A statistical approach to two-particle Bell tests

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Abstract

Extensive experimental tests of the Bell inequality have been conducted over time and the test results are viewed as a testimony to quantum mechanics. In considering the close tie between quantum mechanics and statistical theory, this paper identifies the mistake in previous statistical explanation and uses an elegant statistical approach to derive general formulas for two-particle Bell tests, without invoking any wavefunctions. The results show that, for the special case where the spins/polarizations are in the same, opposite, or perpendicular directions, the general formulas derived in this paper convert to quantum predictions, which are confirmed by numerous experiments. The paper also investigates the linkages between the statistical and quantum predictions and finds that vector decomposition and probability law are at the heart of both approaches. Based on this finding, the paper explains statistically why the local hidden variable theory fails the Bell tests. The paper has important implications for quantum computing, quantum theory in general, and the role of randomism and realism in physics.

Keywords: Bell inequality, probability law, quantum mechanics, realism, local hidden variable theory

1 Introduction

The extensive study on Bell tests originated from the 1935 paper by Einstein et al [26], which claimed that physical reality can be predicted with certainty and that the uncertain nature of quantum prediction is due to incomplete information or the act of local hidden variables. Bohm [4] proposed a thought experiment to test the local hidden variable (LHV) theory and quantum mechanism, but this thought experiment was impractical to implement. In 1964, John Bell [1] developed the Bell inequality from the LHV theory as a testing tool: if the inequality is violated, the LHV theory is disproved. In 1969 Clauser et al [14] extended the Bell inequality to an experimentally testable version. Freedman and Clauser[5], Aspect [23, 24] and many others used this version to test the inequality and convincingly rejected it. Numerous experiments on Bell tests [10, 34, 35, 32, 38, 15, 20, 37, 17, 21, 36, 6, 13, 12, 22] have been conducted to close the 'loopholes' in testing. Since almost all testing results are consistent with the quantum mechanical prediction, they are viewed as a testimony to quantum mechanism.

It is well known that quantum mechanics has a close tie with probability theory. The author suspects that both quantum mechanics and statistics mechanics may essentially be the same in the case of the Bell tests, and therefore identified the mistakes in previous statistical explanation and derived a statistical prediction for two-particle Bell tests. It is revealed that the quantum prediction of the Bell test results is a special case of the statistical prediction. By comparing the statistical and quantum derivations, the author further demonstrates that the essence of quantum prediction is probability law, and that quantum entanglement in two-particle Bell tests is nothing mysterious but an alternative expression for statistical correlation (i.e. there is no difference between statistical and quantum correlations). When the correlated particles are separated and facing different conditions (e.g. polarizers of different orientations), probability law can still maintain their correlation.

The paper is organized as follows: Section 2 demonstrates the deterministic or uncorrelated nature of the Bell inequality and reveals the mistakes in the previous statistical approach. Based on a general case of spin or polarization, Section 3 derives a statistical prediction for Bell tests for all possible uncorrelated and correlated particle pairs. Section 4 explores the linkage between the quantum and statistical predictions, while Section 5 uses the statistical approach to explain the results of representative two-particle Bell tests. Section 6 concludes the paper.

2 Deterministic or uncorrelated nature of the Bell inequality

Realism and localism play a key role in deriving the Bell inequality. The usual assumption for derivation is that at location A, a setting a (e.g. the direction of the spin/polarization analyser) leads to an experimental outcome A(a), while setting b at location B leads to outcome B(b), with the joint outcome being E(a,b)=A(a)B(b). Since a setting leads to an outcome with certainty, the outcome is predetermined by the settings. This fits with the idea of determinism or realism. Moreover, the outcome at a location is determined only by the setting at that location, e.g. A(a) is determined by local setting a at location A, not by setting b at location B. This is localism.

If settings a and b can be changed to a' and b', respectively, we can have joint outcomes: E(a,b')=A(a)B(b'), E(a',b)=A(a')B(b), and E(a',b')=A(a')B(b'). We further assume that the detected outcome at any setting is between -1 and +1, namely $|A| \leq 1$, $|B| \leq 1$. With these assumptions, we can have:

$$
\begin{aligned}
&E(a,b) - E(a,b') = A(a)B(b) - A(a)B(b') \\
&= A(a)B(b) - A(a)B(b') + [A(a)B(b)][A(a')B(b')] - [A(a)B(b)][A(a')B(b')]
\end{aligned}
$$

or

$$E(a,b) - E(a,b') = A(a)B(b)[1 + A(a')B(b')] - A(a)B(b')[1 + A(a')B(b)] \quad (1)$$

In absolute value, we can write:

$$|E(a,b) - E(a,b')| \leq$$
$$|A(a)B(b)| * |(1 + A(a')B(b')| + |A(a)B(b')| * |1 + A(a')B(b)| \quad (2)$$

We have changed the negative sign at the right-hand side of eq. 1 to a positive sign in eq.2 because A(a)B(b') can be negative. Since the values of A(a), B(b), A(a'), and B(b') are all between -1 and 1, we have $|A(a)B(b)| \leq 1$ and $|A(a)B(b')| \leq 1$. As such, the inequality can be written as:

$$|E(a,b) - E(a,b')| \leq |1 + A(a')B(b')| + |1 + A(a')B(b)|$$
$$= 2 \pm |A(a')B(b') + A(a')B(b)|$$

or

$$|E(a,b) - E(a,b')| \leq 2 \pm |E(a',b') + E(a',b)| \quad (3)$$

On the right-hand side of eq.3, we used the '$\pm$' sign because both A(a')B(b') and A(a')B(b) can be negative (leading to negative sign) or positive (leading to positive sign). There are two boundaries in the above inequality. If the lower boundary is satisfied, the inequality holds, so we have arrived at the Bell inequality:

$$|E(a,b) + E(a',b') + E(a'b) - E(a,b')| \leq 2 \quad (4)$$

To incorporate a hidden variable into the inequality, most researchers introduce a random variable. For example, Bell [1, 2] and Clauser et al [14] added to the experiments a hidden variable λ, which has a normalized probability distribution:

$$\int_{-\infty}^{\infty} p(\lambda) d\lambda = 1$$

With the added hidden variable, Bell [1, 2] obtained the expected values of coincidence at the different settings a, a', b and b' as follows:

$$E(a,b) = \int_{-\infty}^{\infty} A(a,\lambda)B(b,\lambda)p(\lambda)d\lambda \quad (5)$$

$$E(a,b') = \int_{-\infty}^{\infty} A(a,\lambda)B(b',\lambda)p(\lambda)d\lambda \quad (6)$$

$$E(a',b) = \int_{-\infty}^{\infty} A(a',\lambda)B(b,\lambda)p(\lambda)d\lambda \quad (7)$$

$$E(a',b') = \int_{-\infty}^{\infty} A(a',\lambda)B(b',\lambda)p(\lambda)d\lambda \quad (8)$$

Using the same procedure that was used to derive the Bell inequality for eq.3 (the deterministic case), Bell ([2] , p178-179) derived (the notations are slightly changed for contemporary readers):

$$
\begin{aligned}
&E(a,b)-E(a,b') = \int_{-\infty}^{\infty} A(a,\lambda)B(b,\lambda)p(\lambda)d\lambda - \int_{-\infty}^{\infty} A(a,\lambda)B(b',\lambda)p(\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} [A(a,\lambda)B(b,\lambda) - A(a,\lambda)B(b',\lambda) + A(a,\lambda)B(b,\lambda)A(a',\lambda)B(b',\lambda) \\
&- A(a,\lambda)B(b,\lambda)A(a',\lambda)B(b',\lambda)]p(\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} A(a,\lambda)B(b,\lambda)[1 + A(a',\lambda)B(b',\lambda)]p(\lambda)d\lambda \\
&- \int_{-\infty}^{\infty} A(a,\lambda)B(b',\lambda)[1 + A(a',\lambda)B(b,\lambda)]p(\lambda)d\lambda
\end{aligned} \tag{9}
$$

In terms of absolute value, we have:

$$
\begin{aligned}
&|E(a,b)-E(a,b')| \leq |\int_{-\infty}^{\infty} A(a,\lambda)B(b,\lambda)[1 + A(a',\lambda)B(b',\lambda)]p(\lambda)d\lambda| \\
&+ |\int_{-\infty}^{\infty} A(a,\lambda)B(b',\lambda)[1 + A(a',\lambda)B(b,\lambda)]p(\lambda)d\lambda| \\
&\leq |\int_{-\infty}^{\infty} [1 + A(a',\lambda)B(b',\lambda)]p(\lambda)d\lambda| + |\int_{-\infty}^{\infty} [1 + A(a',\lambda)B(b,\lambda)]p(\lambda)d\lambda| \\
&= 2 \pm |E(a',b') + E(a',b)|
\end{aligned}
$$

Rearranging the above inequality as before, we can obtain the same inequality as eq. 4.

From the above derivation one may notice that the same term $\int_{-\infty}^{\infty} p(\lambda)d\lambda$ is added to each outcome of the different settings and then this term is filtered out in the end by the definition of expected values in eqs. 7 and 8. As such, the added hidden variable and probability are only additional statistical noise, which does not change the deterministic nature of the resulting inequality.

Later, Bell and others [28, 3, 8] moved on to a version of the Bell inequality based on joint and conditional probabilities. However, they used the same assumption that the distribution of hidden variable λ is UNRELATED to local settings. This assumption apparently contradicts the concept of a local variable. Ironically, this assumption is often regarded as a feature of a local variable. Myrvold et al [29] used a different approach. Instead of concerning the probability distributions of λ conditioned on settings, they conditioned the experimental outcomes on hidden variable λ. Since they assigned no statistical property to λ, its behaviour is unknown, so its role in their derivation is negligible, or not essential at least.

To present a genuine statistical event, one should allow the probability density λ to vary with the local settings. In other words, the probability of value λ must be conditioned on the settings, i.e. for settings a and b, we have the probability p($\lambda|a$) and p($\lambda|b$), respectively. The probability of the joint outcome of settings a and b should be p($\lambda|a,b$). Similarly, we have p($\lambda|a,b'$), p($\lambda|a',b$), p($\lambda|a',b'$) for other joint settings. As such, the expected joint detection should be:

$$E(a,b) = \int_{-\infty}^{\infty} A(a,\lambda)B(b,\lambda)p(\lambda \mid a,b)d\lambda$$

$$E(a,b') = \int_{-\infty}^{\infty} A(a,\lambda)B(b',\lambda)p(\lambda \mid a,b')d\lambda$$

$$E(a',b) = \int_{-\infty}^{\infty} A(a',\lambda)B(b,\lambda)p(\lambda \mid a',b)d\lambda$$

$$E(a',b') = \int_{-\infty}^{\infty} A(a',\lambda)B(b',\lambda)p(\lambda \mid a',b')d\lambda$$

Using this new definition of expected values, the terms for the probability of λ are different for each joint setting and thus cannot be filtered out. As a result, the Bell inequality cannot be derived.

However, one may further assume that the joint probability of outcome at joint setting a and b is the multiplication of probabilities of outcomes at each setting, namely:

$$p(\lambda|a,b) = p(\lambda|a)p(\lambda|b) \tag{10}$$

where $0 \leq p(\lambda|a) < 1$, $0 \leq p(\lambda|b) < 1$, and $\int_{-\infty}^{\infty} p(\lambda|a)d\lambda = \int_{-\infty}^{\infty} p(\lambda|b)d\lambda = 1$.

Applying the same method for joint settings a and b', a' and b, and a' and b', we have:

$$p(\lambda|a,b') = p(\lambda|a)p(\lambda|b')$$

$$p(\lambda|a',b) = p(\lambda|a')p(\lambda|b)$$

$$p(\lambda|a',b') = p(\lambda|a')p(\lambda|b')$$

Based on these joint probabilities, we can calculate E(a,b), E(a,b'), E(a',b) and E(a',b') and, following the same procedure as in deriving eq.9, we can derive Bell inequality eq.4

As we see, eq.10 is crucial for deriving the Bell inequality from a statistical point of view. However, the expression of joint probability as a product of the probability of outcome of two experiments is not without a condition. The well-known but often neglected condition is that the two experiments involved in the joint probability calculation in eq.10 must be totally unrelated, i.e. independent random experiments. Applying this condition to the Bell tests, the requirement is that the probabilities of outcomes at different locations/settings are independent of each other, so 'local' means 'uncorrelated'. This interpretation gives the alternative condition for the Bell inequality. That is, if the outcomes are not deterministic, the outcomes at two different settings should not be correlated.

The common wisdom is that, during a Bell test, the experiments at different locations A and B are apparently independent because the orientations of the polarizers at A and B are changed independently and randomly. However, the independence of settings are not the full condition for independent experiments because local settings are only one element of the polarization experiments. The other element is the light source. In fact, correlated source particles are used all Bell tests conducted so far, the experiments conducted at different locations are not independent. Since the experiments based on different settings are correlated by source particles, the joint probability in a Bell test should be calculated based on conditional probability:

$$p_{a,b} = p_a * p_{b|a}$$

or

$$p_{a,b} = p_b * p_{a|b}$$

Similar mistakes are also commonly made in treating the expected value of joint events as being the multiplication of the expected values of separate events. Due to the statistical nature of the polarization experiments, one needs to allow one setting to generate different results, e.g. experiments based on setting a can have results $A_1(a), A_2(a), \ldots A_n(a)$, so the expected value for results of setting a can be expressed as:

$$E(a) = \frac{1}{n} \sum_i A_i(a) \tag{11}$$

We can also write the expected value for results of setting b as:

$$E(b) = \frac{1}{n} \sum_i B_i(b) \tag{12}$$

Indeed, Bell ([2], p178) realized the importance of introducing eqs.11 and 12 for E(a) and E(b). However, with no precondition being specified, he assumed the following equality as the base for deriving the Bell inequality:

$$E(a, b) = E(a) * E(b) \tag{13}$$

The above equation is used by numerous researchers on Bell tests, but the equation is not unconditional. Statistically, we can expand the expected values as:

$$E(a, b) = \frac{1}{n} \sum_i A_i(a) B_i(b) \tag{14}$$

and

$$E(a) * E(b) = \frac{1}{n^2} \sum_i A_i(a) \sum_i B_i(b) \tag{15}$$

Apparently, $E(a, b) \neq E(a) * E(b)$ in general cases. A special statistical case where $E(a, b) = E(a)*E(b)$ holds is when the outcomes of $A_i(a)$ are independent of (or not correlated to) the outcomes of $B_i(b)$. In this special case, the Bell inequality will hold. If E(a) and E(b) are correlated, we must use the conditional expected values that reflect the correlations between two experiments.

From the above discussion, we can conclude that the Bell inequality does not allow for a probabilistic nature (or correlation, to be exact) because it is based on determinism or realism. To allow for the Bell inequality in a statistical experiment, one must satisfy the condition that $E(a,b) = E(a) * E(b)$, which in turn requires that there is no correlation between $A_i(a)$ and $B_i(b)$. In terms of quantum mechanics terminology, if a and b are in separable (uncorrelated) states, the Bell inequality is valid, otherwise (if a and b are in entangled states), the Bell inequality will be violated.

3 A statistical interpretation of spin/polarization correlation

A statistical presentation of Bell tests seems to be complicated because it involves many random settings, such as random directions of polarizers and random polarization of light or spins of particles. Moreover, spins and polarizations have different features. After trying a number of methods, the author has arrived at a remarkably simple and elegant approach for deriving the statistical prediction.

The difference between polarization and spin is that spins in opposite directions have different values while polarizations in the opposite directions are viewed as being the same. In other words, the spin direction in a plane can have a 360° variation while the polarization direction varies only within 180°, so the case of polarization is a reduced case of spin. For generality, this section focuses on deriving the results for the case of spin, and then shows how the results can be applied to the case of polarization.

There are various types of spin analyzer/detector [31, 27, 7, 16], but all spin detectors rely on a differing scattering cross section for spin polarized particles. During spin detection, the direction of travel of the particle and the detector orientation form a plane, in which the particles are reflected and detected [27]. The spin polarized particles will cause asymmetric reflection, and the asymmetric results indicate the detected spin direction. Essentially, a spin analyser works similarly to a polarizer for light, but the analyser can identify the spin direction along the given detection orientation. Consequently, we use a polarizer with an arrow to represent a spin analyzer.

Fig. 1 shows a general case where the particles of the different spin directions are measured by the two spin analyzers in a Bell test experiment. Two spins, s_1 and s_2, and two spin analyzers, A and B, are positioned in different directions. The spin directions of particles 1 and 2 form an angle of θ_1 and θ_2, respectively, with the x-axis. For simplicity, we assume that s_1 and s_2 are unit vectors, and that spin analyzer A is placed in the direction of the x-axis while spin analyzer B forms an angle of β with the x-axis. Given this setting, the component of s_1 detected by A is $E'_A = E'_{Ax} = cos\theta_1$. Similarly, the angle between s_2 and the spin analyzer B is $\theta_2 - \beta$, so the component of s_2 detected by B is $E'_B = cos(\theta_2 - \beta)$.

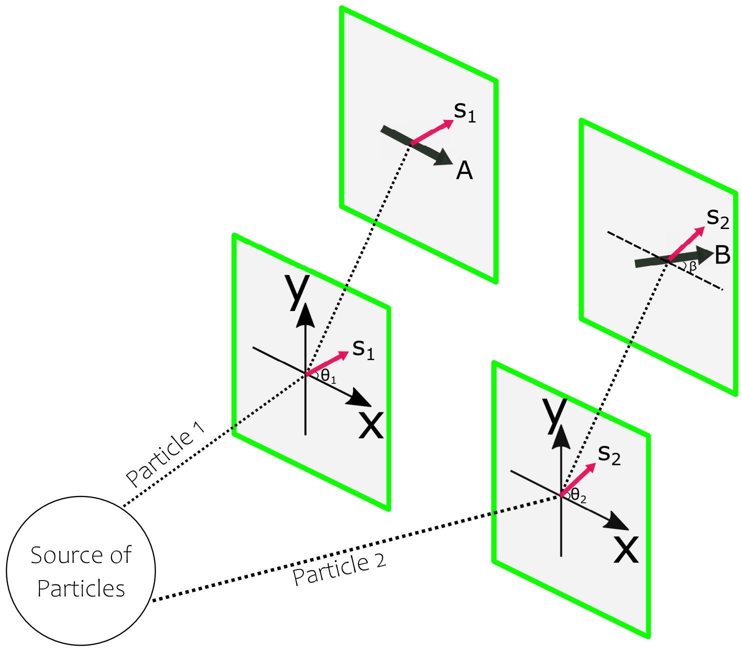

Figure 1: Measuring spin direction

There are two types of correlation measurement in the Bell tests. One is the joint detection counts normalized on the separate detection counts at each settings. The other is the joint detection rate normalized on the emission rate at the particle source. We address them in turn.

3.1 Correlation normalized on outcomes at each setting

This measurement fits with the standard definition of correlation, so we can calculate the expected value, variance and covariance and then obtain correlation. Since the source emits particles of random spin directions, the expected values and variances can be obtained by integrating E'(A) and E'(B) over the spin angles θ_1 and θ_2 in the range of 0 - 2π for particles 1 and 2.

$$E(A) = \frac{\int_0^{2\pi} E'(A)d\theta_1}{\int_0^{2\pi} d\theta_1} = \frac{\int_0^{2\pi} cos\theta_1 d\theta_1}{\int_0^{2\pi} d\theta_1} = \frac{sin\theta_1}{\theta_1}\Bigg|_0^{2\pi} = 0$$

$$E(B) = \frac{\int_0^{2\pi} E'(B)d\theta_2}{\int_0^{2\pi} d\theta_2} = \frac{\int_0^{2\pi} cos(\theta_2 - \beta)d\theta_2}{\int_0^{2\pi} d\theta_2} = \frac{sin(\theta_2 - \beta)}{\theta_2}\Bigg|_0^{2\pi} = 0$$

$$var(A) = \frac{\int_0^{2\pi}[cos\theta_1 - E(A)]^2 d\theta_1}{\int_0^{2\pi} d\theta_1} = \frac{1}{2\pi}\int_0^{2\pi} cos^2\theta_1 d\theta_1$$
$$= \frac{1}{2\pi}\int_0^{2\pi} 0.5(cos2\theta_1 + 1)d\theta_1 = 0.5$$

$$var(B) = \frac{\int_0^{2\pi}[cos(\theta_2 - \beta) - E(B)]^2 d\theta_2}{\int_0^{2\pi} d\theta_2} = \frac{1}{2\pi}\int_0^{2\pi} cos^2(\theta_2 - \beta)d\theta_2$$
$$= \frac{1}{2\pi}\int_0^{2\pi} [0.5cos2(\theta_2 - \beta) + 1]d\theta_2 = 0.5$$

If the two particles are uncorrelated, θ_1 and θ_2 can vary independently, so the covariance can be calculated through a double integral:

$$cov(A, B) = \frac{\iint_0^{2\pi}[cos\theta_1 - E(A)][cos(\theta_2 - \beta) - E(B)]d\theta_1 d\theta_2}{\iint_0^{2\pi} d\theta_1 d\theta_2}$$
$$= \frac{1}{(2\pi)^2}\int_0^{2\pi} cos\theta_1 d\theta_1 \int_0^{2\pi} cos(\theta_2 - \beta)d\theta_2 = 0$$

The zero covariance is expected because of the uncorrelated nature of s_1 and s_2 - the positive and negative joint detection counts will be largely cancelled out. If the two spins are correlated, θ_1 and θ_2 can still change randomly, but the two angles must keep the same difference, i.e. $\theta_2 = \theta_1 + \theta_0$, where θ_0 is the fixed relative angle between two spin directions. In this case, the covariance can be calculated by an integration over θ_1 (or θ_2):

$$cov(A, B) = \frac{\int_0^{2\pi}[cos\theta_1 - E(A)][cos(\theta_1 + \theta_0 - \beta) - E(B)]d\theta_1}{\int_0^{2\pi} d\theta_1}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} 0.5[cos(2\theta_1 + \theta_0 - \beta) + cos(\beta - \theta_0)]d\theta_1 = 0.5cos(\beta - \theta_0)$$

As such, we have the following spin correlation:

$$E(A, B) = \frac{cov(A, B)}{[var(A)]^{1/2}[var(B)]^{1/2}} = \frac{0.5cos(\beta - \theta_0)}{0.5^{0.5} * 0.5^{0.5}} = cos(\beta - \theta_0) \tag{16}$$

Eq.16 is a general result for joint detection for any given orientations of spin detectors. The application of this equation for special occasions can produce quantum predictions. For example, if two particles have the same spin, i.e. entangled particles of the same phase, we have $\theta_0 = 0$, $E_{AB} = cos\beta$. If two particles have the opposite spin, i.e., negatively correlated particles, we have $\theta_0 = \pi$, $E_{AB} = -cos\beta$. If the two spin vectors are perpendicular, $\theta_0 = \pi/2$, $E_{AB} = cos(\pi/2 - \beta) = sin\beta$.

It is worth mentioning that some researchers used light intensity correlation instead of the expected-value correlation for polarization Bell test. For example, Ou and Mandel [18] and Rarity and Tapster [30] regarded the joint detection probability of photons as being proportional to the intensity correlation of light. This approach is misplaced. For polarization experiments, one or more photons (assuming perfect detection for the simplicity of an argument) pass through the polarizer, a positive detection will be recorded, so the intensity is not an appropriate measurement. One may argue that intensity is the square of amplitude so intensity can be used as the proxy of probability of photons passing through the polarizer, based on which the joint probability can be calculated. However, as explained in Section 2, the joint probability cannot be calculated through the multiplication of probabilities of separate detections because of the correlated particles in a Bell test. Since probability measures the average of the squared detection values, the intensity (or probability) correlation approach will produce totally different result from that in this paper. This can be shown in the following expression:

$$p_{AB} = p_A p_B = <E_A^2><E_B^2> \neq <E_A E_B>^2 = E_{AB}^2$$

3.2 Correlation normalized on emissions at the source

For a Bell test, one need to measure many pairs of particles of different spin directions with varied detector orientations. In this case, the joint detection rate is generally normalized on the emission rate at the source and the correlation is calculated based on fixed axes.

Referring to Fig.1, if the correlation is calculated based on x and y axes, the component detected by analyzer B needs to be further decomposed on the x-axis and y-axis:

$$E'_{Bx} = E'_B cos\beta = cos(\theta_2 - \beta)cos\beta$$
$$E'_{By} = E'_B sin\beta = cos(\theta_2 - \beta)sin\beta$$

Since no component on the y-axis is detected by analyzer A, the correlation (joint detection) on the y-axis is zero. On the other hand, both analyzers detect values on the x-axis, so the joint detection value is:

$$E'_{AB} = E'_{Ax} E'_{Bx} = cos\theta_1 cos(\theta_2 - \beta)cos\beta$$

Since the correlation is based on the emissions at source, which are 100% detected (assuming all particles come to and are detected by either detector A or B), the variances are one and thus the correlation is equivalent to co-variance. If particles 1 and 2 are uncorrelated, the joint detection rate will be the value of E'_{AB} integrated over both θ_1 and θ_2:

$$p_{AB} = \frac{\iint_0^{2\pi} E'_{AB} d\theta_1 d\theta_2}{\iint_0^{2\pi} d\theta_1 d\theta_2} = \frac{\iint_0^{2\pi} cos\theta_1 cos(\theta_2 - \beta)cos\beta d\theta_1 d\theta_2}{\iint_0^{2\pi} d\theta_1 d\theta_2}$$
$$= \frac{cos\beta}{(2\pi)^2}\int_0^{2\pi} cos\theta_1 d\theta_1 \int_0^{2\pi} cos(\theta_2 - \beta)d\theta_2 = 0$$

The above result indicates that for uncorrelated particles, the joint detection rate is zero. This makes sense. Due to the uncorrelated random nature, the different detection counts will be washed out by the independent random changes in θ_1 and θ_2.

If two particles are correlated, i.e. $\theta_2 = \theta_1 + \theta_0$, we can obtain correlation by integrating E'_{AB} over θ_1 (or θ_2) in range 0 to 2π:

$$p_{AB} = \frac{\int_0^{2\pi} E'_{AB} d\theta_1}{\int_0^{2\pi} d\theta_1} = \frac{\int_0^{2\pi} cos\theta_1 cos(\theta_1 + \theta_0 - \beta) cos\beta d\theta_1}{\int_0^{2\pi} d\theta_1}$$
$$= \frac{cos\beta}{2\pi} \int_0^{2\pi} 0.5[cos(2\theta_1 + \theta_0 - \beta) + cos(\beta - \theta_0)] d\theta_1$$
$$= 0.5 cos(\beta - \theta_0) cos\beta \quad (17)$$

The above result shows that when the two spin vectors are correlated, i.e., the value of θ_0 is fixed, the joint detection rate is determined only by correlation phase θ_0 and the angle β between the orientations of two spin detectors.

Eqs.16 and 17 can also be applied to light polarization experiments. In the case of polarized light, it is tricky to derive the joint detection because the detected values have to be non-negative and thus are not consistent with the cosine functions for E'_A and E'_B. The common approach (e.g. Aspect et al [14, 23]) is to define the no-detection result as -1, instead of 0. In other words, when the light polarization is perpendicular to the orientation of detector, no photon will be detected and thus a result of -1 with a 90° will be recorded. With this definition, all angles in eqs.16 and 17 should be halved, and then the equation is equally applicable to the Bell tests with polarized light.

Where the two spin vectors are in the same directions (i.e. $\theta_0 = 0$), eq.17 becomes:

$$p_{AB} = 0.5 cos^2\beta = 0.25(cos 2\beta + 1) \quad (18)$$

In this special case, the joint detection rate can also be derived without integration, as shown in Fig. 2.

To present three random directions (i.e. the same direction of spin of the two particles, and the directions of the two spin analyzers A and B), we can fix one of them because only the relative angles between them matter. For convenience of presentation, we assume the spin vector $\overrightarrow{OV}$ to be a unit vector pointing to V($a_x/\sqrt{2}, a_y/\sqrt{2}$), where a_x and a_y are unit vectors at x and y directions, respectively.

The projection of the spin vector $\overrightarrow{OV}$ onto the B axis in Fig.2is:

$$\overrightarrow{OB_2} = \overrightarrow{B_1B_2} + \overrightarrow{OB_1} = [\overrightarrow{a_x} cos(\theta - \theta_b) + \overrightarrow{a_y} sin(\theta - \theta_b)]/\sqrt{2}$$

.

This projection can be further projected onto the x-axis and y-axis and thus decomposed to two components $\overrightarrow{OB_x}$ and $\overrightarrow{OB_y}$, respectively ($\overrightarrow{OB_y}$ is not shown in Fig.2 so as not to complicate the graph):

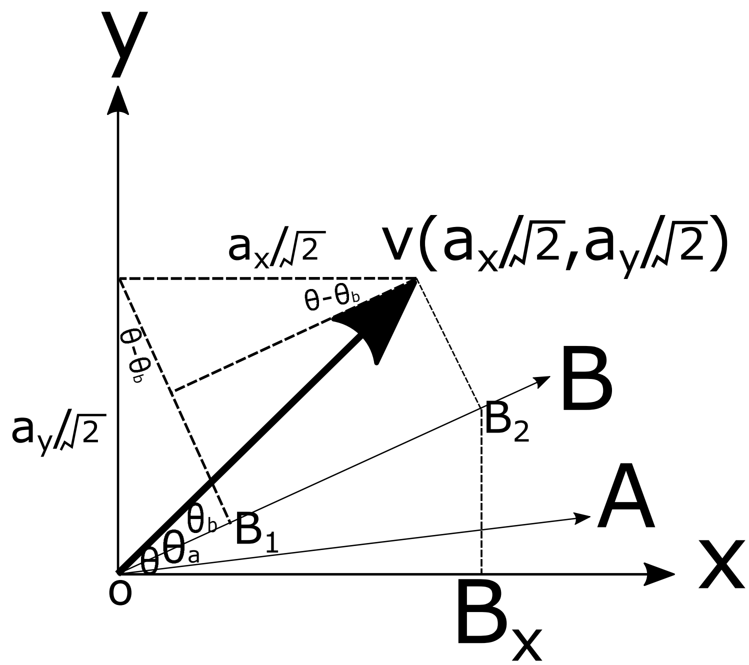

Figure 2: Measuring the correlation of particle pair of the same spin

$$\overrightarrow{OB_x} = cos(\theta - \theta_b)[\overrightarrow{a_x} cos(\theta - \theta_b) + \overrightarrow{a_y} sin(\theta - \theta_b)]/\sqrt{2} \tag{19}$$

$$\overrightarrow{OB_y} = sin(\theta - \theta_b)[\overrightarrow{a_x} cos(\theta - \theta_b) + \overrightarrow{a_y} sin(\theta - \theta_b)]/\sqrt{2} \tag{20}$$

Similarly, the projection of $\overrightarrow{OV}$ onto the A-axis can be decomposed into the x and y components of $\overrightarrow{OA_x}$ and $\overrightarrow{OA_y}$ respectively(not shown in Fig. 2):

$$\overrightarrow{OA_x} = cos(\theta - \theta_a)[\overrightarrow{a_x} cos(\theta - \theta_a) + \overrightarrow{a_y} sin(\theta - \theta_a)]/\sqrt{2}$$

$$\overrightarrow{OA_y} = sin(\theta - \theta_a)[\overrightarrow{a_x} cos(\theta - \theta_a) + \overrightarrow{a_y} sin(\theta - \theta_a)]/\sqrt{2}$$

As such, the joint detection rate can be calculated as:

$$
\begin{aligned}
p_{AB} &= \overrightarrow{OA_x}\overrightarrow{OB_x} + \overrightarrow{OA_y}\overrightarrow{OB_y} \\
&= cos(\theta - \theta_b)[\overrightarrow{a_x}cos(\theta - \theta_b) + \overrightarrow{a_y}sin(\theta - \theta_b)]/\sqrt{2} \\
&\times cos(\theta - \theta_a)[\overrightarrow{a_x}cos(\theta - \theta_a) + \overrightarrow{a_y}sin(\theta - \theta_a)]/\sqrt{2} \\
&+ sin(\theta - \theta_b)[\overrightarrow{a_x}cos(\theta - \theta_b) + \overrightarrow{a_y}sin(\theta - \theta_b)]/\sqrt{2} \\
&\times sin(\theta - \theta_a)[\overrightarrow{a_x}cos(\theta - \theta_a) + \overrightarrow{a_y}sin(\theta - \theta_a)]/\sqrt{2} \\
&= 0.5[\overrightarrow{a_x}cos(\theta - \theta_b) + \overrightarrow{a_y}sin(\theta - \theta_b)] \\
&\times [\overrightarrow{a_x}cos(\theta - \theta_a) + \overrightarrow{a_y}sin(\theta - \theta_a)]cos(\theta_a - \theta_b) \\
&= 0.5cos^2(\theta_a - \theta_b)
\end{aligned}
$$

Or

$$p_{AB} = 0.25[cos2(\theta_a - \theta_b) + 1] \tag{21}$$

Noting that $(\theta_a - \theta_b)$is the angle between the orientations of two detectors A and B, we find that the above result is the same as eq.18. This joint probability of detection is exactly the same as the coincidence rate derived from quantum mechanics. The experiment by Aspect [23] confirmed this result.

The correlation function (eq.16) and the joint detection rate (eq. 17) derived in this section are general results that are applicable to both uncorrelated or correlated polarization/spin of any phase differences. The results can be tested experimentally using the current Bell test techniques. The only change needed is to add a randomly controlled source polarizer for each of the two beams after the collimation lenses but before the traditional Bell test polarizers. If the pair of source polarizers are randomly and separately controlled, i.e. their relative angle of polarizarion θ_0 varies randomly, the source particles are uncorrelated, so the joint detection rate will be zero for a large sample size. If the pair of source polarizers are controlled randomly but jointly, i.e. the relative polarization angle of the pair is fixed at any given value, the joint detection rate should be determined by the relative angle (θ_0) of the first pair of (source) polarizers and that (β) of the second pair, with the quantitative relations determined by eqs.16 amd 17.

4 Linkage between the statistical approach and quantum mechanics

From the previous section, we see that the simple statistical approach gives equivalent but more general results when compared with the predictions from quantum mechanics (QM). This is not a coincidence. This section shows that the statistical approach is at the heart of quantum mechanical prediction on Bell tests.

QM uses wavefunctions to present the different states. For example, a wavefunction of a spin-up (or +1) state can be written in Dirac notation as |0>, while spin-down (or -1) can be written as $|1>$. The spin states can be projected to (or measured on) different axes and may result in different results. If Alice measures a spin state of $|0>$ on the A-axis while Bob measures $|1>$ on the B-axis, we can express this spin state as $|0>\otimes|1>$, or simply $|01>$. A wavefunction $|01> + |10>$ indicates that the measurement on the A-axis is always opposite to the measured results on the B-axis, i.e. the measured results are negatively correlated. Similarly, the stats in wavefunction $|00> + |11>$ are positively correlated. The states in this type of wavefunctions are called entangled states. On the other hand, a wavefunction of $|01> + |00>$ shows that while Alice's measurement is always $|0>$, Bob's measurement can be either $|0>$ or $|1>$, so there is no correlation between the two measurement results. The states in this wavefunction are called separable states. In short, the entangled states are the QM expression for correlation.

Now we consider a normalized entangled state: $\psi = (|00> + |11>)/\sqrt{2}$. If this state is measured by Alice on the A or x axis (both axes coincide, shown in Fig.3), the possible outcome will be $<0|\sigma^A|0> = +1$ or $<1|\sigma^A|1> = -1$. Similarly, if the state is measured by Bob on the B-axis, the possible outcome will be $<0|\sigma^B|0> = +1$ or $< 1|\sigma^B|1> = -1$. Since this is a wavefunction of positively entangled states, Alice and Bob will always obtain the same (positive or negative) measurement outcome. Bob's measurement can be decomposed to two components on the x-axis and y-axis: $\sigma^B = \sigma^B_x cos\beta + \sigma^B_y sin\beta$. Alternatively, we can write: $< 0|\sigma^B_x|0> = cos\beta, <1|\sigma^B_x|1> = -cos\beta, <0|\sigma^B_y|0> = sin\beta, <1|\sigma^B_y|1> = -sin\beta$. Since Alice's measurement is on the x-axis, we have $\sigma^A = \sigma^A_x$.

The correlation between the measurements of Alice and Bob can be calculated by the expected value of joint measurements: $<ab> = <\sigma^A\sigma^B>$. The QM calculation result is as follows:

$$
\begin{aligned}
&<ab> = <\psi|\sigma^A \otimes \sigma^B|\psi> \\
&= 0.5(<00| + <11|)\sigma^A \otimes \sigma^B(|00> + |11>) \\
&= 0.5(<00|\sigma^A \otimes \sigma^B|00> + <11|\sigma^A \otimes \sigma^B|00>) \\
&+ <00|\sigma^A \otimes \sigma^B|11> + <11|\sigma^A \otimes \sigma^B|11>) \\
&= 0.5(<0|\sigma^A|0><0|\sigma^B|0> + <1|\sigma^A|0><1|\sigma^B|0>) + <0|\sigma^A|1><0|\sigma^B|1> \\
&+ <1|\sigma^A|1><1|\sigma^B|1>) = 0.5(<0|\sigma^A|0><0|\sigma^B|0> + <1|\sigma^A|1><1|\sigma^B|1>) \\
&= 0.5(<0|\sigma^A_x|0><0|\sigma^B_x|0> + <1|\sigma^A_x|1><1|\sigma^B_x|1>) = cos\beta
\end{aligned}
$$

The above result is exactly the same as eq.16 (with $\theta_0 = 0$) that was obtained from the much simpler statistical approach. A number of statistical features in the QM approach contribute to this same result. First, the calculation of the expected value in QM (i.e. $<ab> = <\psi|\sigma^A \otimes \sigma^B|\psi>$) is based on a probability-weighted average. Second, the rule of tensor product ($<11|\sigma^A \otimes \sigma^B|00> = <1|\sigma^A|0><1|\sigma^B|0>$) makes an operator (e.g. σ^A or σ^B) work on the wavefunction on its space only. This is exactly the case of measurement (or vector component decomposition) on different axes. Third, the

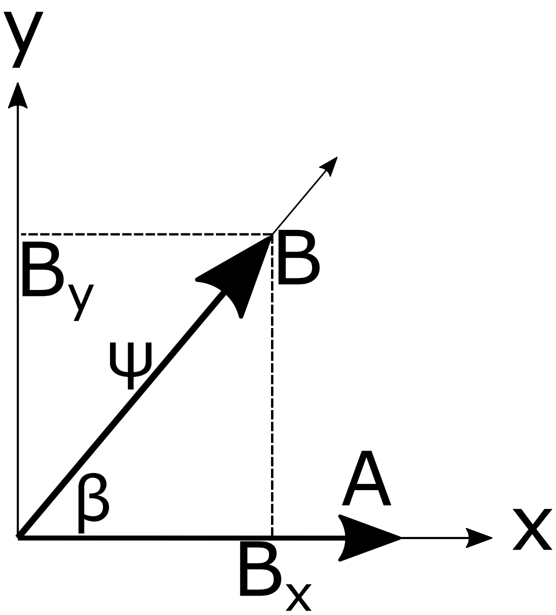

Figure 3: Spin measurement for positively entangled particles

orthogonal condition of basis wavefunctions mimics the measurement of the projection onto the orthogonal axes, e.g. $<0|\sigma^A|0> = +1$, $<1|\sigma^B|1> = -1$, and $<1|\sigma^A|0> = 0$. Fourth, the space (or axis) separation is consistent with the concept of correlation. For example, since Alice measures on the x-axis, only the x-component of the measurement by Bob is relevant to the correlation calculations. This is manifested by $<0|\sigma^A|0><0|\sigma^B|0> =< 0|\sigma_x^A|0><0|\sigma_x^B|0>$. Finally, the normalized wavefunction automatically normalizes the calculated expected value so that it fits the requirement of correlation.

If we use other entangled wavefunctions to perform similar calculations, we would arrive at essentially the same results but with a negative sign for some wavefunctions. For example, with $\phi = (|01> + |10>)/\sqrt{2}$, we find $<ab> =< \phi|\sigma^A \otimes \sigma^B|\phi> = -cos\beta$, which is equivalent to eq.16 with $\theta_0 = \pi$. This is not surprising as this wavefunction indicates a negative correlation. If we use a wavefunction of separable states to calculate the expected joint measurement, we would find that a value of zero. This is the expected because there is no correlation between separable states.

If the measurement axes change randomly, we cannot put a vector on either A or B axis. In this case, the QM derivation of the joint detection rate involves a projection process similar to that used in Fig. 2. Using a matrix presentation, we can express the projection of a vector pointing to (x_1, y_1) onto a specified axis of angle θ as follows:

$$\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \begin{pmatrix} cos\theta & sin\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} cos^2\theta & cos\theta sin\theta \\ cos\theta sin\theta & sin^2\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{22}$$

In the above equation, if we let θ be the angle of B axis with respect with the x-axis, i.e. $\theta - \theta_b$ in Fig. 2, and let $x_1 = \overrightarrow{a_x}/\sqrt{2}$ and $y_1 = \overrightarrow{a_y}/\sqrt{2}$, we can obtain the same result as in eqs.19 and 20.

The matrix in eq.22 is called a projection matrix [28], as it projects a vector onto the axis of angle θ and gives the components of the projection:

$$Q(\theta) = \begin{pmatrix} cos^2\theta & cos\theta sin\theta \\ cos\theta sin\theta & sin^2\theta \end{pmatrix}$$

Using the above projection matrix and an entangled wavefunction (e.g. $\phi = (|01> + |10>)/\sqrt{2}$), we can calculate the probability of joint measurement as:

$$p_{AB} \ = \ <\psi|Q^A \otimes Q^B|\psi> \ = \ 0.5cos^2(\theta_A - \theta_B) \ = \ 0.25[cos2(\theta_A - \theta_B) + 1]$$

Since $\theta_A - \theta_B$ is the angle of the orientations of detectors, the above result is exactly the same as eq.18 or eq.21 that we derived in the statistical approach. The identical result is apparently because the same projection process works in both approaches.

5 Statistical explanation of two-particle Bell tests

Many Bell test experiments are based on the coincidence rate of particle pairs, but a handful of researchers (e.g. [35, 21, 12, 33, 25, 39] have conducted experiments on correlations of 3 or more particles. Multi-particle correlation is generally achieved by special designs of experimental setup to achieve specific quantum states (e.g.[35, 33]) or by exploiting the coherent states of Bose-Einstein condensate (e.g.[21, 12]). The statistical foundation of multi-particle correlation is the same as that for particle pairs, so this paper focus on two-particle correlation. Even though we confine our scope to two-particle Bell tests, there still are copious experiments. This section selects only some representative experiments and puts them into two groups: the polarization experiments of entangled photon pairs and non-polarization experiments based on light phase correlation.

5.1 Polarization experiments

Among numerous Bell test using polarization of photon pairs, we consider only two influential papers by Aspect et al [23, 24]. Like most experiments on the Bell tests, Aspect et al [23, 24] utilized the derivation of Clauser et al [14] for an experimentally applicable quantum mechanical prediction for the counting rates of coincidence. The starting point of their derivation is a probability formula:

$$P(a,b) = w[A(a)_+, B(b)_+] - w[A(a)_+, B(b)_-] \\ - w[A(a)_-, B(b)_+] + w[A(a)_-, B(b)_-]$$

where w means the probability weighting of each outcome in total emission counts R_0, with R_0=$[A(a)_+, B(b)_+]$+$[A(a)_+, B(b)_-]$+$[A(a)_-, B(b)_+]$+$[A(a)_-, B(b)_-]$, so we have $w[A(a)_+, B(b)_+] = [A(a)_+, B(b)_+]/R_0$, etc.

The above equation is a manifest that the net correlation (positive correlation $[A(a)_+, B(b)_+]$+$[A(a)_-, B(b)_-]$ minus negative correlation $[A(a)_+, B(b)_-]$+ $[A(a)_-, B(b)_+]$) in terms of total counts R_0. This equation is consistent with our derivation of joint detection rate presented in Section 2: the net correlation in eq.17 is calculated by integrating E'_{AB} over the angle of $0-2\pi$ while the total counts is obtained by integrating the unit spin vector over the same range. Due to the same foundation for derivation, the resulting eq.18 is unsurprisingly the same as what obtained by Clauser et al [14] and used by Aspect et al [23, 24]. Since the joint detection rate derived from both statistical and quantum approaches is identical, the explanation on the results of Aspect et al [23, 24] will be very similar, so we omit this explanation but examine the maximum violation angle derived from quantum mechanics and confirmed by experiments.

Using the coplanar vectors (shown in Fig.4) introduced by Clauser and Shimony [28] and Aspect et al [23] to present the settings of the Bell test experiments, we can derive the same results as the quantum prediction of the Bell test, but without invoking any wavefunctions.

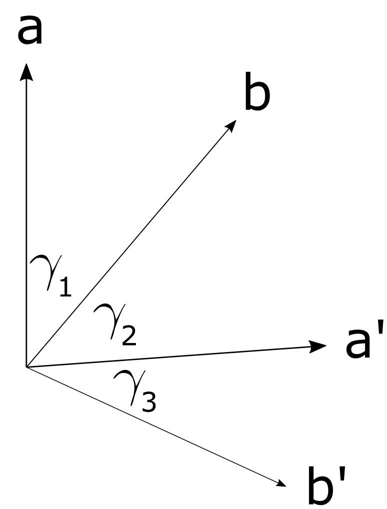

Figure 4: Coplanar vectors presentation of Bell test settings

In Fig.4, vectors a, a', b, and b' represent the direction of the spin detectors, and the angles between them are displayed on the graph. For simplicity of presentation, we assume all vectors are of unit modulus and angles γ_1 , γ_2, and γ_3 are positive and less than π (for any angle θ greater than π, we can rewrite it as $2\pi - \theta$). Applying the spin correlation results in eq.16 derived in Section 3 to a case of positively entangled particles (i.e. $\theta_0 = 0$), we can obtain the experimental results as follows:

$$
\begin{aligned}
&E(a,b) = cos\gamma_1 \\
&E(a,b') = cos(\gamma_1 + \gamma_2 + \gamma_3) \\
&E(a',b) = cos\gamma_2 \\
&E(a',b') = cos\gamma_3
\end{aligned}
$$

The theoretical results for the Bell tests should be:

$$
\begin{aligned}
E_{BT} &= E(a,b) - E(a,b') + E(a',b) + E(a',b') \\
&= cos\gamma_1 - cos(\gamma_1 + \gamma_2 + \gamma_3) + cos\gamma_2 + cos\gamma_3
\end{aligned}
$$

Applying the first and second order conditions of maximization (minimization) for the above equation, we know that E_{BT} reaches the maximum or minimum when sinγ_1= sin γ_2= sinγ_3=sin(γ_1+γ_2+γ_3).

If γ_1, γ_2 and γ_3 are less than $\pi/2$, the condition of maximum/minimum value necessitates that $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ and sinγ=sin3γ. With some trigonometric manipulations, from sinγ=sin3γ we can have sinγ(4cos2γ-1)=sinγ, or γ=$\pi/4$.

Similarly, if γ_1, γ_2 and γ_3 are greater than $\pi/2$ (they are less than π as we assumed before for simplicity), so we can obtain γ=3$\pi/4$.

If some angles are less than $\pi/2$ but some are greater than $\pi/2$, we obtain no satisfying solution. For example, if γ_1 and γ_2 are less than $\pi/2$, but γ_3 is greater than $\pi/2$, from sinγ_1= sinγ_2= sinγ_3, we can infer that $\gamma_1 = \gamma_2$ and γ_3=$\pi - \gamma_2$, so sinγ_1=sin(γ_1+γ_2+γ_3)=sin($\gamma_1 + \pi$)=-sinγ_1, or $\gamma_1 = \gamma_2 = \gamma_3 = 0$. This contradicts our assumption and presents a trivial case where all 4 settings coincide.

To sum up, from the first and second order condition we reveal that the maximum and minimum value of E_{BT} occurs at γ=$\pi/4$ and γ=3$\pi/4$, respectively. If γ=$\pi/4$, we have:

$$
E_{max} = cos\pi/4 - cos3\pi/4 + cos\pi/4 + cos\pi/4 = 2\sqrt{2}
$$

If $\gamma = 3\pi/4$, we have:

$$
E_{min} = cos3\pi/4 - cos9\pi/4 + cos3\pi/4 + cos3\pi/4 = -2\sqrt{2}
$$

As a result, we obtain the same results as the quantum prediction:

$$
|E(a,b) - E(a,b') + E(a',b) + E(a',b')| \leq 2\sqrt{2}
$$

It is worth mentioning that the above derivation shows that the maximum violation of the Bell inequality occurs when γ=$\pi/4$ or γ=3$\pi/4$, E=$\pm 2\sqrt{2}$. This seems in conflict with the results of Aspect et al [23, 24], where the maximum violation of the Bell inequality occurred at θ=$\pi/8$, or θ =3$\pi/8$.

In fact, this difference highlights the difference of the cases of spin and polarization. Our derivation is based on spin detection. As we discussed in Section 3, the angle must be adjusted when applying eqs.16 and 17 to polarization experiments. In most Bell test experiments using light, including Aspect et al [23, 24], a count of photon detection is recorded as +1 and a no detection is recorded as -1. As such, if the angle between the polarizer and the polarization of light is $\theta = \pi/2$, the most likely outcome is no detection or -1. We can express the result as $cos2\theta = \cos\pi = -1$. It is apparent that one needs to double the angle to obtain a result that is consistent with experimental record. On the other hand, our derivation based on spin assumes that a count of photon detection is recorded as +1 and a no detection is recorded as 0. If the angle between the polarizer and the polarization of light is $\gamma = \pi/2$, the most likely outcome is no detection or 0. We can express the result as $cos\gamma = \cos\pi/2 = 0$. This recorded value is equivalent to the case of $\theta = \pi/4$ in Aspect et al [23, 24]. From this we can infer that the angle γ used for spin examples in the present paper is equivalent to twice the angle θ used in Aspect et al [23, 24], i.e. $\gamma = 2\theta$. As a result, the angles for maximum violation of the Bell inequality in Aspect et al [23, 24] will be half the value as in our derivation.

5.2 Interferometry Bell tests

There are Bell tests that examine the correlations between variables other than polarization. One type of research focus on the phase correlation(e.g. [13, 19, 30, 11]). This type of experiments create a pair of photons of the same phase and let them pass through phase shifters and a distance of different length, then detect the phase difference at a Michelson interferometer. The experiments are based on the theoretical prediction of Franson [9] which, based on the phase difference of wavefunctions caused by time difference, developed a similar prediction as eq. 18 in the present paper. Using a classical wave theory of light and intensity correlation, one can also obtain an equivalent result.

For simplicity, we combine the electrical and magnetic components of a light field, so the normalized light field of a photon pair of the same phase at position x and time t can be expressed as:

$$E = cos(\theta + kx - \omega t)$$

where θ is the initial phase of the photon pair at the source, k is wave vector, ω is angular frequency.

Assume that photon A will be added a phase θ_a by a phase shifter and, meanwhile, photon B will be added a phase $\theta_b = \omega\Delta t$ due to the different time or distance travelled (we use only one phase shifter for simplicity). The light fields of the pair become:

$$E_A = cos(\theta + kx - \omega t + \theta_a)$$

$$E_B = cos(\theta + kx - \omega t + \theta_b)$$

Although this type of experiments use the joint intensity as measurement, as we discussed in Section 3, we cannot calculate the correlation of light intensity by directly multiplying the intensities of light field because the changes in intensity are not independent. Since the light phases and thus the light fields are correlated, the joint intensity needs to be calculated from light field correlation:

$$\begin{aligned} E_{AB} &= E_A E_B = cos(\theta + kx - \omega t + \theta_a)cos(\theta + kx - \omega t + \theta_b) \\ &= 0.5[cos(2\theta + 2kx - 2\omega t + \theta_a + \theta_b) + cos(\theta_a - \theta_b)] \end{aligned}$$

The initial phase of photon pair θ can change randomly, so the item related to θ in above equation will net out to zero (by integrating E_{AB} over θ in the range of $0 - 2\pi$). As a result, the above equation becomes:

$$E_{AB} = 0.5cos(\theta_a - \theta_b)$$

As such, the joint intensity can be calculated as:

$$I_{AB} = E_{AB}^2 = 0.25cos^2(\theta_a - \theta_b) = 0.125[cos2(\theta_a - \theta_b) + 1]$$

This result is equivalent to the quantum prediction of eq. 16 in Franson [9] or the eq. 4 used by Brendel et al [19]. From the above derivation we can conclude that the light intensity difference stems from the phase difference caused by phase shifter and by different travel time. Probability law also works in this case because it ensures that the initial random phase of photon pairs have no impact on the interferometry results.

By examining representative experiments, we can conclude that the violation of Bell inequality is caused by the correlation in source particles as well as the physical relationship between the spin/polarization angle and its component on detection axes, or between the phase of electromagnetic wave and the interference outcome. With varying detection conditions (i.e. random changes in detection angles or adding arbitrary phases), probability law can still maintain the correlation of source particles. This leads to the violation of the Bell inequality and the correct statistical predictions, which are consistent with experimental outcomes.

6 Conclusions

The paper presents statistical predictions of two-particle Bell tests, which are equivalent to, but more general than, the QM predictions. By comparing the statistical and QM approaches, the paper shows that probability law is at the heart of both approaches. The statistical presentation of two-particle Bell tests in this paper has far-reaching implications.

First, it can improve our understanding of quantum mechanics and help to demystify it. Although the concepts of superposition and entanglement are widely accepted among physicists, the explanation of these concepts is difficult and thus causes significant misunderstanding. The statistical interpretation of the Bell tests shows that the superposition of entangled states in the two-particle Bell test is nothing more than statistical correlation between states. For the correlated particles at the polarizer or spin detector, probability law maintains the correlation through the expected value, so there is no need for communication (let alone instantaneous or faster-than-light communication) between different locations in the Bell experiments. As quantum entanglement is fully explained by probability law, the Bell test results and quantum mechanics are no longer mysterious.

Second, it has significant implications for quantum computing, which relies on quantum entanglement. Since the quantum entanglement phenomenon results from probability law, statistical noise is a natural and unavoidable part of quantum computing. Understanding the nature of this noise may shed light on how to improve the signal-noise ratio and thus is crucial to the success of quantum computing.

Third, the paper pinpoints the cause for the violation of the Bell inequality and thus explains why the local hidden variable theory is wrong. Although numerous Bell tests reject the local hidden variable theory and support quantum mechanics, they have not shed any light on why the former is wrong and the latter is right. This paper shows that the key lies in probability law, which underpins the Bell test results. Because probability law is universal, if we regard statistical mechanism (which cause statistical variation around the mean) as a 'hidden variable', it is not a local one but a global one. The local hidden variable theory misrepresents this nature and thus fails. It is also this global law that leads to the correct prediction from quantum mechanics.

Last but not the least, the paper may stimulate a reassessment of the role of determinism and realism. Generally speaking, the experimental results on the Bell inequality are interpreted as being a rejection of determinism or local realism, and an embracing of random-ism. While this paper highlights the importance of randomness and probability law, it does not totally reject determinism and realism. In the Bell tests, probability law works only when the particles arrive at and interact with the detector (polarizer or spin analyzer) - it plays no role before and after. When probability law is not in action, it is determinism, realism and logic that describe the behaviour of the particles. In other words, both random-ism and realism play an important role in our understanding of physics.

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