Abstract
In this manuscript, we define a conformal map from the unit disc onto the semi plane. Later, we define a function $f(z) = (s-1)\zeta(s)$. We prove that $f(z)$ belongs to the Hardy space, $H^{1/2}(\mathbb{D})$. We apply Jensen’s formula noting that the measure associated with the singular interior factor of $f$ is zero. Finally, we get

\[ \int_{-\infty}^{\infty} \log \left|\frac{\zeta(\frac{1}{2} + it)}{\frac{1}{4} + t^2}\right| dt = 0 \]

Keywords: Hardy spaces, Jensen’s formula, Schwarz reflection principle, Critical strip, Critical line, Riemann zeta function, Riemann Hypothesis.
Mathematics Subject Classification: 11M26, 11M06

1 Introduction
The Riemann zeta function, $\zeta(s)$ is defined as the analytic continuation of the Dirichlet series

\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \]

which converges in the half plane $\Re(s) > 1$. The Riemann zeta function is a meromorphic function on the whole complex $s$-plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. All the non trivial zeros of the Riemann zeta function lie in the critical strip $0 < \Re(s) < 1$. Riemann Hypothesis states that all the non trivial zeros of the Riemann zeta function lies on the critical line $\Re(s) = \frac{1}{2}$.

Levinson [6], in 1974 proved that more than one third of zeros of Riemann zeta function are on the critical line. Balazard et al.[1] in 1999 proved an equivalent of the Riemann Hypothesis. Shaoji Feng [7], in 2012 proved that at least 41.28 % of the zeros of Riemann zeta function are on the critical line. Pratt et al.[8] in 2020 proved that more than five-twelfths of the zeros are on the critical line.

2 Main Result
Let, $\sum_{\Re(\rho) > \frac{1}{2}}$ be the sum over the hypothetical zeros with real part greater than $\frac{1}{2}$ of the Riemann zeta function, $\zeta(s)$. In the sum, the zeros of multiplicity $n$ are counted $n$ times. Balazard et al.[1] proved that

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \left|\zeta\left(\frac{1}{2} + it\right)\right| \frac{dt}{\frac{1}{4} + t^2} = \sum_{\Re(\rho) > \frac{1}{2}} \log \left|\frac{\rho}{1 - \rho}\right| \]

and the Riemann Hypothesis is true if and only if [1],

\[ \int_{-\infty}^{\infty} \log \left|\zeta\left(\frac{1}{2} + it\right)\right| \frac{dt}{\frac{1}{4} + t^2} = 0 \]
The goal of this paper is to prove the following result.

**Theorem 1:** If \( \zeta(s) \) denotes the Riemann zeta function then

\[
\int_{-\infty}^{\infty} \log \left| \zeta \left( \frac{1}{2} + it \right) \right| \frac{dt}{t^2} = 0
\]

We start the proof of Theorem 1 as follows: Let, \( f \) be a function in the Hardy Space \( H^p(D) \) where \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \) and \( 0 < p < \infty \). Denote by \( f^* \) the function defined almost everywhere on the unit circle \( \partial D = \{ z \in \mathbb{C} \mid |z| = 1 \} \) by,

\[
f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})
\]

Let, \( z \in D \) where \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \). For \( i = \sqrt{-1} \), write

\[
s = s(z) = \frac{1}{2} + \frac{i - z}{2(i + z)} = \frac{i}{i + z}
\]

The formula \( s(z) \) defines an injective, onto and conformal representation of unit disc \( D \) in the semi plane \( \Re(s) > \frac{1}{2} \).

By Jensen’s Formula ([2, Theorem 3.61]) for \( f(0) \neq 0 \) and \( r < 1 \),

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})|d\theta = \log |f(0)| + \sum_{|\alpha| < r, f(\alpha) = 0} \log \frac{r}{|\alpha|}
\]  \hspace{1cm} (3)

where in the sum, \( \sum_{|\alpha| < r, f(\alpha) = 0} \) zeros of multiplicity \( n \) are counted \( n \) times.

Denote the singular interior factor of \( f \) by,

\[
\exp \left\{ -\int_{-\pi}^{\pi} e^{i\theta} + z \over e^{i\theta} - z \; d\mu(\theta) \right\}
\]

As \( r \to 1, r < 1 \), equation (3) becomes ([1] or [3, p. 68]),

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f^*(e^{i\theta})|d\theta = \log |f(0)| + \sum_{|\alpha| < 1, f(\alpha) = 0} \log \frac{1}{|\alpha|} + \int_{-\pi}^{\pi} d\mu(\theta)
\]  \hspace{1cm} (4)

Now we consider the function,

\[
f(z) = (s - 1)\zeta(s)
\]

where \( s = \frac{i}{i + z} \) then,

\[
f(z) = \frac{z}{i + z} \zeta \left( \frac{i}{i + z} \right)
\]

**Lemma 1.1:** \( f \) belongs to the Hardy space, \( H^\frac{2}{3}(D) \) that is \( f \in H^\frac{2}{3}(D) \).

**Proof.** \( \zeta(s) \) has the following property [9, p.95],

\[
|\zeta(s)| = \mathcal{O}(|s|), \quad |s| \to \infty, \quad \Re(s) \geq \frac{1}{2}
\]

If, \( |z| < 1 \) then \( \Re \left( \frac{i}{i + z} \right) > \frac{1}{2} \) so we have,

\[
|f(z)| = \left| \frac{z}{i + z} \zeta \left( \frac{i}{i + z} \right) \right| \leq \frac{c}{|i + z|^2}
\]

for some positive constant \( c \).

\[
|f(re^{i\theta})| \leq \frac{c}{|re^{-i\theta} + r|^2} \leq \frac{c}{\cos^2(\theta)}
\]
Lemma 1.2: Measure $\mu$ associated to the singular interior factor of $f$ is zero.

Proof. To prove that the measure $\mu$ associated to the singular interior factor of $f$ is zero, we adopt the method used by Bercovivi and Foias [10, Proposition 2.1]

Some theorems in Hardy space theory are ([4] and [5]),

Theorem (a): If $f \in H^p(\mathbb{D})$ where $p > 0$, then $f$ has non tangential finite limit on the unit circle almost everywhere denoted by $f^*(e^{i\theta})$, and $\log |f(e^{i\theta})|$ is integrable unless $f(z) \equiv 0$. Also $f(e^{i\theta}) \in L^p$ [4, p.17, Theorem 2.2]

Theorem (b): Every function $f(z) \not\equiv 0$ in $H^p(\mathbb{D})$ ($p > 0$) has a unique factorisation of the form $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function which is determined by a positive singular measure $\mu$ and $F(z)$ is an outer function such that $F \in H^p(\mathbb{D})$ [4, p.24, Theorem 2.8]. Also, $|B(z)| < 1$ in $|z| < 1$ [4, p.19, Theorem 2.4].

Theorem (c): Let $f \in H^p(\mathbb{D})$, $p > 0$, and let $\Gamma$ be an open arc on $\partial \mathbb{D}$. If $f(z)$ is analytic across $\Gamma$, then its inner factor and its outer factor are analytic across $\Gamma$. If $f(z)$ is continuous across $\Gamma$, then its outer factor is continuous across $\Gamma$ [5, p.74, Theorem 6.3]

Theorem (d): If measure $\mu \not\equiv 0$, then there is a point $e^{i\theta}$ for which

$$\lim_{z \to e^{i\theta}} S(z) = 0$$

non tangentially [5, p.73, Theorem 6.2]

Moreover if

$$\lim_{h \to 0} \frac{\mu((\theta - h, \theta + h))}{h \log 1/h} = \infty,$$

then for every $n = 1, 2, \ldots$ [5, p.74, (6.4)]

$$\lim_{z \to e^{i\theta}} \frac{|S(z)|}{(1 - |z|^2)^n} = 0$$

Now, $f(z) = (s - 1)\zeta(s)$ where $s = \frac{1}{1 + z}$

We have proved earlier that $f \in H^2(\mathbb{D})$, so by Theorem (b), $f(z)$ has a decomposition

$$f(z) = B(z)S(z)F(z)$$

Define a set $M = \{z \in \mathbb{C} \mid |z| = 1, z \neq -i\}$

We know that $(s - 1)\zeta(s)$ is analytic across the line $\Re(s) = \frac{1}{2}$. Since $f(z)$ is analytic across $M$, so by Theorem (c) its inner factor and outer factor are analytic across $M$. So, $f(z)$ is analytic across $M$.

By Theorem (d), if $\mu \not\equiv 0$ then

$$\lim_{z \to e^{i\theta}} \frac{|S(z)|}{(1 - |z|^2)^n} = 0$$

Lemma 1.2(a): If $\mu \not\equiv 0$ then

$$\lim_{r \to 1, r < 1} f(-ir) = 0$$

Proof.

$$f(-ir) = B(-ir)S(-ir)F(-ir)$$

$$|f(-ir)| = \left| \frac{(1 - r^2)^3 F(-ir) S(-ir)}{(1 - r^2)^3} \right| \leq \frac{c^\frac{1}{4}}{(\cos^2(\theta))^\frac{3}{4}} \int_{-\pi}^{\pi} |f(re^{i\theta})|^\frac{1}{4} d\theta \leq c^\frac{1}{4} \int_{-\pi}^{\pi} \frac{d\theta}{(\cos^2(\theta))^\frac{3}{4}} = 2c^\frac{1}{4} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) < \infty$$

where $\Gamma$ denotes the Gamma function. Hence, $f \in H^2(\mathbb{D})$.

Now using the above lemma we proceed to prove another lemma.
By Theorem (b) above, \(|B(-ir)| < 1\) for \(r < 1\)

By Theorem (d) above, \(\lim_{r \to 1, r < 1} \frac{|S(-ir)|}{(1-r^2)^{3/2}} = 0\)

Since by Theorem (b), \(F \in H^1_3(D)\) so we get [4, p.36, lemma],

\[|(1 - r)^3 F(z)| \leq 8\|F\|_p^3\]

where \(\|F\|_p = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F(r e^{i\theta})|^p d\theta \right\}^{1/p}\) giving the following inequality,

\[|(1 - r^2)^3 F(-ir)| \leq 8|(1 - r)^3 F(-ir)| \leq 64\|F\|_p^3\]

Using these bounds in \(|f(-ir)| = \left|(1 - r^2)^3 F(-ir) \frac{S(-ir)}{(1-r^2)^{3/2}} B(-ir)\right|\) since the middle term goes to zero by Theorem (d) and the remaining two terms are bounded, so we get

\[\lim_{r \to 1, r < 1} |f(-ir)| = 0\]

Since, \(|.|\) is continuous function, we get , \(\lim_{r \to 1, r < 1} f(-ir) = 0\) so,

\[\lim_{r \to 1, r < 1} f(-ir) = 0\]

In this case ,

\[\lim_{r \to 1, r < 1} \frac{r}{1-r} \zeta \left( \frac{1}{1-r} \right) = 0\]

Let, \(x = \frac{1}{1-r}\)

\[\lim_{x \to \infty, x > 0} (x-1)\zeta(x) = 0\]

which is a contradiction as the above limit is \(\infty\).

Hence our assumption that \(\mu \neq 0\) is wrong. So, we must have

\[\mu \equiv 0\]  \hspace{1cm} (5)

We are ready for another lemma useful in applying Jensen’s formula later.

**Lemma 1.3:**

\[\int_{-\pi}^{\pi} \log |f^*(e^{i\theta})| d\theta = 0\]

**Proof.** Let,

\[I = \int_{-\pi}^{\pi} \log |f^*(e^{i\theta})| d\theta\]

We have, \(f(z) = (s-1)\zeta(s)\) where \(s = \frac{i}{1+i}\)

\[I = \int_{-\pi}^{\pi} \log \left| \frac{e^{i\theta}}{i + e^{i\theta}} \zeta \left( \frac{i}{i + e^{i\theta}} \right) \right| d\theta\]

Write,

\[I = K + L\]

where

\[K = \int_{-\pi}^{\pi} \log \left| \frac{e^{i\theta}}{i + e^{i\theta}} \right| d\theta\]

and

\[L = \int_{-\pi}^{\pi} \log \left| \zeta \left( \frac{i}{i + e^{i\theta}} \right) \right| d\theta\]

**Lemma 1.3(b):**

\[K = 0\]
Proof.

\[ K = -\int_{-\pi}^{\pi} \log |i + e^{i\theta}| \, d\theta \]

By Jensen’s formula, since \( m(z) = i + z \) is analytic in \( |z| \leq 1 \) so we have \( K = 0 \)

**Lemma 1.3(c):**

\[ \int_{-\pi}^{\pi} \log \left| (\frac{i}{i + e^{i\theta}}) \right| \, d\theta = 0 \]

Proof.

\[ \frac{i}{i + e^{i\theta}} = \frac{1}{2} + \frac{i}{2} \tan \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \]

\[ L = \int_{-\pi}^{\pi} \log \left| \left( \frac{1}{2} + \frac{i}{2} \tan \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \right) \right| \, d\theta \]

Substitute \( \phi = \frac{\pi}{4} - \frac{\theta}{2} \) then,

\[ L = 2 \int_{-\pi/4}^{3\pi/4} \log \left| \left( \frac{1}{2} + \frac{i}{2} \tan \phi \right) \right| \, d\phi \]

Define

\[ L_1 = 2 \int_{-\pi/4}^{\pi/2} \log \left| \left( \frac{1}{2} + \frac{i}{2} \tan \phi \right) \right| \, d\phi \]

and

\[ L_2 = 2 \int_{\pi/2}^{3\pi/4} \log \left| \left( \frac{1}{2} + \frac{i}{2} \tan \phi \right) \right| \, d\phi \]

In \( L_1 \), substitute \( t = \frac{\tan \phi}{2} \) which is a valid substitution as \( t = \frac{\tan \phi}{2} \) is injective on \((-\pi/4, \pi/2)\)

\[ L_1 = \int_{-1}^{\infty} \log \left| \left( \frac{1}{4} + it \right) \right| \, dt \]

In \( L_2 \), substitute \( p = \frac{\tan \phi}{2} \) which is a valid substitution as \( p = \frac{\tan \phi}{2} \) is injective on \((\pi/2, 3\pi/4)\)

\[ L_2 = \int_{-1}^{\infty} \log \left| \left( \frac{1}{4} + ip \right) \right| \, dp \]

Hence,

\[ L = L_1 + L_2 = 0 \]

\[ \Rightarrow \int_{-\pi}^{\pi} \log |f^*(e^{i\theta})| \, d\theta = 0 \quad (6) \]

Next, we proceed to another lemma.

**Lemma 1.4:** \( f(z) = -\frac{z}{1+z^2} \zeta \left( \frac{i}{1+z^2} \right) \) is analytic in \( |z| \leq r, \ r < 1 \) and \( \log |f(0)| = 0 \)
Proof. Let, \( w(z) = \frac{1}{1+z} \) and define

\[
h(z) = (z-1)\zeta(z)
\]

\( h(z) \) is entire function and \( w(z) \) is analytic in \( |z| \leq r, \; r < 1 \) so the composition \( h(w(z)) = f(z) \) is analytic in \( |z| \leq r, \; r < 1 \). Hence, \( f \) is continuous at zero.

\[
f(0) = \lim_{z \to 0} f(z)
\]

\[
f(0) = \lim_{z \to 0} \frac{-z}{1+z} \zeta\left(\frac{i}{1+z}\right)
\]

Let, \( \eta = \frac{1}{1+z} \) then

\[
f(0) = \lim_{\eta \to 1} (\eta - 1)\zeta(\eta) = 1
\]

\[
\log |f(0)| = 0
\]

(7)

(8)

Now, we proceed to next lemma.

Since, \( f(0) \neq 0 \) and \( f(z) = -\frac{z}{1+z} \zeta\left(\frac{i}{1+z}\right) \) so \( f(\alpha) = 0 \) corresponds to \( \zeta\left(\frac{i}{1+i}\right) = 0 \).

Let, \( \rho \) denote non trivial zeros of Riemann zeta function then,

\[
\rho = \frac{i}{1+\alpha}
\]

**Lemma 1.8:**

\[
\sum_{|\alpha|<1, f(\alpha) = 0} \log \frac{1}{|\alpha|} = \sum_{\Re(\rho) > \frac{1}{2}, \zeta(\rho) = 0} \log \left| \frac{\rho}{1 - \rho} \right|
\]

Proof. \( \rho = \frac{i}{1+\alpha} \) gives \( \alpha = i\left(1 - \rho \right) \rho \) so \( |\alpha| < 1 \) corresponds to \( \Re(\rho) > \frac{1}{2} \) and \( f(\alpha) = 0 \) corresponds to \( \zeta(\rho) = 0 \). Hence we get

\[
\sum_{|\alpha|<1, f(\alpha) = 0} \log \frac{1}{|\alpha|} = \sum_{\Re(\rho) > \frac{1}{2}, \zeta(\rho) = 0} \log \left| \frac{\rho}{1 - \rho} \right|
\]

(9)

Using equation (5),(6),(8) and (9) in equation (4), we get,

\[
\sum_{\Re(\rho) > \frac{1}{2}, \zeta(\rho) = 0} \log \left| \frac{\rho}{1 - \rho} \right| = 0
\]

(10)

Using equation (1) and (10) gives,

\[
\int_{-\infty}^{\infty} \log |\zeta(\frac{i}{1+it})| \frac{1}{\frac{i}{1+it}} \, dt = 0
\]

This proves equation (2) and completes the proof of Theorem 1. Hence the Riemann Hypothesis is true.

## 3 Acknowledgements

We are thankful to all the Professors for helpful and useful conversations.
4 References


6. Norman Levinson, More than one third of zeros of Riemann zeta function are on $\sigma = 1/2$, Advances in Mathematics (1974).


