z"\}, \text{BoxRatios} \rightarrow \{1, 1, 1\}];\)

Coarse Grid band runs from latitude (\(\frac{\pi}{2} - \text{fineGridMAX}\)) to latitude (\(\frac{\pi}{2} - \text{southOfEquator}\))

In[72]:= (*KEEP this cell - DO NOT DELETE*)
    (*The coarse grid band. *)
    \[ \text{do2} = \frac{2 \pi}{360} \] \(\text{coarseGridSpacing} \text{ (Convert grid spacing to radians*)}; \)
    \(\text{coarseGrid} = {};\)
    \(\text{idB} = 1 + \text{Length}[\text{fineGrid}];\) (*ID for the coarse band grid points*)
    \[ \text{For}\[\delta j = 0., \delta j < \frac{\text{coarseEnd} - \text{coarseStart}}{\text{do2}}, \delta j ++, \text{deltaH} = \frac{\pi}{2} - \text{coarseStart} - \delta j \text{ do2} - \frac{\text{do2}}{3.1/2}; \]
    (*\text{Print}["(\delta j,\text{deltaH}) = \{\},\{\delta j,\text{deltaH}\}];\*)
    \[ \text{For}\[\text{ai} = 0., \text{ai} < \text{Ceiling}\left[\frac{2 \pi}{\text{deltaH}}(\text{Cos}[\text{deltaH}] + 0.01)\right], \text{ai} ++, \text{apointH} = \text{ai} \text{ do2}/(\text{Cos}[\text{apointH}] + 0.01); \]
    (*\text{Print}["\{\text{ai},\text{apointH}\} = \{\},\{\text{ai},\text{apointH}\}\];\*)
    \[ \text{AppendTo}\[\text{coarseGrid}, \{\text{idB}, \text{ai}, \delta j, \text{apointH}, \text{deltaH}, \text{er}[\text{apointH}, \text{deltaH}]\}\]; \]
    \[ \text{idB} = \text{idB} + 1 \]
]\]
  \(\text{lpCoarse1} = \text{ListPointPlot3D}[\text{Table}[\text{coarseGrid}[[i, 6]], \{i, 1, \text{Length}[\text{coarseGrid}], 10\}], \text{PlotRange} \rightarrow \{(\text{-1.2, 1.2}), (\text{-1.2, 1.2}), (\text{-1.2, 1.2})\}, \text{AxesLabel} \rightarrow \{"x", "y", "z"\}, \text{BoxRatios} \rightarrow \{1, 1, 1\}];\)
  \(\text{Length}[\text{coarseGrid}];\) (*\text{Show}[\{\text{lpFine},\text{lpCoarse}\}];\*)
\end{verbatim}
Now we need to rotate the combined fine/coarse grid ‘gridN’ so that it is centered on the sample, the `sourceCenter`.

```math
\text{In}[78]= \text{rotzToSample} = \text{RotationMatrix}[{{0, 0, 1}, \text{sourceCenter}}];
\%
\text{In}[81]= \text{gridN} = \text{Join}[\text{fineGrid}, \text{coarseGrid}];
\text{grid} = \text{Table}[\text{gridN}[i, 1], \text{gridN}[i, 2], \text{gridN}[i, 3], \text{gridN}[i, 4],
\text{gridN}[i, 5], \text{rotzToSample}.\text{gridN}[i, 6]], \{i, \text{Length[gridN]}}];
\text{nGrid} = \text{Length[grid]};
\text{lpgrid} = \text{ListPointPlot3D[Table[grid[[i, 6]], \{i, 1, \text{Length[grid]}, 10\}],}
\text{PlotRange} \to \{\{-1.2, 1.2\}, \{-1.2, 1.2\}, \{-1.2, 1.2\}},
\text{AxesLabel} \to \{"x", "y", "z"}, \text{BoxRatios} \to \{1, 1, 1\}}];
```

![Image of grid](image)

**Figure 10:** The grid. The grid is centered on the source sample, with a finely spaced cap. The grid covers one hemisphere, centered on the sample. The fine and coarse grids are off-set, so they do not share any grid points. There are 12485 grid points on the hemisphere.

```math
\text{In}[87]= \text{αGrid} = \text{Table}[\alpha\text{FROMr[grid[[j, 6]]}, \{j, \text{Length[grid]}\];
\text{δGrid} = \text{Table}[\delta\text{FROMr[grid[[j, 6]]}, \{j, \text{Length[grid]}\];
\text{rGrid} = \text{Table}[grid[[j, 6]], \{j, \text{Length[grid]}\};
```

4b. Section Summary
The fine grid on the 'cap' has 7459 grid points.
The grid points on the cap are separated by gridSpacing = 0.6° in latitude and longitude.

On the entire hemisphere, there is a second set of grid points that are separated by gridSpacing = 2.° in latitude and longitude. The two sets do not share any grid points.
The second set has 5026 grid points.
The total grid, 'grid', has 7459 + 5026 = 12485 grid points.

5. The alignment function $\eta(H)$ for the sample of sources

"Best" means we use the $\psi_{\text{Src}}$ that were listed in the catalog. We calculate the alignment function $\eta(H)$ at the grid points $H$.

Given the alignment function $\eta(H)$, one can find the smallest alignment angle $\eta_{\text{min}}$ and the largest avoidance angle $\eta_{\text{max}}$.

5a. Determine the alignment angle $\eta(H)$

First find $\eta(H_j)$ on the grid and find the smallest and largest values of the alignment function on the grid. Then use the function $\eta_{\text{BarAtHwithAny}}$ derived in Secs. 2 and 3 to go between grid points and locate the smallest and largest angles, $\eta_{\text{min}}$ and $\eta_{\text{max}}$, and their locations, the hubs $H_{\text{min}}$ and $H_{\text{max}}$. These are the extremes for convergence and divergence of the polarization directions.

Definitions:

$v\psi_{\text{Src}}$ unit vectors along the polarization directions $\psi$ in the tangent planes of the sources
$eN$ local unit vectors along local North
$eE$ local unit vectors along local East
grid$\eta_{\text{BarHj}}$ \{j, $\eta(H_j)$ \}, where j is the index for grid point $H_j$ and $\eta(H)$ is the average alignment angle at $H_j$. See Eq. (1).
sortgrid$\eta_{\text{BarHj}}$ \{j, $\eta(H_j)$ \}, with smallest angles $\eta(H)$ first.
grid$\eta_{\text{BarMin}}$ index j for the grid point $H$ with the smallest value of $\eta(H)$
grid$\eta_{\text{BarMin index}}$ j for the grid point $H$ with the smallest value of $\eta(H)$
grid$\eta_{\text{BarMax}}$ largest $\eta(H)$ on grid
grid$\eta_{\text{BarMax index}}$ j for the grid point $H$ with the largest value of $\eta(H)$
\( \eta_{\text{min}} \) \( \Omega_{\text{obs}} \) smallest \( \eta(H) \) and \( H \), local min near grid \( j \) \( \eta_{\text{BarMin}} \) use \( \eta_{\text{BarAtHwithAny}} \) off-grid

\( \eta_{\text{max}} \) \( \Omega_{\text{obs}} \) largest \( \eta(H) \) and \( H \), local max near grid \( j \) \( \eta_{\text{BarMax}} \)

funcDataObs off-grid data for extreme alignment angles \( \eta \) and their hubs \( H \)

\( \eta_{\text{BarMin}} \) \( \text{funDataObs} \) \( \eta_{\text{BarMax}} \) \( \text{funDataObs} \) \( \eta_{\text{min}} \) \( \eta_{\text{max}} \)

\( H_{\text{min}} \) location RA \( \alpha \) in radians
\( H_{\text{min}} \) location dec \( \delta \) in radians
\( H_{\text{min}} \) location (RA,dec) = (\( \alpha \), \( \delta \)) in radians

\( H_{\text{max}} \) location RA \( \alpha \) in radians
\( H_{\text{max}} \) location dec \( \delta \) in radians
\( H_{\text{max}} \) location (RA,dec) = (\( \alpha \), \( \delta \)) in radians

\( v_{\psi} \) \( e_{N} \), \( e_{E} \) unit vectors in the tangent plane of each source \( S_{i} \), pointing along the polarization direction, local North, and local East, respectively. See Fig. 2.*

\( v_{\psi} \) \( \text{Src} \) = Table[\( \cos[\psi_{\text{Src}}[[i]]] \) \( e_{N} \)[\( \alpha_{\text{Src}}[[i]] \), \( \delta_{\text{Src}}[[i]] \] ] + \( \sin[\psi_{\text{Src}}[[i]]] \) \( e_{E} \)[\( \alpha_{\text{Src}}[[i]] \), \( \delta_{\text{Src}}[[i]] \] ], \{i, \text{nSrc}\}];

\( \psi_{\text{Src}} \) \( \eta_{\text{BarHj}} \) = Table[\( 1/n\text{Src} \) Sum[ArcCos[\( \text{Abs} \)[\( r_{\text{Grid}}[[j]] \) \( v_{\psi} \) \( \text{Src}[[i]] \) ] / (\( (r_{\text{Grid}}[[j]]) - (r_{\text{Grid}}[[j]]. \( r_{\text{Src}}[[i]] \) \( r_{\text{Src}}[[i]] \) (r_{\text{Grid}}[[j]]) - (r_{\text{Grid}}[[j]]. \( r_{\text{Grid}}[[j]] \) \( r_{\text{Grid}}[[j]] \) ) \( r_{\text{Src}}[[i]] \) )]^{1/2} ] - 0.000001 \), \{i, n\text{Src}\}], \{j, n\text{Grid}\}];

sortgridj\( \eta_{\text{BarHj}} \) = Sort[gridj\( \eta_{\text{BarHj}} \), \#1[[2]] < \#2[[2]] &];
gridj\( \eta_{\text{BarMin}} \) = sortgridj\( \eta_{\text{BarHj}} [[1]] \) ; (* \{j, \( \eta(H_{j}) \)\} for smallest \( \eta(H_{j}) \) *)
grid\( \eta_{\text{BarMin}} \) = gridj\( \eta_{\text{BarMin}} [[2]] \);
gridj\( \eta_{\text{BarMax}} \) = sortgridj\( \eta_{\text{BarHj}} [[-1]] \) ; (* \{j, \( \eta(H_{j}) \)\} for largest \( \eta(H_{j}) \) *)
grid\( \eta_{\text{BarMax}} \) = gridj\( \eta_{\text{BarMax}} [[2]] \);

The results just found on the grid should be close to the results. Use FindMinimum and FindMaximum to go off-grid and get closer.
\[ \eta_{\text{min}} \text{Hobs} = \text{FindMinimum}[[\eta \text{BarAtWithAny}[\alpha H, \delta H, \psi \text{Src}],
{(\alpha H, \alpha \text{Grid}[\text{gridj} \eta \text{BarMin}[1])}], 
(\delta H, \delta \text{Grid}[\text{gridj} \eta \text{BarMin}[1])])]];
\]
\[ \eta_{\text{max}} \text{Hobs} = \text{FindMaximum}[[\eta \text{BarAtWithAny}[\alpha H, \delta H, \psi \text{Src}],
{(\alpha H, \alpha \text{Grid}[\text{gridj} \eta \text{BarMax}[1])]], 
(\delta H, \delta \text{Grid}[\text{gridj} \eta \text{BarMax}[1])])]];
\]
\[ \text{funcDataObs} = \{1, \{\eta_{\text{min}} \text{Hobs}[1], \{\alpha H, \delta H\} /. \eta_{\text{min}} \text{Hobs}[2]\},
\{ \eta_{\text{max}} \text{Hobs}[1], \{\alpha H, \delta H\} /. \eta_{\text{max}} \text{Hobs}[2]\}\};
\]
\[ \text{... FindMinimum: The function value 0.367378 + 1.56099 \times 10^{-9} \text{i} is a real number at} \ (\alpha H, \delta H) = (3.29794, -0.0045653). \]
\[ \text{... FindMaximum: The line search decreased the step size to within the tolerance specified by} \ \text{AccuracyGoal and PrecisionGoal}
\]
\[ \text{but was unable to find a sufficient increase in the function. You may need more than MachinePrecision digits of working}
\]
\[ \text{precision to meet these tolerances.} \]
\]
\[ \text{Out}[104] = \{1, \{0.368159, \{3.30455, -0.0153025\}\}, \{1.16344, \{2.56781, -0.438086\}\}\};
\]
\[ \text{In}[105] =
\]
\[ \eta_{\text{BarMin}} \text{funcDataObs} = \text{funcDataObs}[2, 1];
\text{eta}_{\text{BarMax}} \text{funcDataObs} = \text{funcDataObs}[3, 1];
\text{Hmin}_{\text{bar}} \text{funcDataObs} = \text{funcDataObs}[2, 2, 1];
\text{Hmax}_{\text{bar}} \text{funcDataObs} = \text{funcDataObs}[2, 2, 2];
\text{Hmin}_{\text{bar}} \text{funcDataObs} = \text{funcDataObs}[2, 2, 1];
\text{Hmax}_{\text{bar}} \text{funcDataObs} = \text{funcDataObs}[3, 2, 1];
\text{Hmax}_{\text{bar}} \text{funcDataObs} = \text{funcDataObs}[3, 2, 2];
\text{Hmax}_{\text{bar}} \text{funcDataObs} = \{\text{funcDataObs}[3, 2, 1], \text{funcDataObs}[3, 2, 2]\};
\]
\[ \text{In}[112] =
\]
\[ \text{Print} \{\text{"When moving off-grid, check that the hubs \ Hmin and \ Hmax did not move more than a grid spacing:"}\};
\]
\[ \text{Print} \{\text{"When we found a local minimum, the hub \ Hmin moved off-grid by ",}
\text{ArcCos} \{\text{er} \{\text{Hmin}_{\text{bar}} \text{funcDataObs}, \text{Hmin}_{\text{bar}} \text{funcDataObs}\},
\text{er} \{\alpha \text{Grid}[\text{gridj} \eta \text{BarMin}[1]]], \delta \text{Grid}[\text{gridj} \eta \text{BarMin}[1]]]\} \{360. / 2.\pi\}, \text{"°.}"\};
\]
\[ \text{Print} \{\text{"When we found a local maximum, the hub \ Hmax moved off-grid by ",}
\text{ArcCos} \{\text{er} \{\text{Hmax}_{\text{bar}} \text{funcDataObs}, \text{Hmax}_{\text{bar}} \text{funcDataObs}\},
\text{er} \{\alpha \text{Grid}[\text{gridj} \eta \text{BarMax}[1]]], \delta \text{Grid}[\text{gridj} \eta \text{BarMax}[1]]\} \{360. / 2.\pi\}, \text{"°.}"\};
\]
\[ \text{Print} \{\text{"The alignment hub \ Hmin is ",}
\text{ArcCos} \{\text{er} \{\text{Hmin}_{\text{bar}} \text{funcDataObs}, \text{Hmin}_{\text{bar}} \text{funcDataObs}\}. \text{sourceCenter} \} \{360. / 2.\pi\}, \text{"° from the source center."}\};
\]
\[ \text{Print} \{\text{"The avoidance hub \ Hmax is ",}
\text{ArcCos} \{\text{er} \{\text{Hmax}_{\text{bar}} \text{funcDataObs}, \text{Hmax}_{\text{bar}} \text{funcDataObs}\}. \text{sourceCenter} \} \{360. / 2.\pi\}, \text{"° from the source center."}\};
\]
\[ \text{Print} \{\text{"Now compare that with the grid: The fine grid spacing close to the sources is ",}
\text{gridSpacing}, \text{"°. If the hub is more than ",} \text{fineCapRadius} \{360. / 2.\pi\}, \text{"° from the sample center, then the grid spacing is ",} \text{coarseGridSpacing}, \text{"°.}"\};
\]
When moving off-grid, check that the hubs $H_{\text{min}}$ and $H_{\text{max}}$ did not move more than a grid spacing:

When we found a local minimum, the hub $H_{\text{min}}$ moved off-grid by $0.0328513^\circ$.

When we found a local maximum, the hub $H_{\text{max}}$ moved off-grid by $0.052382^\circ$.

The alignment hub $H_{\text{min}}$ is $15.3924^\circ$ from the source center.

The avoidance hub $H_{\text{max}}$ is $45.8139^\circ$ from the source center.

Now compare that with the grid: The fine grid spacing close to the sources is $0.6^\circ$. If the hub is more than $28.6479^\circ$ from the sample center, then the grid spacing is $2.0^\circ$.

5b. Plot the Alignment Angle Function $\eta(H)$

Definitions

- $\alpha_{\text{HminDegrees}}$, $H_{\text{min}}$ location RA $\alpha$ in degrees
- $\alpha_{\text{HminHours}}$, $H_{\text{min}}$ location RA $\alpha$ in hours
- $\delta_{\text{HminDegrees}}$, $H_{\text{min}}$ location Dec $\delta$ in degrees
- $\alpha_{\text{HmaxDegrees}}$, $H_{\text{max}}$ location RA $\alpha$ in degrees
- $\alpha_{\text{HmaxHours}}$, $H_{\text{max}}$ location RA $\alpha$ in hours
- $\delta_{\text{HmaxDegrees}}$, $H_{\text{max}}$ location Dec $\delta$ in degrees
- $r_{\text{Hmin}}, r_{\text{Hmax}}$ radial unit vectors to the alignment and avoidance hubs $H_{\text{min}}$ and $H_{\text{max}}$
- $r_{\text{PerpHmin}}$, $r_{\text{PerpHmax}}$ a unit vector in the plane of the great circle combining $r_{\text{Center Src}}$ and $r_{\text{Hmin}}$, $r_{\text{Hmax}}$
- $r_{\text{GreatMinCircle}(\theta)}$, $r_{\text{GreatMaxCircle}(\theta)}$ radial unit vector to a point on the great circle
- $\alpha_{\text{GreatMin}}$, $\alpha_{\text{GreatMax}}$, longitude at the point for $\theta$
- $\delta_{\text{GreatMin}}$, $\delta_{\text{GreatMax}}$, latitude at the point for $\theta$
- $x_{\text{yAitoffGreatMin}}$, $x_{\text{yAitoffGreatMax}}$ Aitoff plot coordinates for the great circles
- $r_{\text{crossMin}}$, $r_{\text{crossMax}}$ unit vector perpendicular, normal to the plane of the great circle
- $\theta_{\text{minMAXgreatcircles}}$, angle between the vectors normal to the planes of the two great circles

- $\{\alpha_j, \delta_j, \eta(H)\}$ at each grid point $H = H_j$, in degrees
- $\{x, y, \eta(x,y)\}$, where $x, y$ are Aitoff coordinates and $\eta(x,y)$ is the alignment angle on grid
- $\{x,y\}$ Aitoff coordinates for the sources’ locations on the sphere
- $d_j$ContourPlot, separation of successive contour lines, in degrees
- listCP, list contour plot of $\eta(H)$ from $x\eta_{\text{BarAitoffTable}}$
- $r_{\text{Plus}}\psi$, unit vector in the polarization directions $\psi$
- $\psi$ polarLines, lines from each source along its polarization direction $\psi$
- mapOf$\eta_{\text{Bar}}$, contour plot of the alignment angle $\eta(H)$, adorned with source locations and labels
- mapOf$\eta_{\text{BarLocal}}$, magnified, local view of the map
(* Equatorial coordinates (α, δ) for the hubs H_{min} and H_{max} in other units.*)

\[ α_{H_{min}Degrees} = H_{min} α \text{funDataObs} \left( \frac{360}{2 \pi} \right) \]
\[ α_{H_{min}Hours} = H_{min} α \text{funDataObs} \left( \frac{24}{2 \pi} \right) \]
\[ δ_{H_{min}Degrees} = H_{min} δ \text{funDataObs} \left( \frac{360}{2 \pi} \right) \]
\[ α_{H_{max}Degrees} = H_{max} α \text{funDataObs} \left( \frac{360}{2 \pi} \right) \]
\[ α_{H_{max}Hours} = H_{max} α \text{funDataObs} \left( \frac{24}{2 \pi} \right) \]
\[ δ_{H_{max}Degrees} = H_{max} δ \text{funDataObs} \left( \frac{360}{2 \pi} \right) \]

\[ r_{H_{min}} = r_\pi \left( \frac{2 \pi}{360} \right) + \pi, -δ_{H_{min}Degrees} \left( \frac{2 \pi}{360} \right) \]
\[ r_{PerpH_{min0}} = r_{H_{min}} - (r_{H_{min}}.sourceCenter) \]
\[ r_{PerpH_{min}} = \left( r_{PerpH_{min0}}.r_{PerpH_{min0}} \right)^{1/2} ; \]
\[ r_{GreatMinCircle}[\theta_] := \cos[\theta] \text{sourceCenter} + \sin[\theta] r_{PerpH_{min}} \]
\[ α_{GreatMin}[\theta_] := αFROMr[r_{GreatMinCircle}[\theta]] \]
\[ δ_{GreatMin}[\theta_] := δFROMr[r_{GreatMinCircle}[\theta]] \]
\[ xyAitoffGreatMin = Table[[xH180[ α_{GreatMin}[\theta] (360 / (2 π)), δ_{GreatMin}[\theta] (360 / (2 π)) ]], \]
\[ yH180[ α_{GreatMin}[\theta] (360 / (2 π)), δ_{GreatMin}[\theta] (360 / (2 π)) ]], \{\theta, 1, 360\}] ; \]

\[ r_{H_{max}} = r_\pi \left( \frac{2 \pi}{360} \right) + \pi, -δ_{H_{max}Degrees} \left( \frac{2 \pi}{360} \right) \]
\[ r_{PerpH_{max0}} = r_{H_{max}} - (r_{H_{max}}.sourceCenter) \]
\[ r_{PerpH_{max}} = \left( r_{PerpH_{max0}}.r_{PerpH_{max0}} \right)^{1/2} ; \]
\[ r_{GreatMaxCircle}[\theta_] := \cos[\theta] \text{sourceCenter} + \sin[\theta] r_{PerpH_{max}} \]
\[ α_{GreatMax}[\theta_] := αFROMr[r_{GreatMaxCircle}[\theta]] \]
\[ δ_{GreatMax}[\theta_] := δFROMr[r_{GreatMaxCircle}[\theta]] \]
\[ xyAitoffGreatMax = Table[[xH180[ α_{GreatMax}[\theta] (360 / (2 π)), δ_{GreatMax}[\theta] (360 / (2 π)) ]], \]
\[ yH180[ α_{GreatMax}[\theta] (360 / (2 π)), δ_{GreatMax}[\theta] (360 / (2 π)) ]], \{\theta, 1, 360\}] ; \]

\[ crossMin0 = \text{Cross}[r_{H_{min}}, sourceCenter] ; \]
\[ crossMin = \left( crossMin0 . crossMin0 \right)^{1/2} ; \]
\[ crossMax0 = \text{Cross}[r_{H_{max}}, sourceCenter] ; \]
\[ crossMax = \left( crossMax0 . crossMax0 \right)^{1/2} ; \]
\[ θ_{minMAXgreatcircles} = \text{ArcCos}[crossMax . crossMin] \left( \frac{360}{2 \pi} \right) ; \]
(* The following table α j δ j η BarHjTable is created to generate a map of the alignment angle η(H) over the sphere. *)
(* Table α j δ j η BarHjTable entries: 1. α 2. δ 3. alignment angle η BarRgnkj at grid point (all in degrees) *)

α j δ j η BarHjTable = α j δ j η BarHjTable0 = {};

For [ j = 1, j ≤ Length [ gridj BarHj ], j ++,
    AppendTo [ α j δ j η BarHjTable0, { α Grid [ [ j ] ] * (360. / (2. π)),
        δ Grid [ [ j ] ] * (360. / (2. π)),
        gridj η BarHj [ [ j, 2 ] ] * (360. / (2. π)) } ];

    If [ 180. > α Grid [ [ j ] ] * (360. / (2. π)) > 0.,
        AppendTo [ α j δ j η BarHjTable0,
            { α Grid [ [ j ] ] * (360. / (2. π)) + 180.,
            δ Grid [ [ j ] ] * (360. / (2. π)),
            gridj η BarHj [ [ j, 2 ] ] * (360. / (2. π)) } ];

    If [ 360. ≥ α Grid [ [ j ] ] * (360. / (2. π)) ≥ 354.,
        AppendTo [ α j δ j η BarHjTable0,
            { α Grid [ [ j ] ] * (360. / (2. π)) - 360.,
            δ Grid [ [ j ] ] * (360. / (2. π)),
            gridj η BarHj [ [ j, 2 ] ] * (360. / (2. π)) } ];

α j δ j η BarHjTable0 = {};

lpCheckCoverage = ListPlot [ Table [ { α j δ j η BarHjTable [ [ i, 1 ] ], α j δ j η BarHjTable [ [ i, 2 ] ] }, { i, Length [ α j δ j η BarHjTable ] } ] ];

(* The grid does not cover the sphere. Check that the α j δ j η BarHjTable table covers the entire Celestial Sphere. *)

lpCheckCoverage = ListPlot [ Table [ { α j δ j η BarHjTable [ [ i, 1 ] ], α j δ j η BarHjTable [ [ i, 2 ] ] }, { i, Length [ α j δ j η BarHjTable ] } ] ];

Figure 11: Check. Since the grid does not cover the sphere, only half, we should check that the α j δ j η BarHjTable table covers the entire Celestial Sphere.
(* Transcribe the alignment function $\eta(H)$, the location of the sources, and the Celestial Equator onto an Aitoff plot.*)

```
xxyBarAitoffTable = Table[{
xH180[αjHjTable[[k, 1]], αjδjHjTable[[k, 2]]],
yH180[αjδjHjTable[[k, 1]], αjδjHjTable[[k, 2]]],
{k, Length[αjδjHjTable]}}, (* The alignment angle function $\eta(H)$ on the grid, mapped onto a 2D Aitoff projection of the sphere. *)
```

```
xyAitoffSources = Table[{
xH180[αSrc[n] (360/(2 π)), δSrc[n] (360/(2 π))],
yH180[αSrc[n] (360/(2 π)), δSrc[n] (360/(2 π))]}, {n, nSrc}]; (* The Aitoff coordinates for the sources' locations. *)
```

(* Contour plot of the alignment angle function $\eta(H)$ on the grid. *)

```
dηContourPlot = 6;
```

```
(*, in degrees. *)
```

```math
\text{listCP} = \text{ListContourPlot}[\text{Union}[xxyBarAitoffTable,\{\{xH180[αHminDegrees, δHminDegrees], yH180[αHminDegrees, δHminDegrees], \etaBarMin×(360./ (2. π )) - 1.0\}\}],
\{\{xH180[αHmaxDegrees, δHmaxDegrees], yH180[αHmaxDegrees, δHmaxDegrees], \etaBarMax×(360./ (2. π )) + 1.0\}\}], \text{AspectRatio} \rightarrow 1/2, \text{Contours} \rightarrow \text{Table}[\eta, \{\eta, \text{Floor}[\text{gridj}\etaBarMin[2] \times (360./ (2. π ))] + 1, \text{Ceiling}[\text{gridj}\etaBarMax[2] \times (360./ (2. π ))] - 1, \text{dηContourPlot}\}],
\text{ColorFunction} \rightarrow \text{"TemperatureMap"}, \text{PlotRange} \rightarrow \{\{-4.0, 3.5\}, \frac{7.5}{11.0} \{-3, 3\}\}, \text{Axes} \rightarrow \text{False}, \text{Frame} \rightarrow \text{False}, \text{PlotLegends} \rightarrow \text{Placed}[\text{BarLegend}[\text{Automatic, LegendMargins} \rightarrow \{(0, 0), (10, 5)\}, \text{LegendLabel} \rightarrow \text{"$\eta(H)$, °"}, \text{LabelStyle} \rightarrow \{\text{Plain, FontFamily} \rightarrow \text{"Times"}\}, \text{Right}]];
(*Construct the map of $\eta(\eta)$.*)

```math
\text{mapOf\etaBar} = \\
\text{Show}[\{\text{listCP}, \text{Table}[\text{ParametricPlot}[\{xH180[\alpha, \delta], yH180[\alpha, \delta]\}, \{\delta, -90, 90\}, \text{PlotStyle} \to \{\text{Black, Thickness[0.002]}\}, (*\text{Mesh}\to\{11,5,0\} (*)\text{MeshStyle}\to\text{Thick,*})\text{PlotPoints} \to 60], \{\alpha, 0, 360, 30\}\}], \\
\text{Table}[\text{ParametricPlot}[\{xH180[\alpha, \delta], yH180[\alpha, \delta]\}, \{\alpha, 0, 360\}, \text{PlotStyle} \to \{\text{Black, Thickness[0.002]}\}, (*\text{Mesh}\to\{11,5,0\} (*)\text{MeshStyle}\to\text{Thick,*})\text{PlotPoints} \to 60], \{\delta, -60, 60, 30\}\}], \\
\text{Graphics}[\{\text{PointSize[0.004]}, \text{Text}[\text{StyleForm["N", \text{FontSize} \to 14, \text{FontWeight} \to "Bold"], \{0, 0\} \}, \text{Text}[\text{StyleForm["Equatorial Coordinate System", \text{FontSize} \to 14, \text{FontWeight} \to "Plain"], \{0, -1.85\} \}, (*\text{Sources S:}*)\text{PointSize[0.006]}, \text{Green, Point[ xyAitoffSources ]}, \text{Gray, PointSize[0.002]}, \text{Point[ xyAitoffGreatMin ]}, \text{Point[ xyAitoffGreatMax ]}, \text{Black, Text[StyleForm["H_min", \text{FontSize} \to 12, \text{FontWeight} \to "Bold"], \{-3.3, +1.0\}], \{\text{Arrow[BezierCurve[\{-3.3, +1.2\}, \{-1.3, +3.0\}], \{xH180[\alphaHmaxDegrees, \deltaHmaxDegrees], yH180[\alphaHmaxDegrees, \deltaHmaxDegrees]\}]}}\}], \\
\text{Text}[\text{StyleForm["H_max", \text{FontSize} \to 12, \text{FontWeight} \to "Bold"], \{3.3, 1.0\}], \{\text{Arrow[BezierCurve[\{3.3, 1.2\}, \{0.3, 3.0\}], \{xH180[\alphaHminDegrees, \deltaHminDegrees], yH180[\alphaHminDegrees, \deltaHminDegrees]\}]}}\}], \\
\text{Text}[\text{StyleForm["H_min", \text{FontSize} \to 12, \text{FontWeight} \to "Bold"], \{-3.3, -1.0\}], \{\text{Arrow[BezierCurve[\{-3.3, -1.2\}, \{-2.3, -2.5\}, \{xH180[\alphaHminDegrees - 180, -\deltaHminDegrees], yH180[\alphaHminDegrees - 180, -\deltaHminDegrees]\}]}}\}], (*\text{(**)}) \\
\text{Text}[\text{StyleForm["H_max", \text{FontSize} \to 12, \text{FontWeight} \to "Bold"], \{3.3, -1.0\}], \{\text{Arrow[BezierCurve[\{3.3, -1.2\}, \{2.3, -2.0\}, \{xH180[\alphaHmaxDegrees + 180, -\deltaHmaxDegrees], yH180[\alphaHmaxDegrees + 180, -\deltaHmaxDegrees]\}]}}\}], \text{ImageSize} \to 0.9 \times 432];
```

5c. Section Summary

Equatorial Coordinate System

\(\eta(H)\)°
The arc along the Celestial Sphere from the sample’s center and the closest alignment hub $H_{\text{min}}$ is 15.3924°. The arc along the Celestial Sphere from the sample’s center and the closest avoidance hub $H_{\text{max}}$ is 45.8139°.

To guide the eye, two Great Circles are plotted, one through the sources’ center and the avoidance hubs $H_{\text{max}}$ and $-H_{\text{max}}$. The other connects the center of the sources’ locations with the alignment hubs $H_{\text{min}}$ and $-H_{\text{min}}$. The Great Circles are shaded Gray, □.

The angle between the normals to the planes of the two great circles is 88.4516°.

Note: Although somewhat obscured by the distortion needed to plot a sphere on a flat surface, the function $\eta(H)$ is symmetric across diameters: Diametrically opposite points $-H$ and $H$ have the same alignment angle $\eta$.

\[ \eta(H_1, H_2) = \eta(-H_1, -H_2) \]

\[ \eta(H_1, -H_2) = \eta(-H_1, H_2) \]

For a more detailed analysis, the code snippet below (in Mathematica) can be used to plot the alignment function $\eta(H)$:

```mathematica
(* Local contour plot of the alignment function $\eta(H)$, *).
d\etaContourPlot = 6;(*, in degrees. *)
framelinks = {(*}, \{yH[135, 24], 30°}, \{yH[135, 0], 0°}\}, None},
{\{xH180[150, 0], "10h"}, \{xH180[180, 0], "12h"}, \{xH180[190, 0],
  StyleForm["H_{\text{max}}", FontSize -> 12, FontWeight -> "Bold"], \{xH180[210, 0], "14h"}, \{None\}];
listCPlocal = ListContourPlot[Union\[xy\etaBarAitoffTable\(*,\{\{xH180[\alphaH\text{minDegrees},\deltaH\text{minDegrees}],
  yH180[\alphaH\text{minDegrees},\deltaH\text{minDegrees}],\etaBarMin+(360./(2.\pi)-1.0)\},
  \{\{xH180[\alphaH\text{maxDegrees},\deltaH\text{maxDegrees}],yH180[\alphaH\text{maxDegrees},\deltaH\text{maxDegrees}],
  \etaBarMax+(360./(2.\pi)-1.0)\}\}, AspectRatio -> 1/2,
  Contours -> Table[\[\eta, \{\{\deltaHBarMin[\text{gridMin}][2]\} * (360./(2.\pi)) + 1,
  Ceiling[\text{gridMin}\text{barMax}[\text{gridMin}][2]\} * (360./(2.\pi)) - 1, \etaBarContourPlot\}],
  ColorFunction -> "TemperatureMap", PlotRange -> \{\{xH180[145, 0], xH180[215, 0], \}
  \{yH180[180, -5], yH180[180, 32]\}], Axes -> False, Frame -> True,
  FrameLabel -> \{\"\alpha\", \"\delta\"\}, Close-Up View\}], FrameTicks -> framelinks,
  PlotLegends -> Placed[BarLegend[Automatic, LegendMargins -> \{0, 0\}, \{10, 5\}],
  LegendLabel -> \("\eta(H)\", \"\"\}], LabelStyle -> \{Plain, FontFamily -> "Times\}\], Right]];
In[167]:= (*Plot polarization directions*)

\[\text{rPlus}[i, d] := \frac{(r\text{Src}[[i]] + d \psi \text{Src}[[i]]) - (r\text{Src}[[i]] + d \psi \text{Src}[[i]])}{\sqrt{(r\text{Src}[[i]] + d \psi \text{Src}[[i]]) \cdot (r\text{Src}[[i]] + d \psi \text{Src}[[i]])}}\]

\[\text{polarLines}[d] := \text{Table}\left\{\text{Line}\left[\text{xH180}, \text{yH180} \left\{\frac{360.}{2. \pi}, \text{\deltaFROMr[ rPlus[i, d]] \left\{\frac{360.}{2. \pi}\right\}}\right\}\right]\right\}, \{i, n\text{Src}\}\]

In[169]:= (*Construct the map of \(\eta(H)\).*)

\[\text{mapOf\text{BarLocal} = Show[ \text{Table}\left[\text{ParametricPlot}\left[\text{xH180}, \text{yH180} \left\{\frac{360.}{2. \pi}, \text{\deltaFROMr[ rPlus[i, d]] \left\{\frac{360.}{2. \pi}\right\}}\right\}\right], \{\delta, -5, 60\}, \{\alpha, 120, 240, 30\}\right]\}, \text{PlotStyle} \rightarrow \text{Black, Thickness[0.002]}, \text{PlotPoints} \rightarrow 60, \{\alpha, 90, 270\}, \{\delta, 0, 90, 30\}\right]\]

\[\text{Graphics}\left[\{\text{PointSize[0.009]}, \text{Black, Thick, polarLines[0.03]}\right\}, \{\text{Sources S:}\}
\text{Green, PointSize[0.012]}, \text{Point[ xyH180Sources]}, \text{Gray, PointSize[0.007]}, \text{Point[ xyH180GreatMin]}, \text{Point[ xyH180GreatMax]}, \text{Black, Text[StyleForm["X", FontSize\rightarrow 12, FontWeight\rightarrow \"Bold\"]}, \{\text{xH180[\alpha, \delta, \text{\deltaFROMr[ rPlus[i, d]] \left\{\frac{360.}{2. \pi}\right\}}\right\]}, \{\text{Arrow[BezierCurve[\{(0.17, -0.1), (-0.1, 0.1)\}]}, \{\text{yH180[\alpha, \delta, \text{\deltaFROMr[ rPlus[i, d]] \left\{\frac{360.}{2. \pi}\right\}}\right\}\right\}\right\}]\right\}, \text{ImageSize} \rightarrow 0.9 \times 432;\]

Figure 13: Map of the alignment angle function \(\eta(H)\) in the neighborhood of the sources. The polarization directions display parallax, generally pointing toward the alignment hub \(H_{\text{min}}\). Note how close the hub \(H_{\text{min}}\) is to the sources.
6. Uncertainty Runs

6a. Creating and Storing Uncertainty Runs

For each “uncertainty run”, the polarization direction $\psi$ for each source is allowed to differ from the best value $\psi_{\text{Src}}$ by an amount $\delta \psi$ chosen according to a Gaussian distribution with a mean equal to the best value $\psi_{\text{Src}}$ and half-width $\sigma \psi_{\text{Src}}$, $\psi = \psi_{\text{Src}} + \delta \psi$. Both values $\psi_{\text{Src}}$ and $\sigma \psi_{\text{Src}}$ are taken from the JVAS1450 catalog.

The notebook .nb version generates new uncertainty runs. The pdf version uses old uncertainty runs that are uploaded from previously saved files that are not publically available. Thus both versions have some cells commented out: (* comments are not processed by Mathematica*).

Definitions:

$r_{\text{Src}} \times r_{\text{Grid}}$ unit vector $S_i \times H_j$, the cross product of the radial unit vector to source $S_i$ with the radial unit vector to grid point $H_j$

$n_R$ number of uncertainty runs

$n_{\text{Run}}$ sequential index labeling the runs

$\psi_{\text{Data}}$ table $\{n_{\text{Run}}, \psi\}$ of polarization directions $\psi = \psi_{\text{Src}} + \delta \psi$ for each run

$\text{runData}$ collection of data to save from the uncertainty runs, see below for content list

$n_{\text{RunPrint}}$ dummy index controlling when current TimeUsed and MemoryInUse are printed

$\psi_{\text{SrcU}}$ the polarization direction $\psi$ for the run.

$r_{\text{Src}} \times \psi_{\text{Src}}$ unit vector, $S_i \times \psi_i$, cross product of the radial vector $S_i$ to the source with the vector $\psi_i$ in the direction of the polarization

$\eta_{\text{BarToGridU}}\{j, \eta(H_j)\}$, where $j$ is the index for the grid point $H_j$ and $\eta(H_j)$ is the alignment angle function, (1), at $H_j$

$\text{sort} \eta_{\text{BarToGridU}}\{j, \eta(H_j)\}$, with the smaller angle $\eta(H)$ first.

$\eta_{\text{BarMinU}}\{j, \eta(H_j)\}$ for the smallest value of $\eta(H)$, best alignment

$\eta_{\text{BarMaxU}}\{j, \eta(H_j)\}$, for the largest value of $\eta(H)$, most avoided

$\eta_{\text{min}}^{\text{HU}}$ off-grid local min data $\{\eta_{\text{min}}, \{\alpha, \delta\} \text{ at } H_{\text{min}}\}$

$\eta_{\text{max}}^{\text{HU}}$ off-grid local max data $\{\eta_{\text{max}}, \{\alpha, \delta\} \text{ at } H_{\text{max}}\}$

$\text{funcDataU}$ off-grid, superior values of $\{n_{\text{Run}}, \eta_{\text{min}}^{\text{HU}}, \eta_{\text{max}}^{\text{HU}}\}$ collected results

$H_{\text{min}}^{\text{funDataU}}$ values of $\alpha = \alpha$ for hub $H_{\text{min}}$ from uncertainty runs, alignment

$H_{\text{min}}^{\text{funDataU}}$ values of $\delta = \delta$ for hub $H_{\text{min}}$ from uncertainty runs, alignment

$H_{\text{max}}^{\text{funDataU}}$ values of $\alpha = \alpha$ for hub $H_{\text{max}}$ from uncertainty runs, avoidance

$H_{\text{max}}^{\text{funDataU}}$ values of $\delta = \delta$ for hub $H_{\text{max}}$ from uncertainty runs, avoidance

Tables:

$\psi_{\text{Data}}$ entries: 1. Run # 2. $\psi_{\text{SrcU}},$ list of polarization position angles $\psi$

$\text{gridDataU}$ on-grid, entries: 1. Run # 2. $\{\eta_{\text{min}}, \{\alpha, \delta\} \text{ at } H_{\text{min}}\}$ 3. $\{\eta_{\text{max}}, \{\alpha, \delta\} \text{ at } H_{\text{max}}\}$

$\text{funcDataU}$ off-grid, (better) entries: 1. Run # 2. $\{\eta_{\text{min}}, \{\alpha, \delta\} \text{ at } H_{\text{min}}\}$ 3. $\{\eta_{\text{max}}, \{\alpha, \delta\} \text{ at } H_{\text{max}}\}$

To generate your own Uncertainty Runs:

First calculate “$r_{\text{Src}} x r_{\text{Grid}}$” and then evaluate the “For” statement in the following two cells.

One can save the results with the “Put[]” statements.

Once saved, there is no need to repeat the runs. Comment out the “$r_{\text{Src}} x r_{\text{Grid}}$” and “For” statements by enclosing them in (*comment
brackets*).

The data can be retrieved with the “Get” statements.

\[\text{Clear}[\text{rSrc}\times\text{rGrid}];\]

\[
\text{gridDataUn} = \{\}; \text{psiData} = \{\}; \text{funcDataU} = \{\}; \text{nRunPrint} = 0;
\]

For [nRun = 1, nRun = nR, nRun++],

If [nRun > nRunPrint, Print["At the start of run ", nRun, ", the time is ", TimeUsed[]];

nRunPrint = nRunPrint + 500];

\[
\psi_{\text{SrcU}} = \text{Table}[\text{RandomVariate}[\text{NormalDistribution}[\psi_{\text{SrcU}}[[i]], \psi_{\text{SrcU}}[[i]]]],\{i, \text{nSrc}\}];
\]

(* {table of PPA angles \(\psi\) for the sources in region j0, in radians}*)

\[
\text{rSrcxpsi} = \text{Table}[\text{Sin}[\psi_{\text{SrcU}}[[i]]] e_{\text{ESrc}}[[i]]] -
\]

\[
\text{Cos}[\psi_{\text{SrcU}}[[i]]] e_{\text{ESrc}}[[i]], \{i, \text{nSrc}\}];
\]

(* {table of the cross product of rSrc and vector in direction of \(\psi_{\text{SrcU}}\), a unit vector} *)

\[
\psi_{\text{BarToGrid}} = \text{Table}[[i, nSrc] \text{Sum[ArcCos[}
\]

\[
\text{Abs[} \text{rSrcxpsi}[[i]].\text{rSrcxGrid}[[i, j]] - 0.000001 \text{]}, \{i, \text{nSrc}\}]},\{j, \text{nGrid}\}];
\]

(* {grid point i, value of the alignment angle \(\eta_{\text{H}}[j]\) averaged over all sources, in radians} *)

\[
\text{BarMinU} = \text{Sort}[\text{BarToGridU}]; \text{BarMaxU} = \text{Sort}[\text{BarToGridU}];
\]

(* {\(\eta_{\text{BarMinU}}[j]\), at the grid point \(H_j\) with smallest \(\eta_{\text{BarMaxU}}[j]\)} *)

\[
\text{BarMinU} = \text{Sort}[\text{BarToGridU}]; \text{BarMaxU} = \text{Sort}[\text{BarToGridU}];
\]

(* {\(\eta_{\text{BarMinU}}[j]\), at the grid point \(H_j\) with maximum \(\eta_{\text{BarMaxU}}[j]\)} *)

\[
\text{AppendTo[}\text{gridDataUn},\{\text{nRun}, \eta_{\text{BarMinU}}[2]\},
\]

\[
\{\eta_{\text{BarMinU}}[2], \eta_{\text{BarMaxU}}[2]\}]\}; \text{BarMaxU} = \text{Sort}[\text{BarToGridU}];
\]

(* {\(\eta_{\text{BarMaxU}}[j]\), at the grid point \(H_j\) with maximum \(\eta_{\text{BarMaxU}}[j]\)} *)

\[
\text{AppendTo[}\text{psiData},\{\text{nRun}, \psi_{\text{psi}}\}];
\]

(* {collect discrete (on-grid) data} *)

\[
\eta_{\text{max}} = \text{FindMaximum}[\eta_{\text{BarAtWithAny}}[\alpha, \delta, \psi_{\text{Data}}[\text{nRun}, 2]],
\]

\[
\{(\alpha, \text{gridDataUn}[\text{nRun}, 2, 2]), (\delta, \text{gridDataUn}[\text{nRun}, 2, 2])\}];
\]

(* {collect continuous (function-based) data} *)

\[
\text{t2 = TimeUsed[]};
\]

Print["Time used to compute psiData, gridDataUn, and funcDataU: t2 - t1 = ", t2 - t1]
Hint: You can save memory if you do not get the "ψData". The table ψData is needed to reconstruct the exact values of the gridDataUn table, but it is not needed in any following calculation.


Hint: Saving data files avoids the time it takes to complete the “For” statement. You can make the above “For” statement into a remark so that it doesn’t evaluate.

In[175]:= SetDirectory[homeDirectory]; (*Retrieve an old data file*)  (*ψData=Get["20211031PsiDataUqsoClump1U10000.dat"] ; *) (*gridDataUn=Get["20211031gridDataUnqsoClump1U10000.dat"] ; *) (*Get the funcDataU file for the pdf version:* )

funcDataU = Get["20211031funcDataQSON27U10000.dat"];

In[177]= (*If needed, edit the following to collect data files together.*)  (*ψData=Join[ψData4000,ψData6000]; Length[ψData] ψData[[1]] gridDataUn=Join[gridDataUn4000,gridDataUn6000]; nR=Length[gridDataUn] gridDataUn[[1]] *)

In[178]= (*nR may not be previously defined, depending on what cells have been processed.*)  (*Define nR for the pdf version:* )

nR = Length[funcDataU]

Out[178]= 10000
(*Define quantities based on the function continuous results. The continuous results should be better than the on-grid quantities.*)

\[ \eta_{\text{BarMin}} \text{funDataU} = \text{Table}[\text{funcDataU}[i1, 2, 1], \{i1, \text{Length[funcDataU]}\}] ; \]

\[ \eta_{\text{BarMax}} \text{funDataU} = \text{Table}[\text{funcDataU}[i1, 3, 1], \{i1, \text{Length[funcDataU]}\}] ; \]

\[ H_{\min} \alpha \text{funDataU} = \text{Table}[\text{If[funcDataU}[i1, 2, 2, 1] < \pi,\text{funcDataU}[i1, 2, 2, 1] + \pi,\text{funcDataU}[i1, 2, 2, 1]], \{i1, \text{Length[funcDataU]}\}] ; \]

\[ H_{\min} \delta \text{funDataU} = \text{Table}[\text{If[funcDataU}[i1, 2, 2, 1] < \pi, -\text{funcDataU}[i1, 2, 2, 2], \text{funcDataU}[i1, 2, 2, 2]], \{i1, \text{Length[funcDataU]}\}] ; \]

\[ H_{\min} \alpha \delta \text{funDataU} = \text{Table}[\{\text{Hmin}\alpha \text{funDataU}[i1], \text{Hmin}\delta \text{funDataU}[i1]\}, \{i1, \text{Length[funcDataU]}\}] ; \]

\[ H_{\max} \alpha \text{funDataU} = \text{Table}[\text{If[funcDataU}[i1, 3, 2, 1] > \pi, \text{funcDataU}[i1, 3, 2, 1] - \pi, \text{funcDataU}[i1, 3, 2, 1]], \{i1, \text{Length[funcDataU]}\}] ; \]

\[ H_{\max} \delta \text{funDataU} = \text{Table}[\text{If[funcDataU}[i1, 3, 2, 1] > \pi, -\text{funcDataU}[i1, 3, 2, 2], \text{funcDataU}[i1, 3, 2, 2]], \{i1, \text{Length[funcDataU]}\}] ; \]

\[ H_{\max} \alpha \delta \text{funDataU} = \text{Table}[\{\text{Hmax}\alpha \text{funDataU}[i1], \text{Hmax}\delta \text{funDataU}[i1]\}, \{i1, \text{Length[funcDataU]}\}] ; \]

(*Check to make sure that the hubs are collected together and not diametrically across from one another.*)

\[ \text{lpHubs} = \text{ListPlot}\{\text{Hmin}\alpha \delta \text{funDataU}, \text{Hmax}\alpha \delta \text{funDataU}\}, \text{PlotRange} \rightarrow \text{All}, \]

\[ \text{PlotStyle} \rightarrow \{\{\text{Blue}, \text{PointSize[0.01]\}}, \{\text{Red}, \text{PointSize[0.01]\}}\}, \]

\[ \text{PlotLabel} \rightarrow \text{"The hubs from the uncertainty runs"}, \text{AxesLabel} \rightarrow \{\text{"\(\alpha\) (rad)"}, \text{"\(\delta\) (rad)"}\}] ; \]

Figure 14: Uncertainty run hubs. The alignment hubs \(H_{\min}\) are in blue, the avoidance hubs \(H_{\max}\) are in red. Symmetry across a diameter means there are hubs diametrically opposed to these. Including any diametrically opposed hubs would ruin the statistical calculations for the hubs.

6b. The Effects of Uncertainty on the Smallest Alignment Angle \(\eta_{\text{min}}\)

This section fits a Gaussian distribution to the \(\eta_{\text{min}}\) from the uncertainty runs.

Definitions

\[ \text{sort}\eta_{\text{BarMin}} \quad \text{sort the list of \(\eta_{\text{min}}\) from the uncertainty runs} \]

\[ \eta_{0\text{minU}} \quad \text{estimated mean of the Gaussian fit} \]
σminU  estimated half-width of the Gaussian fit
hlminU0, hlminU  histogram \{η, bin height\} tables needed to set up the NonlinearModelFit
nlmminU  non-linear model fit of a Gaussian to the ηmin histogram
showNLMB  plot of Gaussian and histogram
pTableNLMminU  table of parameter attributes, including standard error
σηBarminUFit, ηBarminUFit  - half-width, and mean of the Gaussian fit

```math
In[190]:=
Print["The number of uncertainty runs is ", Length[funcDataU], "."]
```

```
The number of uncertainty runs is 10 000.
```

```math
In[191]:=
sortηBarMinU = Sort[ηBarMinFunDataU];
η0minU = mean[ηBarMinFunDataU]; (*Guess the mean for the Gaussian.*)
σminU = stanDev[ηBarMinFunDataU]; (*Guess the half-width.*)
hlminU0 = HistogramList[sortηBarMinU, {η0minU − 5 σminU, η0minU + 5 σminU, 0.4 σminU}];
hlminU = Table[{(1/2) hlminU0[[1, i1]] + hlminU0[[1, i1 + 1]], hlminU0[[2, i1]]},
{i1, Length[ hlminU0[[2]] ]}];
nlmminU = NonlinearModelFit[hlminU, a Exp[-(1/2.) ((x − xθ) / b)^2]],
{\{a, Length[sortηBarMinU/6]\}, \{b, σminU\}, \{xθ, η0minU\}}, x]; (*x is ηBarMin*)
```

```math
In[190]:=
pTableNLMminU = nlmminU["ParameterTable"]
{σηBarminUFit, ηBarminUFit} = {b, xθ} /. nlmminU["BestFitParameters"]; (*radians*)
```

```
<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1596.69</td>
<td>11.3899</td>
<td>140.185</td>
</tr>
<tr>
<td>b</td>
<td>0.014963</td>
<td>0.00012325</td>
<td>121.404</td>
</tr>
<tr>
<td>xθ</td>
<td>0.377741</td>
<td>0.00012325</td>
<td>3064.85</td>
</tr>
</tbody>
</table>
```

```math
In[198]:=
showNLMB = Show[
{Histogram[sortηBarMinU, {η0minU − 5 σminU, η0minU + 5 σminU, 0.4 σminU}],
PlotLabel \rightarrow "Uncertainty run \(\eta_{min}\) ",
AxesLabel \rightarrow \{"\eta_{min}, radians", "\Delta R"\}],
Plot[Normal[nlmminU], \{x, η0minU − 5 σminU, η0minU + 5 σminU\}, PlotLabel \rightarrow "\eta_{min}\]",
ListPlot[hlminU, PlotLabel \rightarrow "\eta_{min}\]]];
```

```
Out[199]=
```

```
```

```
```

```
```

```
```

```
```
Figure 15: The Gaussian fit to the alignment angle $\eta_{\text{min}}$ histogram. The height is the number of runs $\Delta R$ in each bin. Note how nicely symmetric this is.

The total number of runs is $R = \sum (\Delta R) = 10000$.

6c. The Effects of Uncertainty on the Largest Avoidance Angle $\eta_{\text{max}}$

This section fits a Gaussian distribution to the $\eta_{\text{max}}$ returned by the uncertainty runs.

Definitions: Similar to the definitions in Sec. 6b.

In[202]:= sort$\eta$BarMaxU = Sort[$\eta$BarMaxfunDataU];
$\eta$0maxU = mean[$\eta$BarMaxfunDataU]; (*Guess the mean for the Gaussian. *)
$\sigma$maxU = stanDev[$\eta$BarMaxfunDataU]; (*Guess the half-width.*)
histogramrangemaxU = ($\eta$0maxU - 5 $\sigma$maxU, $\eta$0maxU + 5 $\sigma$maxU, 0.4 $\sigma$maxU);
hlmaxU = Table[{{1/2} hl0maxU[[1, i]], hl0maxU[[1, i] + 1]], hl0maxU[[2, i]]},
{i, Length[hl0maxU[[2]]]}];
nlmmaxU = NonlinearModelFit[hlmaxU, a Exp[-(1/2.) ((x - x0)/b)^2],
{{a, 300.}, {b, $\sigma$maxU}, {x0, $\eta$maxU}}, x]; (*x is $\eta$BarmaxU.*)
nlmBmaxU = NonlinearModelFit[hlmaxU, {a (1 + $e^{-4(x-x0)/b}$)^-1} Exp[-(1/2.) ((x - x0)/b)^2],
{{a, nR/12}, {b, $\sigma$maxU}, {x0, $\eta$maxU}}, x];

In[209]:= pTableNLmmaxU = nlmBmaxU["ParameterTable"]
{($\sigma$BarBmaxFitU, $\eta$BarBmaxFitU) =
ParametersNLmmaxU = {a, x0} /. nlmBmaxU["BestFitParameters"]; (*radians*)

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1568.65</td>
<td>17.1524</td>
<td>91.4541</td>
</tr>
<tr>
<td>b</td>
<td>0.0166647</td>
<td>0.000210408</td>
<td>79.2016</td>
</tr>
<tr>
<td>x0</td>
<td>1.15348</td>
<td>0.000210408</td>
<td>5482.09</td>
</tr>
</tbody>
</table>

Out[209]=

In[211]:= showNLmmaxU = Show[Histogram[sort$\eta$BarMaxU,
histogramrangemaxU, PlotLabel -> "$\eta_{\text{max}}$", AxesLabel -> {"$\eta_{\text{max}}$, radians", "$\Delta R$"}],
Plot[Normal[nlmBmaxU], {x, $\eta$0maxU - 5 $\sigma$maxU, $\eta$0maxU + 5 $\sigma$maxU}, PlotLabel -> "$\eta_{\text{max}}$"],
ListPlot[hlmaxU, PlotLabel -> "$\eta_{\text{max}}$"]];
Figure 16: The Gaussian fit to the avoidance angle $\eta_{\text{max}}$ histogram. Each bin has a height equal to the number of runs $\Delta R$ in the bin. Like the distribution for $\eta_{\text{min}}$, Fig. 15, this one is well fit by a Gaussian.

6d. The Effects of Uncertainty on the Locations $(\alpha, \delta)$ of the Alignment Hubs $H_{\text{min}}$

Each uncertainty run returns an alignment hub $H_{\text{min}}$. In this section, we investigate the distribution of the locations the alignment hubs $H_{\text{min}}$.

There are two hubs, $H_{\text{min}}$ and $-H_{\text{min}}$ for each uncertainty run, by the symmetry across a diameter. So we collect the data together by moving the $-H_{\text{min}}$ hubs across a diameter to join the $H_{\text{min}}$ hubs. See Fig. 14.

```mathematica
In[214]:= sortHminαfunDataU = Sort[Union[HminαfunDataU]];
lpHminU = ListPlot[Union[HminαfunDataU], PlotRange -> All, PlotStyle -> {Blue, PointSize[0.01]}, PlotLabel -> "The alignment hubs from the uncertainty runs", AxesLabel -> {"\(\alpha\) (rad)", "\(\delta\) (rad)"}];

In[215]:= sortHminα = Sort[HminαfunDataU];
x0Hmin = mean[HminαfunDataU]; (*Guess the mean for the Gaussian.*)
dx0Hmin = stanDev[HminαfunDataU]; (*Guess the half-width.*)
histogramrangeRAHminU = {x0Hmin - 5 dx0Hmin, x0Hmin + 5 dx0Hmin, 0.4 dx0Hmin};
h10xHmin = HistogramList[sortHminα, histogramrangeRAHminU];
h1xHmin = Table[{{1/2}, {1/2}, h10xHmin[1, i1] + h10xHmin[[1, i1 + 1]], h10xHmin[[2, i1]]}, {i1, Length[h10xHmin[[2]]]}];

In[216]:= nlmxHmin = NonlinearModelFit[h1xHmin, a Exp[-(1/2) ((x - x0)/b)^2], {{a, Length[sortHminα]/6}, {b, dx0Hmin}, {x0, x0Hmin}}]; (*x is Hminα*)
\text{Out[238]} = \text{Out[234]} = \text{Out[224]} = \text{Out[222]} =
\text{In[228]} := \text{In[222]} := \text{In[226]} := \text{In[227]} :=

\begin{align*}
\text{Histogram, histogramrangeDecHminU, } & \delta \\
\text{Normal[nlmxHmin]} & \\
\text{expOfnlmxHmin[x_] := } - \left( \frac{1}{2} \right) (y - x0) / b^2 / . \text{nlmxHmin["BestFitParameters"]} \\
\text{expOfnlmxHmin[x]} & \\
\begin{array}{lllll}
\text{Estimate} & \text{Standard Error} & \text{t-Statistic} & \text{P-Value} \\
\hline
a & 2711.63 & 180.691 & 15.007 & 4.86638 \times 10^{-13} \\
b & 0.022115 & 0.00170162 & 12.9964 & 8.46543 \times 10^{-12} \\
x0 & 3.2964 & 0.00170162 & 1937.22 & 4.75292 \times 10^{-59} \\
\end{array}
\end{align*}

\text{Out[224]} = 2711.63 e^{-1022.35 (-3.2964 + x)^2}

\text{Out[226]} = -1022.35 (-3.2964 + x)^2

\begin{align*}
\text{Show[nlmxHmin} & = \text{Show[}\{\text{Histogram[}\text{sortHmin}, \text{histogramrangeRAHminU,} \\
\text{PlotLabel } \rightarrow \text{"Hmin"}, \text{AxesLabel } \rightarrow \{\text{"Hmin, radians"}, \text{"\(\Delta R\)"}, \text{PlotRange } \rightarrow \text{All}],} \\
\text{Plot[}\text{Normal[nlmxHmin]}], \{x, 3, 3.51\}, \text{PlotRange } \rightarrow \text{All, PlotLabel } \rightarrow \text{"Hmin"}],} \\
\text{ListPlot[}\text{hlxHmin, PlotLabel } \rightarrow \text{"Hmin"} \}];
\end{align*}

\begin{align*}
\text{nlmyHmin} & = \text{NonlinearModelFit[}\text{hlxHmin, a Exp[- (1/2.) ((y - y0) / b) ^ 2]},} \\
& \{\{a, \text{Length[}\text{sortHmin} / 6\}], \{b, \text{dy0Hmin}, \{y0, \text{y0Hmin}\}\}, y\}; (* y \text{ is Hmin}\delta *)
\end{align*}

\text{Out[234]} = \text{Out[230]} = \text{Out[238]} = \text{Out[239]} =

\begin{align*}
\text{Show[nlmyHmin} & = \text{Show[}\{\text{Histogram[}\text{sortHmin}, \text{histogramrangeDecHminU,} \\
\text{PlotLabel } \rightarrow \text{"\(\delta\)Hmin"}, \text{AxesLabel } \rightarrow \{\text{"\(\delta\)Hmin, radians"}, \text{"\(\Delta R\)"}, \text{PlotRange } \rightarrow \text{All}],} \\
\text{Plot[}\text{Normal[nlmyHmin]}], \{y, -0.3, 0.2\}, \text{PlotRange } \rightarrow \text{All, PlotLabel } \rightarrow \text{"\(\delta\)Hmin"}],} \\
\text{ListPlot[}\text{hlyHmin, PlotLabel } \rightarrow \text{"\(\delta\)Hmin"} \}];
\end{align*}
In[240]:= histsForHminRAdec = GraphicsRow[{shownlmxHmin, shownlmyHmin}];

Figure 17: The Gaussian fits to the Hmin RA and DEC histograms, where the height is the number of runs $\Delta R$ in each bin. In both graphs, the total number of runs is $R = \sum (\Delta R) = 10000$. These are not symmetric distributions and would be better fit by the functions used in Sec. 7 for Random run $\eta_{\min}$ and $\eta_{\max}$ results. Keep this in mind when looking at Fig. 19.

In[244]:= expoHminU[x_, y_] := - (expOfnlmxHmin[x] + expOfnlmyHmin[y])

Print["The exponent of the probability distribution for Hmin, i.e. the negative log of the distribution: ", expoHminU[\[Alpha], \[Delta]]]

The exponent of the probability distribution for Hmin, i.e. the negative log of the distribution: $1022.35 (-3.2964 + \alpha)^2 + 519.592 (0.00726195 + \delta)^2$

Figure 18: The negative log of the likelihood of (RA,dec) for Hmin, as a function of RA and dec. Where the likelihood is down by a factor $e^{-1/2}$, the negative log is 0.5 and that defines the half-width $\sigma$ of the distribution.
(*Find the curve for the intersection in Fig. 18*)

frθHmin[r_, θ_] := Simplify[(expoHminU[x, y] - 0.5 /. {x -> HminαFit + r Cos[θ], y -> HminδFit + r Sin[θ]})
frθHmin[r, θ];
solverHmin[θ_] := Solve[frθHmin[r, θ] == 0, r];
solverHmin[θ];
rHmin[θ_] := Abs[r /. solverHmin[θ][[2]]]
rHmin[θ];
Plot[rHmin[θ], {θ, 0, 2. π}];

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

uncertRunHmins = Show[{lpHminU, ParametricPlot[{HminαFit + rHmin[θ] Cos[θ], HminδFit + rHmin[θ] Sin[θ]}, {θ, 0, 2. π}, PlotStyle -> Orange, PlotRange -> All]}];

The alignment hubs from the uncertainty runs

Figure 19: All of the alignment hubs Hmin from uncertainty runs.
The ellipse encloses the most likely locations of the hubs. Symmetry across diameters means there is another set diametrically opposite those displayed here.

6e. The Effects of Uncertainty on the Locations (α,δ) of the Avoidance Hubs Hmax

Each uncertainty run returns an avoidance hub Hmax. In this section, we investigate the distribution of the locations the avoidance hubs Hmax.

There are two hubs, Hmax and −Hmax for each uncertainty run, by the symmetry across a diameter. So we collect all the hubs together by moving the −Hmax hubs across a diameter to join the Hmax hubs. See Fig. 14.
\[Out[269]= \]
\[In[273]= \]
\[In[277]= \]
\[Out[267]= \]
\[Out[269]= \]
\[Out[274]= \]
\[In[272]= \]
\[Out[273]= \]
\[In[275]= \]
\[Out[276]= \]
\text{In[279]}:= \text{pTablenlmyHmax} = \text{nlmyHmax}[["ParameterTable"]]
\text{(\text{\omega Hmax}\text{\delta Fit, Hmax}\text{\delta Fit})} = \text{ParametersnlmyHmax} = \{b, y0\} /. \text{nlmyHmax}[["BestFitParameters"]];
\text{(*radians*)}
\text{Normal[nlmyHmax]}
\text{expOfnlmyHmax[y_]} := -\left(\frac{1}{2}\right) \left(\frac{(y-y0)}{b}\right)^2 /. \text{nlmyHmax}[["BestFitParameters"]]
\text{expOfnlmyHmax[y]}
\begin{tabular}{lcccc}
\text{Estimate} & \text{Standard Error} & \text{t-Statistic} & \text{P-Value} \\
\hline
a & 1457. & 171.974 & 8.47226 & 2.24872 \times 10^{-8} \\
b & 0.282644 & 0.0385221 & 7.33719 & 2.40543 \times 10^{-7} \\
y0 & -0.39549 & 0.0385221 & -10.2666 & 7.4637 \times 10^{-10} \\
\hline
\end{tabular}
\text{Out[279]}= \text{Estimate Standard Error t-Statistic P-Value}
\text{Out[281]}= 1457. \text{e}^{-6.25878 (0.39549 \cdot y)^2}
\text{Out[283]}= -6.25878 (0.39549 + y)^2
\text{In[284]}= \text{shownlmyHmax} = \text{Show}[\{\text{Histogram[sortHmax}, \text{histogramrange,}
\text{PlotLabel} \rightarrow "\text{\delta Hmax }", \text{AxesLabel} \rightarrow \{"\text{\delta Hmax, radians", "}\Delta R\}",
\text{Plot[Normal[nlmyHmax],\{y,-2.,0.8\},PlotRange \rightarrow All,PlotLabel} \rightarrow "\text{\delta Hmax"]},
\text{ListPlot[hlyHmax,PlotLabel} \rightarrow "\text{\delta Hmax"]});
\text{histsForHmaxRAdec} = \text{GraphicsRow}[\{\text{shownlmxHmax, shownlmyHmax}]\}
\text{Figure 20: The Gaussian fits to the Hmax RA and DEC histograms, where the height is the number of runs \Delta R in each bin.}
In both graphs, the total number of runs is \(R = \Sigma (\Delta R) = 10000\). These are not well-fit by Gaussians since they slant left and right. Keep this in mind when viewing Fig. 22.
\text{In[288]}= \text{expoHmaxU[x_, y_] := -(expOfnlmxHmax[x] + expOfnlmyHmax[y])}
\text{Print["The exponent of the probability distribution for Hmax, i.e. the negative log of the distribution: ", expoHmaxU[a, \delta]]}
The exponent of the probability distribution for Hmax, i.e. the negative log of the distribution: 13.9429 (-2.56174 + a)^2 + 6.25878 (0.39549 + \delta)^2
\text{In[291]}= \text{findHmaxUncertainty} = \text{Plot3D[\{expoHmaxU[x, y], 0.5\}, \{x, x0 - 0.3, x0 + 0.3\} /. \text{nlmxHmax}[["BestFitParameters"]],
\text{\{y, y0 - 0.5, y0 + 0.5\} /. \text{nlmyHmax}[["BestFitParameters"]],}
\text{PlotLabel} \rightarrow "\text{Negative log of the probability of (a,\delta) for Hmax", AxesLabel} \rightarrow \{"\text{\alpha (rad)", "\delta (rad")}]\);
Figure 21: The negative log of the likelihood of \((RA, \text{dec})\) for \(H_{\text{max}}\) as a function of RA and dec. Where the likelihood is down by a factor \(e^{-1/2}\), the negative log is +0.5 and that defines the half-width \(\sigma\) of the distribution.

\[\text{In[292]}=\]
\]

\(\text{fr}\theta H_{\text{max}}[r, \theta] :=\)
\[\text{Simplify}[\left(\text{expoHmaxU}[x, y]\right) - 0.5 / \{x \rightarrow H_{\text{max}}\alpha \text{Fit} + r \cos[\theta], y \rightarrow H_{\text{max}}\delta \text{Fit} + r \sin[\theta]\}]\]
\[\text{fr}\theta H_{\text{max}}[r, \theta];\]
\[\text{solverHmax}[\theta] := \text{Solve}[\text{fr}\theta H_{\text{max}}[r, \theta] = 0, \theta];\]
\[\text{solverHmax}[\theta];\]
\[r_{\text{Hmax}}[\theta] := \text{Abs}[r / . \text{solverHmax}[\theta][[2]]];\]
\[r_{\text{Hmax}}[0.8];\]
\[\text{Plot}[r_{\text{Hmax}}[\theta], \{\theta, 0, 2.\pi\}];\]

\(\text{Solve}:\) Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

\(\text{Solve}:\) Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

\(\text{uncertRunHmaxs} =\)
\[\text{Show}[\left\{\text{lpHmaxU}, \text{ParametricPlot}[\{H_{\text{max}}\alpha \text{Fit} + r_{\text{Hmax}}[\theta] \cos[\theta], H_{\text{max}}\delta \text{Fit} + r_{\text{Hmax}}[\theta] \sin[\theta]\}, \{\theta, 0, 2.\pi\}, \text{PlotStyle} \rightarrow \text{Orange}, \text{PlotRange} \rightarrow \text{All}\}\];\]
The avoidance hubs from the uncertainty runs

The ellipse encloses the most likely locations of the hubs. Symmetry across diameters means there is another set diametrically opposite those displayed here.

6f. The Effects of Uncertainty on the angle $\theta$ between the planes of the Sample to $H_{\text{min}}$ Great Circle and the Sample to $H_{\text{max}}$ Great Circle.

These are the Gray lines in Figs. 3, 4, 12, 13. Starting at the sources, these Great Circles run through the hubs, the locations of best convergence and most divergence for the polarization directions.

Definitions:

"uRuns" prefix results from the uncertainty runs
uRunsCrossMin unit vector normal to the Great Circle connecting the center of the source region with the alignment hub $H_{\text{min}}$
uRunsCrossMax unit vector normal to the Great Circle connecting the center of the source region with the alignment hub $H_{\text{max}}$
uRuns$\theta_{\text{minmaxU greatcircles}}$ angle between the two normals in degrees
sort$\theta_{\text{minmaxU greatcircles}}$ sort "uRuns$\theta_{\text{minmaxU greatcircles}}"$, smallest $\theta$ first

See Definitions above in Secs. 6a,6b for other quantities below. There you should find similarly named quantities.
(*Fit two peaks for θ:*)

\[\text{sortθminmaxU} = \text{Sort}[\text{uRunsθminmaxU}];\]

\[\text{x0θ} = \text{mean}[\text{uRunsθminmaxU}];\]

\[\text{dx0θ} = 0.3 \times \text{stanDev}[\text{uRunsθminmaxU}];\]

\[\text{histogramrange} = \{70, 103, 1.5\};\]

\[\text{hl0} = \text{HistogramList}[\text{sortθminmaxU}, \text{histogramrange}];\]

\[\text{hl} = \text{Table}[\{\text{hl0}[1, i1] + \text{hl0}[2, i1], \text{hl0}[2, i1]\}, \{i1, \text{Length[hl0][2]}\}];\]

\[\text{nlmθ} = \text{NonlinearModelFit}[\text{hl}, a3 \exp[-(1/2.) \times \{(x - x03)/b3\}^2], \{a3, \text{Length[sortθminmaxU]} / 5\}, \{b3, \text{dx0θ}, \{x03, x0θ\}\}];\]

\[\text{pTableNLMθ} = \text{nlmθ["ParameterTable"]};\]

\[\text{showNLMθ} = \text{Show}[\{\text{Histogram}[\text{sortθminmaxU}, \text{histogramrange}, \text{PlotLabel} \rightarrow \text{"Angle θ between the Two Gray Great Circles in Figs. 3, 4, 12, 13."}, \text{AxesLabel} \rightarrow \{\"θ, degrees\", \"ΔR\"\}], \text{Plot[Normal[nlmθ], \{x, 0, 250\}, PlotRange} \rightarrow \text{All}], \text{ListPlot[hl]} \}];\]

6g. Map of the Hubs for the Uncertainty Runs

In this subsection, we map the locations of the many alignment hubs \(H_{\text{min}}\) and the avoidance hubs \(H_{\text{max}}\) that are found in the uncertainty runs.

Definitions:

\(\text{vψSrcBig, Small}\) unit vectors, \(\text{v(ψ ± σψ)}\), large & small, the one-sigma range of polarization directions \(ψ\).
In[327]:= (*The Aitoff coordinates for the hubs $H_{\text{min}}$ locations.*)
xyAitoffHminU = Table[ {
  xH180[ HminαfunDataU[[n]] (360/(2π)),
  HminαfunDataU[[n]] (360/(2π)), yH180[ HminαfunDataU[[n]] (360/(2π)),
  HminαfunDataU[[n]] (360/(2π)) ]}, {n, Length[HminαfunDataU]]};

In[322]:= (*The Aitoff coordinates for the hubs $H_{\text{max}}$ locations.*)
xyAitoffHmaxU = Table[ {
  xH180[ HmaxαfunDataU[[n]] (360/(2π)),
  HmaxαfunDataU[[n]] (360/(2π)), yH180[ HmaxαfunDataU[[n]] (360/(2π)),
  HmaxαfunDataU[[n]] (360/(2π)) ]}, {n, Length[HmaxαfunDataU]]};

In[323]:= (*The Aitoff coordinates for the hubs $-H_{\text{min}}$ locations.*)
xyAitoffOppositeHminU = Table[ {
  xH180[ If[0 ≤ HminαfunDataU[[n]] (360/(2π)) < +180,
    HminαfunDataU[[n]] (360/(2π)), +180, If[360 > HminαfunDataU[[n]] (360/(2π)) > 180,
    HminαfunDataU[[n]] (360/(2π)) - 180]], -HminαfunDataU[[n]] (360/(2π)) ],
  yH180[ If[0 ≤ HminαfunDataU[[n]] (360/(2π)) < +180,
    HminαfunDataU[[n]] (360/(2π)), +180, If[360 > HminαfunDataU[[n]] (360/(2π)) > 180,
    HminαfunDataU[[n]] (360/(2π)) - 180]], -HminαfunDataU[[n]] (360/(2π)) ]}, {n, Length[HminαfunDataU]]};

In[324]:= (*The Aitoff coordinates for the hubs $-H_{\text{max}}$ locations.*)
xyAitoffOppositeHmaxU = Table[ {
  xH180[ If[0 ≤ HmaxαfunDataU[[n]] (360/(2π)) < +180,
    HmaxαfunDataU[[n]] (360/(2π)), +180, If[360 > HmaxαfunDataU[[n]] (360/(2π)) > 180,
    HmaxαfunDataU[[n]] (360/(2π)) - 180]], -HmaxαfunDataU[[n]] (360/(2π)) ],
  yH180[ If[0 ≤ HmaxαfunDataU[[n]] (360/(2π)) < +180,
    HmaxαfunDataU[[n]] (360/(2π)), +180, If[360 > HmaxαfunDataU[[n]] (360/(2π)) > 180,
    HmaxαfunDataU[[n]] (360/(2π)) - 180]], -HmaxαfunDataU[[n]] (360/(2π)) ]}, {n, Length[HmaxαfunDataU]]};

In[325]:= (*ψφ unit vectors pointing along the polarization direction, have an experimental uncertainty. These are their plus/minus values.*)
ψφSrcBig = Table[ Cos[ψφSrc[i]], eN[ aSrc[i]], δSrc[i] ] +
  Sin[ψφSrc[i]], eE[ aSrc[i]], δSrc[i] ]}, {i, nSrc}];
ψφSrcSmall = Table[ Cos[ψφSrc[i], eN[ aSrc[i]], δSrc[i] ] +
  Sin[ψφSrc[i], eE[ aSrc[i]], δSrc[i] ]}, {i, nSrc}];

In[327]:= (*Plot polarization direction Uncertainty in Sec. 6*)
\[ r_{\text{Plus}}\psi\text{Big}[i, d_] := (r\text{Src}[i] + d \psi\text{SrcBig}[i]) /\]
  \[ ((r\text{Src}[i] + d \psi\text{SrcBig}[i]).(r\text{Src}[i] + d \psi\text{SrcBig}[i]))^{1/2} \]
polarLinesBig[d_] := Table[ Line[Table[ {
  xH180[ aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)),
  aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)), yH180[ aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)),
  aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)) ]}, {xH180[ aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)),
  aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)), yH180[ aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)),
  aFROMr[r\text{Plus}\psi\text{Big}[i, d]] (360./(2\pi)) ]}, {i, nSrc}];
In[320]:= (*Plot polarization direction Uncertainty in Sec. 6*)
    rPlusSmall[i_, d_] := (rSrc[[i]] + d*ψSrcSmall[[i]]) / 
        (((rSrc[[i]] + d*ψSrcSmall[[i]])*(rSrc[[i]] + d*ψSrcSmall[[i]]))^(1/2))

    polarLinesSmall[d_] := Table[Line[{{xH180[αFROMr[rPlusSmall[i, d]] (360. / (2. π)], 
            δFROMr[rPlusSmall[i, d]] (360. / (2. π))}, yH180[αFROMr[rPlusSmall[i, d]]] 
            (360. / (2. π)), δFROMr[rPlusSmall[i, d]] (360. / (2. π))}],
        {xH180[αFROMr[rPlusSmall[i, -d]] (360. / (2. π)), δFROMr[rPlusSmall[i, -d]] 
            (360. / (2. π))}, yH180[αFROMr[rPlusSmall[i, -d]]] 
            (360. / (2. π)), δFROMr[rPlusSmall[i, -d]] (360. / (2. π))}], {i, nSrc}]

In[331]:= (* Local contour plot of the alignment angle function *η(H) on the grid. *)
    (*d*ηContourPlot = 6 ;*) (*, in degrees. *)
    frameticks = {{{yH[135, 24], 30 °}, {yH[135, 0], 0 °}}}, None},
        {{{xH180[150, 0], "10h"}, {xH180[180, 0], "12h"}, {xH180[190, 0], StyleForm["Hmin", 
            FontSize -> 12, FontWeight -> "Bold"]}, {xH180[210, 0], "14h"}}}, {None}};
    (*frameticks=*{{yH[150, 22.5], 30°},{yH[150, 48.5], 60°}}),None},
        {{{xH180[150,(±15) 30],"10h"}, 
            {xH180[180,15],"12h"},{xH180[210,15],"14h"}},{None}};*)

In[332]:= listPlocalU = Show[ Table[ParametricPlot[{{xH180[α, δ], yH180[α, δ]}, 
            {δ, -5, 60}, PlotStyle -> {Black, Thickness[0.002]}, PlotPoints -> 60, 
            PlotRange -> {{xH180[145, 0], xH180[215, 0]}, {yH180[180, -5], yH180[180, 32]}}, Axes -> False, 
            Frame -> True, FrameLabel -> {"α", "δ", "Close-Up View"}, FrameTicks -> frameticks], 
            {α, 120, 240, 30}], Table[ParametricPlot[{{xH180[α, δ], yH180[α, δ]}, {α, 90, 270}, 
            PlotStyle -> {Black, Thickness[0.002]}, PlotPoints -> 60}, {δ, 0, 90, 30}], 
            Graphics[{{PointSize[0.01], Red, (*Hmax*) Point[xAitoffHmaxU], 
                Point[xAitoffOppositeHmaxU], PointSize[0.009], Gray, {Thick, polarLines[0.03]}, 
                {Thick, polarLinesBig[0.03]}, {None}}, (*Sources S:*), 
            Green, PointSize[0.01], Point[xAitoffSources], PointSize[0.01], Blue, {Hmin:*}, 
            Point[xAitoffHminU], Point[xAitoffOppositeHminU], Gray, PointSize[0.005]}], 
            ParametricPlot[{{xH180[(HminFit+rHmin[θ] Cos[θ]) (360. / (2. π)), 
                HminFit + rHmin[θ] Sin[θ]) (360. / (2. π))}, yH180[ 
                HminFit + rHmin[θ] Cos[θ] (360. / (2. π)), HminFit + rHmin[θ] Sin[θ] (360. / (2. π))]}, 
            {θ, 0., 2. π}, PlotStyle -> {Orange, Thickness[0.01]}], 
            ParametricPlot[{{xH180[(HmaxFit + rHmax[θ] Cos[θ]) (360. / (2. π)), 
                HmaxFit + rHmax[θ] Sin[θ]) (360. / (2. π))}, yH180[ 
                HmaxFit + rHmax[θ] Cos[θ] (360. / (2. π)), HmaxFit + rHmax[θ] Sin[θ] (360. / (2. π))]}, 
            {θ, 0., 2. π}, PlotStyle -> {Orange, Thickness[0.005]}], ImageSize -> 0.9 × 3432;
In[333]:= listCPlocalU
Print["Figure 24: Uncertainty plot. The sources are shaded green, ",

Green, ". Three polarization directions are plotted for each source: the
reported value ψ and the one-sigma values ψ ± σψ are plotted as gray, ", Gray,
", line segments through the sources. All of the alignment hubs H_{min} from the uncertainty
runs are plotted as overlapping blue dots, ", Blue, ", with the orange ellipse, ",
Orange, ", denoting the highest density of uncertainty-run hubs. "]

Out[333]=

Figure 24: Uncertainty plot. The sources are shaded green, 

■. Three polarization directions are plotted for each source: the
reported value ψ and the one-sigma values ψ ± σψ are plotted as gray, 

■, line segments through the sources. All of the alignment hubs H_{min} from the uncertainty
runs are plotted as overlapping blue dots, 

■, with the orange ellipse, 

■, denoting the highest density of uncertainty-run hubs.

6h. Section Summary

In[335]=
Print["To estimate the effects of experimental uncertainty, there were ",

Length[funcDataU], " uncertainty runs."]
Print["Uncertainty runs have polarization directions ψ = ψSrc + δψ, ",

"where δψ is chosen with a normal
distribution of half-width σψ about the best value ψSrc." ]
Print["The uncertainty runs determine the smallest alignment angle to be \bar{η}_{min} = ",

\etaBarminUFit\left(360./\left(2.\pi\right)\right), "\circ ± ", \sigma\eta\etaBarminUFit\left(360./\left(2.\pi\right)\right), "\circ. " ]
Print["The uncertainty runs determine the largest avoidance angle to be \bar{η}_{max} = ",

\etaBarmaxUFit\left(360./\left(2.\pi\right)\right), "\circ ± ", \sigma\eta\etaBarmaxUFit\left(360./\left(2.\pi\right)\right), "\circ. " ]
Print["The uncertainty runs determine the angle θ between

the two grey Great Circles in Figs. 3, 4, 12, 13, to be \theta = ",

θminmaxUFit3, "\circ ± " ,Abs[dx@θminmaxUFit3[ ], "\circ. " ]
To estimate the effects of experimental uncertainty, there were 10,000 uncertainty runs. Uncertainty runs have polarization directions \( \psi = \psi_{\text{Src}} + \delta\psi \), where \( \delta\psi \) is chosen with a normal distribution of half-width \( \sigma_{\psi} \) about the best value \( \psi_{\text{Src}} \). The uncertainty runs determine the smallest alignment angle to be \( \eta_{\text{min}} = 21.643^\circ \pm 0.857315^\circ \). The uncertainty runs determine the largest avoidance angle to be \( \eta_{\text{max}} = 66.0895^\circ \pm 0.954815^\circ \). The uncertainty runs determine the angle \( \theta \) between the two grey Great Circles in Figs. 3, 4, 12, 13, to be \( \theta = 88.1025^\circ \pm 3.74787^\circ \).

7. Probability and Significance

The problem of "significance" is to determine the likelihood that random polarizations directions would produce better alignment or avoidance than the observed polarization directions.

To determine the probability distributions and related formulas, we made many runs with random data and fit the results. One finds that the probability distributions for the smallest alignment angle \( \eta_{\text{min}} \) and the largest avoidance angle \( \eta_{\text{max}} \) are not well-described by Gaussian functions. Better fits have the Gaussian multiplied by a step-function. The fitting functions are based on the following distribution,

\[
f(y) = \frac{1}{\sqrt{2\pi}} \left(1 + e^{4(y-1)}\right)^{-1} e^{-\frac{y^2}{2}} \tag{4}
\]

More discussion appears below when the function (4) is needed.

Applied to the probability distribution for the smallest alignment angle \( \eta_{\text{min}} \) the fitting function takes the form

\[
P_{\text{min}}(\eta) = \left( \frac{\text{norm}}{\sigma \sqrt{2\pi}} \right) \left(1 + e^{4 \left(\frac{\eta - \eta_0}{\sigma}\right)}\right)^{-1} e^{-\frac{(\eta - \eta_0)^2}{2\sigma^2}} \tag{5}
\]

where norm makes the integral equal to unity, \( \eta_0 \) and \( \sigma \) are parameters that are adjusted to fit the random run results.

7a. Probability and Significance Formulas

Definitions:

- norm a constant used to normalize the distribution so the integral of probability is 1.
- probMIN0, probMAX0 probability distributions for alignment (MIN) and avoidance (MAX), functions of \( \eta, \eta_0, \sigma \)
- signiMIN0, signiMAX0 significance as a function of \( \eta, \eta_0, \sigma \)
\begin{verbatim}
In[340]:= (* y = (η - η0)/σ; dy = dη/σ *)
(* The normalization factor "norm" is needed for the probability density *)
norm = \left[ \left( \frac{1}{(2\pi)^{1/2}} \right) \text{NIntegrate}\left[ \left( 1 + e^{\frac{(y-\eta0)}{\sigma}} \right)^{-1} \text{e}^{-\frac{y^2}{2}}, \{y, -\infty, \infty\} \right] \right]^{-1};

norm;(*Constant needed to make the integral
of the probability distribution equal to unity.*)

In[342]:= probMIN0[η_, η0_, σ_] := \left( \frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left( 1 + e^{-\frac{(\eta-\eta0)}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left( \frac{\eta-\eta0}{\sigma} \right)^2}

signiMIN0[η_, η0_, σ_] := \text{NIntegrate}[probMIN0[η1, ηθ, σ], \{η1, -\infty, η\}]
probMAX0[η_, η0_, σ_] := \left( \frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left( 1 + e^{-\frac{4(\eta-\eta0)}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left( \frac{\eta-\eta0}{\sigma} \right)^2}

signiMAX0[η_, η0_, σ_] := \text{NIntegrate}[probMAX0[η1, ηθ, σ], \{η1, η, \infty\}]

The significance signiMIN0[η, ηθ, σ] is the Integral of probMIN0, i.e. signiMIN0 = \int_{-\infty}^{\eta} P_{MIN}(\eta) \, d\eta.
The significance signiMAX0[η, ηθ, σ] is the Integral of probMAX0, i.e. signiMAX0 = \int_{\eta}^{\infty} P_{MAX}(\eta) \, d\eta.

7b. Generating random \(\psi\) runs

The notebook .nb version generates new random runs. The pdf version uses old random runs that are uploaded from previously saved files that are not publically available.

Definitions:

nRunMax number of random runs to be generated
ρRgnRadius distance to furthest source from sourceCenter, radians
minGridCenterToHmin, max - minimum number of grid spaces between Hmin, Hmax and sources’ center
gridjηBarMinRand
iSmimmas parameters for center to hub distance
nRunPar dummy index to control printing frequency
rSrcxrGrid unit vector perpendicular to the plane of rSrc for Si and rGrid to point Hj
ψSrcRand random polarization directions for the sources
rSrcxψSrc cross product of rSrc and the vector in direction of ψSrcR, both are unit vectors
jηBarToGrid \{j, ηj\} = \{grid point \#, value of the alignment angle Eq. (1) averaged over all sources Si, in radians\}
sortjηBarToGrid - sort jηBarToGrid, smallest alignment angles ηj first
gridjηBarMinRand - \{j, ηj\} for the grid point Hj with the smallest alignment angle ηj, not counting Hj that are too close to the sample
gridjηBarMaxRand - \{j, ηj\} for the grid point Hj with the largest avoidance angle ηj, not counting Hj that are too close to the sample
niSnrData 1. run # 2. iSmin 3. iSmax 4. nSrc 6. ρRgnRadius
ψDataRand 1. run # 2. ψSrcRand table
runData 1. run # 2. sourceCenter 3. \{j, ηj\} at point Hj where smallest ηj 4. \{j, ηj\} at point Hj where largest ηj 5. nSrc 6. ρRgnRadius
\end{verbatim}
In[346]=

(* Remove comment marks, "(*" and ")", below to generate your own table "runData". *)
(* Evaluate this cell for the notebook .nb version *)
(*
 nRunMax = 500;
 nISnrData = {};
 ψDataRand = {};
 runData = {};
 times = {};
 (* Set up the For statement.*)
 nRunPrint = 0;
 minGridCenterToHmin = 2;
 (* minimum number of grid spaces between Hmin and sources' center *)
 minGridCenterToHmax = 2;
 (* minimum number of grid spaces between Hmax and sources' center *)
 *)

In[347]=

(* Evaluate this cell for the notebook .nb version. You may have found rSrcxrGrid already with uncertainty. Here it is again: *)
(* rSrcxrGrid1 = Table[ Cross[ rSrc[[i]], rGrid[[j]] ] , {i,nSrc},{j,nGrid}] *)
(* first step: raw cross product, not unit vectors *)
 rSrcxrGrid = Table[ rSrcxrGrid1[[i,j]]/
     (rSrcxrGrid1[[i,j]].rSrcxrGrid1[[i,j]] + 0.000001)^1/2. , {i,nSrc},{j,nGrid}];*)
\[348\] (* Evaluate this cell for the notebook .nb version *)
(* \(t[1]=\text{TimeUsed[]}\); *)
For[\(nRun=1\),\(nRun\leq nRunMax\),\(nRun++\),
    If[\(nRun>nRunPrint\),Print["At the start of run ",nRun," the time is ",\(\text{TimeUsed[]}\)," seconds and the memory in use is ",\(\text{MemoryInUse[]}\)," bytes.");
    nRunPrint=nRunPrint+100];
\(\psi\)SrcRand=Table[\(\text{RandomReal}[\{0.001,\pi-0.001\}]\),\{i,nSrc\}];
\(r\)Srcx\(\psi\)Src = Table[\(\text{Sin}[\psi\)SrcRand[[i]]] \(e_{N\text{Src}}[[i]]\)-\(\text{Cos}[\psi\)SrcRand[[i]]] \(e_{ESrc}[[i]]\), \{i,nSrc\}]; (*table of the cross product of \(r\)Src and vector in direction of \(\psi\)SrcRand, a unit vector*)
\(j\)\(\eta\)BarToGrid = Table[\(\text{Abs}[r\)Srcx\(\psi\)Src[[i,j]] \(e_{N\text{Src}}[[i]]\)-\(\text{Cos}[\psi\)SrcRand[[i]]] \(e_{ESrc}[[i]]\)]\),\{i,nSrc\},\{j,\text{nGrid}\}];
sortj\(\eta\)BarToGrid=Sort[\(j\)\(\eta\)BarToGrid,\#1[[2]]\(<\#2[[2]]\)&];
iSmin=Catch[Do[If[\(\text{ArcCos}[\text{sourceCenter.rGrid[[sortj\(\eta\)BarToGrid[[i,1]]]]}\) \(-0.000001\]/\(\text{d}\_\theta\text{minGridCenterToHmin,Throw[i],\{i,100\}}\)];]
gridj\(\eta\)BarMinRand=sortj\(\eta\)BarToGrid[[iSmin]]; (* \(\{\text{j},\eta\}\), at the grid point \(H_j\) with minimum \(\eta\), not counting the center \(j_0\)*)
iSmax=Catch[Do[If[\(\text{ArcCos}[\text{sourceCenter.rGrid[[sortj\(\eta\)BarToGrid[[-i,1]]]]}\) \(-0.000001\]/\(\text{d}\_\theta\text{maxGridCenterToHmax,Throw[i],\{i,100\}}\)];]
gridj\(\eta\)BarMaxRand=sortj\(\eta\)BarToGrid[[iSmax]]; (* \(\{\text{j},\eta\}\), at the grid point \(H_j\) with maximum \(\eta\), not counting the center \(j_0\)*)
AppendTo[\(\text{niSnrData},\{nRun,iSmin,iSmax,nSrc,}\(\rho\)RgnRadius\}]];
AppendTo[\(\psi\)DataRand,\{nRun,\(\psi\)SrcRand\}];
AppendTo[\(\text{runData},\{nRun,\text{sourceCenter},\{\text{grid[[gridj}\(\eta\)BarMinRand[[1]]]],gridj\(\eta\)BarMinRand[[2]]\},
\{\text{grid[[gridj}\(\eta\)BarMaxRand[[1]]]],gridj\(\eta\)BarMaxRand[[2]]\},\text{nSrc,}\(\rho\)RgnRadius\}\}\}] (*collect data for saving in a file.*)]
\[349\] (* Evaluate this cell for the notebook .nb version *)
(* \(t[2]=\text{TimeUsed[]}\); *)
Print["Computer time needed to generate random runs: ",\(t[2]-t[1]\)," seconds."\)*)
\[350\] (*Save a new table*)
SetDirectory[\(\text{homeDirectory}\)]; (*\(\text{Put[niSnrData,\"20211012niSnrDataQSON27Random2000e.dat\"]}\)*)
(*\(\text{Put[\(\psi\)DataRand,\"20211012\(\psi\)DataRandQSON27Random2000e.dat\"]}\)*)
(*\(\text{Put[runData,\"20211012runDataQSON27Random2000e.dat\"]}\)*)
In[351]:= (*Get an old table*)
SetDirectory[homeDirectory];
(*niSnrData=Get["20211012niSnrDataQSON27Random2000e.dat"]*)
(*ψDataRand=Get["20211012ψDataRandQSON27Random2000e.dat"]*)
(*Get the runData files for the pdf version:*)
runData2000a = Get["20210905runDataQSON27Random2000a.dat"];
runData2000b = Get["20210906runDataQSON27Random2000b.dat"];
runData2000c = Get["20210906runDataQSON27Random2000c.dat"];
runData2000d = Get["20210906runDataQSON27Random2000d.dat"];
runData2000e = Get["20210906runDataQSON27Random2000e.dat"];

In[357]:= (*Edit the following statements to Join separate data files, if needed*)
(*Join the runData files for the pdf version:*)
runData = Join[runData2000a, runData2000b, runData2000c, runData2000d, runData2000e];
nRunMax = Length[runData];

7c. Analyzing random $\psi$ runs

Definitions:

- $\eta_{BarminData}$: $\eta_{\min}$ in order of random runs
- sort$\eta_{Barmin}$: sorted
- $\eta_{0Bmin}$, $\sigma_{Bmin}$: mean and standard deviation of $\eta_{BarminData}$
- hlmin, hlmin0: histogram data
- nlmBmin: fit to $\eta_{\min}$ histogram
- $\{a,b,x0\}$: best fit parameters
- showNlmBmin: figure displaying the fit to the $\eta_{\min}$ from random runs
- nlmBminPtable: Parameter table for the fit

- $\eta_{BarmaxData}$: $\eta_{\max}$
- sort$\eta_{Barmax}$: sorted
- $\eta_{0Bmax}$, $\sigma_{Bmax}$: mean and standard deviation of $\eta_{BarmaxData}$
- hlmax, hlmax0: histogram data
- nlmBmax: fit to $\eta_{\max}$ histogram
- $\{a,b,x0\}$: best fit parameters
- showNlmBmax: figure displaying the fit to the $\eta_{\max}$ from random runs
- nlmBmaxPtable: Parameter table for the fit

- rHminR: rGrid at $H_{\min}$
- anglerHminToCenter: $\theta$ from $H_{\min}$ to sourceCenter
- $\theta_{HminToCenter}$, $\sigma_{\theta_{HminToCenter}}$: mean and standard deviation of $\theta$
runData
1. nRun  2. r at Region Center  3a. grid data for Hmin  3b. \( \eta_{\text{min}} \)
4a. grid data for Hmax  4b. \( \eta_{\text{max}} \)  5. nSrc  6. radius \( \rho \) RgnRadius

"fitData" table
1a. nSrc, number of sources  1b. rgnRadius, nominal radius of region  1c. RMS radius
2a. x0min: \( x_0 = \eta_0 \) align (min)  2b. dx0min error: \( dx_0 - \sigma \) for \( x_0 = \eta_0 \) align (min)
3a. bmin: \( b = \sigma \) align (min)  3b. dbmin: \( \sigma \) for \( b = \sigma \) align (min)
4a. amin: \( a = \sigma \) align (min)  4b. damin: \( \sigma \) for \( a = \sigma \) align (min)
5a. x0max: \( x_0 = \eta_0 \) avoid (max)  5b. dx0max error: \( dx_0 - \sigma \) for \( x_0 = \eta_0 \) avoid (max)
6a. bmax: \( b = \sigma \) avoid (max)  6b. dbmax: \( \sigma \) for \( b = \sigma \) avoid (max)
7a. amax: \( a = \sigma \) avoid (max)  7b. damax: \( \sigma \) for \( a = \sigma \) avoid (max)
8a. \( \sigma_{\theta} \) RminToCenter: \( \text{stanDev[anglerHminToCenter]} - \sigma \) to H  8b. \( \theta \) RminToCenter: \( \text{mean[anglerHminToCenter]} - \theta \) to H
9a. \( \theta \) RmaxToCenter: \( \text{mean[anglerHmaxToCenter]} - \theta \) to H  9b. \( \sigma_{\theta} \) RmaxToCenter: \( \text{stanDev[anglerHmaxToCenter]} - \theta \) to H

In[359]:= Print["There are ", Length[runData], " random runs to analyze."]
There are 10000 random runs to analyze.

In[360]:= \( \eta_{\text{BarminData}} \) = Table[runData[[i1, 3, 2]], {i1, Length[runData]}];
\( \eta_{\text{BarmaxData}} \) = Table[runData[[i1, 4, 2]], {i1, Length[runData]}];
\( r_{\text{HminR}} \) = Table[runData[[i1, 3, 1, 6]], {i1, Length[runData]}];
\( r_{\text{HmaxR}} \) = Table[runData[[i1, 4, 1, 6]], {i1, Length[runData]}];
sort\( \eta_{\text{Barmin}} \) = Sort[\( \eta_{\text{BarminData}} \)];
\( \theta_{\text{Barmin}} \) = mean[\( \eta_{\text{BarminData}} \)]; (*Guess the mean for the Gaussian.*)
\( \sigma_{\text{Bmin}} \) = stanDev[\( \eta_{\text{BarminData}} \)]; (*Guess the half-width.*)
hlmin0 = HistogramList[sort\( \eta_{\text{Barmin}} \), {\( \eta_{\text{Barmin}} - 5 \sigma_{\text{Bmin}}, \eta_{\text{Barmin}} + 5 \sigma_{\text{Bmin}}, 0.4 \sigma_{\text{Bmin}} \)}];
hlmax0 = Table[{{1/2} hlmin0[[1, i1]] + hlmin0[[1, i1 + 1]]}, hlmin0[[2, i1]]],
{\( \eta_{\text{Barmin}} \) [2] }];

nlmBmin = NonlinearModelFit[hlmin, \( a \left( 1 + \frac{e^{-4 (x - \theta) / b}}{b} \right)^{-1} \) Exp[\( -(1/2.)(x - \theta) / b \)^2]
\( (*,b>\theta*) \), {{a, Length[runData] / 12}, {b, \sigma_{\text{Bmin}}}, {\theta, \eta_{\text{Barmin}}}}, x];

In[370]:= {amin, bmin, x0min} = {a, b, \theta} /. nlmBmin["BestFitParameters"];
damin, dbmin, dx0min = nlmBmin["ParameterErrors"]; (*x is \( \eta_{\text{Barmin}} \)*
In[368]:= sortηBarmax = Sort[ηBarmaxData];
ηBmax = mean[ηBarmaxData]; (*Guess the mean for the Gaussian. *)
σBmax = stanDev[ηBarmaxData]; (*Guess the half-width.*)
hlmax0 = HistogramList[sortηBarmax, {η0Bmax - 5 σBmax, η0Bmax + 5 σBmax, 0.4 σBmax}];
hlmax = Table[{{1/2} hlmax0[[1, i1]] + hlmax0[[1, i1 + 1]], {i1, Length[ hlmax0[[2]] ]}}];
nlmBmax = NonlinearModelFit[hlmax, {a (1 + e^(-((x-x0)/b)^2))}, {{a, nRunMax/12}, {b, σBmax}, {x0, η0Bmax}}, x];

In[377]:= {amax, bmax, x0max} = {a, b, x0} /. nlmBmax["BestFitParameters"]; {damax, dbmax, dx0max} = nlmBmax["ParameterErrors"]; (*x is ηBarmax*)

In[379]:= anglerHminToCenter = Table[ArcCos[Abs[rHminR[[i]].sourceCenter] - 0.00001], {i, Length[rHminR]}];
θrHminToCenter = mean[anglerHminToCenter];
σθrHminToCenter = stanDev[anglerHminToCenter];
anglerHmaxToCenter = Table[ArcCos[Abs[rHmaxR[[i]].sourceCenter] - 0.00001], {i, Length[rHmaxR]}];
θrHmaxToCenter = mean[anglerHmaxToCenter];
σθrHmaxToCenter = stanDev[anglerHmaxToCenter];
t[6] = TimeUsed[];
fitData = {{nSrc, ρRgnRadius, ρRMS}, {x0min, dx0min}, {bmin, dbmin}, {amin, damin}, {x0max, dx0max}, {bmax, dbmax}, {amax, damax}, {♂rHminToCenter,
♂rHmaxToCenter}, {♂rHmaxToCenter}} (*collect data for saving in a file.*)

In[386]= ListPlot[{{sortηBarmin, sortηBarmax}}];
ListPlot[hlmin];
Normal[nlmBmin];
Print["The parameter table for the fit to η_min: "]
nlmBminPtable = nlmBmin["ParameterTable"]
Normal[nlmBmax];
Print["The parameter table for the fit to η_max: "]
nlmBmaxPtable = nlmBmax["ParameterTable"]

The parameter table for the fit to η_min:

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1616.71</td>
<td>15.0943</td>
<td>107.107</td>
</tr>
<tr>
<td>b</td>
<td>0.0571078</td>
<td>0.000595943</td>
<td>95.8277</td>
</tr>
<tr>
<td>x0</td>
<td>0.609515</td>
<td>0.000498685</td>
<td>1222.24</td>
</tr>
</tbody>
</table>

The parameter table for the fit to η_max:

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1611.24</td>
<td>16.0725</td>
<td>100.248</td>
</tr>
<tr>
<td>b</td>
<td>0.0572295</td>
<td>0.000638084</td>
<td>89.6895</td>
</tr>
<tr>
<td>x0</td>
<td>0.963426</td>
<td>0.00053394</td>
<td>1804.37</td>
</tr>
</tbody>
</table>
7d. Significance of the alignment and avoidance Hub Test metrics for the sample studied in this work

Definitions

fitting function parameters from random runs:

$\eta_{0\text{min}}$ mean of probability distribution for smallest alignment angle $\eta_{\text{min}}$

$\delta\eta_{0\text{min}}$ standard error in the mean as reported by Mathematica

$\sigma_{\text{min}}$ half-width of probability distribution for smallest alignment angle $\eta_{\text{min}}$

$\delta\sigma_{\text{min}}$ standard error in the half-width as reported by Mathematica

$\eta_{0\text{max}}$ mean of probability distribution for largest avoidance angle $\eta_{\text{max}}$

$\delta\eta_{0\text{max}}$ standard error in the mean as reported by Mathematica

$\sigma_{\text{max}}$ half-width of probability distribution for largest avoidance angle $\eta_{\text{max}}$

$\delta\sigma_{\text{max}}$ standard error in the half-width as reported by Mathematica

probmin probability distribution for smallest alignment angle $\eta_{\text{min}}$. This depends on the random runs.
signimin significance, integral of probmin over smaller values of $\eta_{\min}$

probmax probability distribution for largest avoidance angle $\eta_{\max}$

signimax significance, integral of probmax over larger values of $\eta_{\max}$

$\text{signiBarMinfunDataObs}$ Significance of the smallest alignment angle $\eta_{\min}$

$\text{signiBarMinfunDataObs}$ standard errors in $\eta_{\min}$ and $\sigma_{\min}$, i.e. $d\eta_{\min}$ and $d\sigma_{\min}$, give the significances plus/minus values

$\text{sigSmallyBarMinfunDataObs}$, Big extremes of significance assuming one standard error

$\text{signiBarMaxfunDataObs}$ Significance of the largest avoidance angle $\eta_{\max}$

$\text{signiBarMaxfunDataObs}$ standard errors in $\eta_{\max}$ and $\sigma_{\max}$, i.e. $d\eta_{\max}$ and $d\sigma_{\max}$, give the significances plus/minus values

$\text{sigSmallyBarMaxfunDataObs}$, Big extremes of significance assuming one standard error

In[398]:= (*Parameters $\eta$ from random runs, together with their standard errors.*)

$\eta_{\min} = x_{\min}$; $d\eta_{\min} = dx_{\min}$;
$\eta_{\max} = x_{\max}$; $d\eta_{\max} = dx_{\max}$;
$\sigma_{\min} = b_{\min}$; $d\sigma_{\min} = db_{\min}$;
$\sigma_{\max} = b_{\max}$; $d\sigma_{\max} = db_{\max}$;

In[402]:= probmin[$\eta_$] := probMIN0[$\eta$, $\eta_{\min}$, $\sigma_{\min}$]

signimin[$\eta_$] := signiMIN0[$\eta$, $\eta_{\min}$, $\sigma_{\min}$]

probmax[$\eta_$] := probMAX0[$\eta$, $\eta_{\max}$, $\sigma_{\max}$]

signimax[$\eta_$] := signiMAX0[$\eta$, $\eta_{\max}$, $\sigma_{\max}$]

In[406]=

Print["For this sample, but with random polarization directions $\psi$, the random runs give the mean value $\eta_{\min}$ and the half-width $\sigma_{\min}$ of the fitting function of random runs for the smallest alignment angle $\overline{\eta_{\min}}$:"
]

Print[" $\eta_{\min} = \eta_{\min}(360.\,(2\pi)), "^\circ \pm " , d\eta_{\min}(360.\,(2\pi)), "^\circ \text{ and } \sigma_{\min} = \sigma_{\min}(360.\,(2\pi)), "^\circ \pm " , d\sigma_{\min}(360.\,(2\pi)), "^\circ$. (Random $\psi$ distribution)"
]

Print[" "]

Print["For this sample, but with random polarization directions $\psi$, the random runs give the mean $\eta_{\max}$ and the half-width $\sigma_{\max}$ for the distributions for avoidance:"]

Print[" $\eta_{\max} = \eta_{\max}(360.\,(2\pi)), "^\circ \pm " , d\eta_{\max}(360.\,(2\pi)), "^\circ \text{ and } \sigma_{\max} = \sigma_{\max}(360.\,(2\pi)), "^\circ \pm " , d\sigma_{\max}(360.\,(2\pi)), "^\circ$. (Random $\psi$ distribution)"
]

For this sample, but with random polarization directions $\psi$, the random runs give the mean $\eta_{\min}$ and the half-width $\sigma_{\min}$ of the fitting function of random runs for the smallest alignment angle $\overline{\eta_{\min}}$:

$\eta_{\min} = 34.9226^\circ \pm 0.0285725^\circ$ and $\sigma_{\min} = 3.27204^\circ \pm 0.034145^\circ$. (Random $\psi$ distribution)

For this sample, but with random polarization directions $\psi$, the random runs give the mean $\eta_{\max}$ and the half-width $\sigma_{\max}$ for the distributions for avoidance:

$\eta_{\max} = 55.2002^\circ \pm 0.0305925^\circ$ and $\sigma_{\max} = 3.27901^\circ \pm 0.0365595^\circ$. (Random $\psi$ distribution)
(* Significance of the smallest alignment angle $\eta_{\text{min}}$ *)

\[ \text{sig} \eta \eta_{\text{BarMinfunDataObs}} = \text{signimin}[\eta \eta_{\text{BarMinfunDataObs}}]; \]

\[ \text{sigrange} \eta \eta_{\text{BarMinfunDataObs}} = \text{Sort}[\text{Partition}[\text{Flatten}[\text{Table}[\{\text{signiMIN0}[\eta \eta_{\text{BarMinfunDataObs}}, \eta_{\text{0min}} + \gamma_1 \eta_{\text{0min}}, \sigma_{\text{min}} + \gamma_2 \sigma_{\text{domin}}, \gamma_1, \gamma_2], \{\gamma_1, -1, 1\}, \{\gamma_2, -1, 1\}\}, 3]]; \]

\[ \text{sigSmall} \eta \eta_{\text{BarMinfunDataObs}} = \text{sigrange} \eta \eta_{\text{BarMinfunDataObs}}[[1, 1]]; \]

\[ \text{sigBig} \eta \eta_{\text{BarMinfunDataObs}} = \text{sigrange} \eta \eta_{\text{BarMinfunDataObs}}[[-1, 1]]; \]

(* Experimental uncertainties and the Significance of the smallest alignment angle $\eta_{\text{min}}$ *)

\[ \text{sig} \eta \eta_{\text{BarMinfunDataObs}}; \]

\[ \text{sigrange} \eta \eta_{\text{BarMinfunDataObsU}} = \text{Sort}[\text{Table}[\{\text{signiMIN0}[\eta \eta_{\text{BarMinfunDataObs}} + \gamma_1 \sigma_{\text{etaBarminUFit}, \eta_{\text{0min}}, \sigma_{\text{min}}}, \gamma_1\}, \{\gamma_1, -1, 1\}\]]; \]

\[ \text{sigSmall} \eta \eta_{\text{BarMinfunDataObsU}} = \text{sigrange} \eta \eta_{\text{BarMinfunDataObsU}}[[1, 1]]; \]

\[ \text{sigBig} \eta \eta_{\text{BarMinfunDataObsU}} = \text{sigrange} \eta \eta_{\text{BarMinfunDataObsU}}[[-1, 1]]; \]

(* Significance of the largest avoidance angle $\eta_{\text{max}}$ *)

\[ \text{sig} \eta \eta_{\text{BarMaxfunDataObs}} = \text{signimax}[\eta \eta_{\text{BarMaxfunDataObs}}]; \]

\[ \text{sigrange} \eta \eta_{\text{BarMaxfunDataObs}} = \text{Sort}[\text{Partition}[\text{Flatten}[\text{Table}[\{\text{signiMAX0}[\eta \eta_{\text{BarMaxfunDataObs}}, \eta_{\text{0max}} + \gamma_1 \eta_{\text{0max}}, \sigma_{\text{max}} + \gamma_2 \sigma_{\text{domax}}, \gamma_1, \gamma_2], \{\gamma_1, -1, 1\}, \{\gamma_2, -1, 1\}\}, 3]]; \]

\[ \text{sigSmall} \eta \eta_{\text{BarMaxfunDataObs}} = \text{sigrange} \eta \eta_{\text{BarMaxfunDataObs}}[[1, 1]]; \]

\[ \text{sigBig} \eta \eta_{\text{BarMaxfunDataObs}} = \text{sigrange} \eta \eta_{\text{BarMaxfunDataObs}}[[-1, 1]]; \]

(* Experimental uncertainties and the Significance of the smallest alignment angle $\eta_{\text{max}}$ *)

\[ \text{sig} \eta \eta_{\text{BarMaxfunDataObs}}; \]

\[ \text{sigrange} \eta \eta_{\text{BarMaxfunDataObsU}} = \text{Sort}[\text{Table}[\{\text{signiMAX0}[\eta \eta_{\text{BarMaxfunDataObs}} + \gamma_1 \sigma_{\text{etaBarmaxFitU}, \eta_{\text{0max}}, \sigma_{\text{max}}}, \gamma_1\}, \{\gamma_1, -1, 1\}\]]; \]

\[ \text{sigSmall} \eta \eta_{\text{BarMaxfunDataObsU}} = \text{sigrange} \eta \eta_{\text{BarMaxfunDataObsU}}[[1, 1]]; \]

\[ \text{sigBig} \eta \eta_{\text{BarMaxfunDataObsU}} = \text{sigrange} \eta \eta_{\text{BarMaxfunDataObsU}}[[-1, 1]]; \]

(* The names "gridj$\eta$BarMinRan", "j$\eta$BarMax" are, or perhaps were, similar to quantities below, so save the current values labeled by "Best".*)

\[ \{\text{j$\eta$BarMinBest}, \text{j$\eta$BarMaxBest}\} = \{\eta \text{BarMinfunDataObs}, \eta \text{BarMaxfunDataObs}\}; \]
The smallest alignment angle is \( \eta_{\text{min}} = 21.094^\circ \), which has a significance of \( \text{sig.} = 0.000014494 \), plus/minus = + 0.0000304195 and - 0.0000101141, giving a range from \( \text{sig.} = 4.37992 \times 10^{-6} \) to 0.0000449135. (Very Significant: < 1%).

The largest avoidance angle is \( \eta_{\text{max}} = 66.6604^\circ \), which has a significance of \( \text{sig.} = 0.000289222 \), plus/minus = + 0.000538206 and - 0.000195894, giving a range from \( \text{sig.} = 0.0000933282 \) to 0.000827428. (Very Significant: < 1%).

These uncertainties are due to the experimental uncertainty in the observed polarization directions.
More Statistics of the Alignment Function $\eta(H)$:

The min alignment angle, $\eta_{\text{min}} = 21.094^\circ$, is $\Delta \eta = 13.8287^\circ$ below the most likely value, $34.9226^\circ$, for random runs. Since the half-width $\sigma$ is $3.27204^\circ$, the difference, $\Delta \eta = 13.8287^\circ$, makes $\eta_{\text{min}}$ separated from the most likely random run value by $4.22631 \sigma_s$. Thus, the smallest alignment angle $\eta_{\text{min}}$ is $4.22631 \sigma_s$ below the most likely random run value. ($4 \sigma_s$ is a very high level of confidence.)

The max avoidance angle, $\eta_{\text{max}} = 66.6604^\circ$, is $\Delta \eta = 11.4602^\circ$ above the most likely value, $55.2002^\circ$, for random runs. Since the half-width $\sigma$ is $3.27901^\circ$, the difference $\Delta \eta = 11.4602^\circ$ makes $\eta_{\text{max}}$ separated from the most likely random run value by $3.49502 \sigma_s$. Thus, the smallest avoidance angle $\eta_{\text{max}}$ is $3.49502 \sigma_s$ above the most likely random run value. ($3.5 \sigma_s$ is a high level of confidence.)

The computer time expended so far is $77.97$ seconds.