Discrete Noether’s Theorem

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June 12, 2021

Abstract
Conservation law for discrete symmetry

I try to write a generalization of Noether’s theorem to the discrete case. If a Lagrangian is invariant for a discrete symmetry S then there is a conservation law for each point (accessible) in the space:

\[
\mathcal{L}[R(q)] = \mathcal{L}(q)
\]

0 = \frac{d\mathcal{L}[S(q)]}{dt} - \frac{d\mathcal{L}(q)}{dt} = \frac{d\mathcal{L}(Q)}{dt} - \frac{d\mathcal{L}(q)}{dt} = \]

\[
= \frac{\partial\mathcal{L}(Q)}{\partial Q} \frac{\partial Q}{\partial q} + \frac{\partial\mathcal{L}(Q)}{\partial \dot{Q}} \frac{\partial \dot{Q}}{\partial q} - \frac{\partial\mathcal{L}(q)}{\partial q} - \frac{\partial\mathcal{L}(q)}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q} =
\]

\[
= \frac{d}{dt} \left[ \frac{\partial\mathcal{L}(Q)}{\partial Q} \frac{\partial Q}{\partial q} - \frac{\partial\mathcal{L}(q)}{\partial q} \right]
\]

then along the trajectory (where the Eulero-Lagrange equation \( \frac{d}{dt} \frac{\partial\mathcal{L}(q)}{\partial \dot{q}} = \frac{\partial\mathcal{L}(q)}{\partial q} \) is true) there is a conservation law.

This conservation law is true for each point of the trajectory, and for each discrete symmetry \( Q(t) = S[q(t)] \).

The symmetry can be a linear transformation \( Q_n = \pm Aq \mp nB \) that are reflection, discrete spatial translation, discrete spatial rotation, etc.

I write some example of discrete Noether’s theorem:

- **discrete translation** \( Q_n = S^n(q) = q + n\Delta \): the derivative is \( \partial_q Q = \partial_q S(q) = 1 \), then the generalized momentum is constant \( p(Q_n) = p(q + n\Delta) \) and \( q \in [0, \Delta] \): the generalized momentum is an arbitrary function in an interval \([0, \Delta]\), and it is replicated indefinitely in \([n\Delta, (n+1)\Delta]\). If \( \Delta \to 0 \) then there is an invariant momentum in each point of the space, the usual Noether’s theorem.

- **discrete rotation** \( \theta_n = S^n(\theta) = \theta + n2\pi/M \): the derivative is \( \partial_\theta S(\theta) = 1 \), the angular momentum is an arbitrary function in an interval \([0, 2\pi/M]\) and \( p(\Theta_n) = p(\theta + n2\pi/M) \)

- **reflection** \( Q_n = (-1)^n q \): the derivative is \( \partial_q S(q) = -1 \), then the generalized momentum reverses with every reflection, but this happen only if the reflection point is on the trajectory (to satisfy the Eulero-Lagrange equation). It seem that there is an invariant in quantum mechanics (for example parity), then there is an invariant in classical mechanics.

For example the classical Hamiltonian \( E = \frac{m}{2} \left[ \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right] + V(r) \) where \( \mathcal{L}(r) = \mathcal{L}(-r) \) has the invariant \( m\dot{r}(\vec{r}) = \pm m\dot{r}(\vec{r}) \), that is the invariant obtained from the infinitesimal transformation \( R = r - \epsilon r \) transformed in the discrete with \( \epsilon = 2 \). The discrete Noether’s theorem give the correct sign.