A new derivation of Euler-Bohlin invariant of linearly damped Harmonic oscillator with constant frequency

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Abstract :

[In this paper a simple derivation of Euler-Bohlin invariant is given without any kind of symmetry analysis].

Key words : Invariant, Damped harmonic oscillator, Euler-Bohlin Invariant.

1. Introduction.

The standard equation of damped harmonic oscillator with constant frequency and damping force proportional to velocity is :

$$\ddot{x} + 2K\dot{x} + \omega^2 x = 0 \text{ (overhead dots represent time derivative)} \qquad \ldots(1.1)$$

A well known time-independent invariant of damped harmonic oscillator is Euler-Bohlin invariant [1] which is

$$\frac{(\dot{x}+\lambda_1 x)^{\lambda_1}}{(\dot{x}+\lambda_2 x)^{\lambda_2}} = \text{constant} \qquad \ldots(1.2)$$

where λ_1 and λ_2 are given by

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= 2K \\ \text{and} \quad \lambda_1 . \lambda_2 &= \omega^2 \end{aligned}\right] \qquad \ldots(1.3)$$

2. New derivation of invariant (1.2) of damped harmonic oscillator.

To derive we use a basic result of integrability of a first order nonlinear differential equation of the form :

$$y'(x) - s(x) + \frac{R(x)}{y} = 0, \qquad y' = \frac{dy}{dx} \qquad \ldots(2.1)$$

This integrability condition [2] of differential equation (2.1) is

$$\frac{d}{dx}\left(\frac{R}{S}\right) = \frac{(n-1)}{n^2} S ; \qquad n = \text{constant} \qquad \ldots(2.2)$$

And when the above condition is satisfied an integrating factor of (2.1) is given by [2]

$$\mu = \frac{y}{[y+f(x)]^n} \qquad \ldots(2.3)$$

$$\text{where } f(x) = -\frac{nR(x)}{S(x)} \qquad \ldots(2.4)$$

Now to use the above result, let

$$\dot{x} = y(x) \qquad \ldots(2.5)$$

Therefore $\ddot{x} = y'(x)\, y(x)\ ;\ y' = \frac{dy}{dx}$ …(2.6)

using (2.5) and (2.6) equation (1.1) can be recasted as

$$y'(x) + 2K + \frac{\omega^2 x}{y} = 0 \qquad \ldots(2.7)$$

Equation (2.7) is of the form (2.1). Hence the integrability condition for (1.7) is, using (2.2)

$$\frac{d}{dx}\left(-\frac{\omega^2 x}{2K}\right) = \frac{(n-1)}{n^2}(-2K)$$

$$\text{i.e., } \frac{\omega^2}{2K} = \frac{(n-1)}{n^2}(2K)$$

$$\text{i.e., } n^2\omega^2 - 4K^2 n + 4K^2 = 0 \qquad \ldots(2.8)$$

Equation (2.8) is a quadratic in n, have two values of n given by n_1 and n_2, where

$$\left.\begin{aligned} n_1 + n_2 &= \frac{4K^2}{\omega^2} \\ \text{and } n_1.n_2 &= \frac{4K^2}{\omega^2} \end{aligned}\right| \qquad \ldots(2.9)$$

Then two independent integrating factors of (2.7) are [using (2.3) and (2.4)]

$$\mu_1 = \frac{y}{\left[y + \frac{n_1\omega^2 x}{2K}\right]^{n_1}} \qquad \ldots(2.10)$$

$$\mu_2 = \frac{y}{\left[y + \frac{n_2\omega^2 x}{2K}\right]^{n_2}} \qquad \ldots(2.11)$$

Now, theory of first order ordinary differential equation asserts [3] that the ratio of two linearly independent integrating factors is constant and is the solution of differential equation concerned. It is an easy check that μ_1 and μ_2 are linearly independent.

Hence
$$\frac{\mu_1}{\mu_2} = c = \frac{\left[y + \frac{n_2\omega^2 x}{2K}\right]^{n_2}}{\left[y + \frac{n_1\omega^2 x}{2K}\right]^{n_1}} \qquad \ldots(2.12)$$

Equation (2.12) is thus the solution of (2.7). And it is clear that a solution of (2.7) is an invariant of (1.1).

Therefore, it turns out that (2.12) is an invariant of (1.1).

Now, a little manipulation of (2.12) gives

$$\frac{\left[y + \frac{n_2\omega^2 x}{2K}\right]^{\frac{\omega^2}{2K}n_2}}{\left[y + \frac{n_1\omega^2 x}{2K}\right]^{\frac{\omega^2}{2K}n_1}} = \text{Constant, because } \quad c = \text{Const.} \qquad \ldots(2.13)$$

Equation (2.13) is exactly Euler-Bohlin invariant of (1.1). This may be verified as follows :

Let $\theta_1 = \frac{\omega^2}{2K} n_2$ and $\theta_2 = \frac{\omega^2}{2K} n_1$

Then (2.13) can be rewritten as

$$\frac{(y+\theta_1 x)^{\theta_1}}{(y+\theta_2 x)^{\theta_2}} = \text{Const} = \frac{(\dot{x}+\theta_1 x)^{\theta_1}}{(\dot{x}+\theta_2 x)^{\theta_2}} \quad \text{, using (2.5)} \qquad \ldots(2.14)$$

$$\left.\begin{aligned} &\text{and} \quad \theta_1 + \theta_2 = \frac{\omega^2}{2K} n_2 + \frac{\omega^2}{2K} n_1 = \frac{\omega^2}{2K}(n_2 + n_1) = \frac{\omega^2}{2K}\frac{4K^2}{\omega^2} = 2K \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{using (2.9)} \\ &\text{and} \quad \theta_1 . \theta_2 = \frac{\omega^4}{4K^2} n_2\, n_1 = \frac{\omega^4}{4K^2}\frac{4K^2}{\omega^2} = \omega^2 \end{aligned}\right| \qquad \ldots(2.15)$$

A comparison of (1.3) and (2.15) asserts that θ_1 and θ_2 are identical with λ_1 and λ_2.

Finally a comparison of (1.2) and (2.14) asserts that equation (2.14) is the Euler-Bohlin time independent invariant of damped harmonic oscillator equation (1.1).

3. Conclusion :

Euler-Bohlin invariant is a well known time independent invariant of damped harmonic oscillator. In above a new derivation of the invariant is given. The derivation is simple and uses no symmetry methods.

References

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