A straightforward and Lagrangian proof of the Einsteinian equivalence between the mass and the inernal energy (i.e. rest energy) V2 : Additional analysis

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Abstract

This article contains additional analysis to my article viXra: 2006.0022 “A straightforward and Lagrangian proof of the Einsteinian equivalence between the mass and the internal energy V2."

Sommaire

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1. A small reminder of the conclusion of "A straightforward and Lagrangian proof of the Einsteinian equivalence between the mass and the inernal energy (i.e. rest energy) V2"

We have a way to demonstrate the famous Einstein formula $E^*=Mc^2$ directly from an appropriate Lagrangian function selecting the correct variables.

Instead of $L\left(\{\boldsymbol{r_a}\},\left\{\frac{dr_a}{dt}\right\}\right)$, we use $L'\left(\{\boldsymbol{r_a^*}\},\left\{\frac{dr_a^*}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)=\frac{L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{dr_a^*}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$.

Instead of $L\left[\{\varphi\},\left\{\frac{\partial\varphi}{\partial r}\right\},\left\{\frac{\partial\varphi}{\partial t}\right\}\right]$, we use $L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial r^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right]\equiv\frac{\iiint\Lambda^*\left(\varphi^*,\frac{\partial\varphi^*}{\partial r^*},\gamma(V_\mathrm{C})\frac{\partial\varphi^*}{\partial t}\right)dV^*}{\gamma(V_\mathrm{C})}$.

In the two cases we've calculated directly that $\boldsymbol{P_c}\equiv\frac{\partial L'}{\partial v_\mathrm{C}}=\gamma\frac{E^*}{c^2}\boldsymbol{V}_\mathrm{C}$

In this article, we also showed:

- The strong link with this law and the time dilation formula that highlight the crucial role of the Einstein's requirement of non-universality of time;
- A discussion on the meaning of the new set of variables chosen with an amusing modified velocity addition formula that does not contradict the of Einstein-Poincaré one;
- A discussion of the origin of the energy scale and the link to mass as stated by Landau-Lifchitz;
- Why in Newtonian mechanic Einstein's law is hidden;
- I also add some elements for a Hamiltonian analysis and a discussion about the model of electron that allows the formalism to be applied to a concrete example.

Erratum from the previous article: some small corrections (Lagrangian correction) have been made but one of them important (in the modified velocity addition formula) as I show that even with the new set of variable used, the speed of light (Einstein constant) is again constant.

2. Annex

2.1. Elements of Hamiltonian analysis for a material system free

- **Hamiltonian map**

The 4-momentum is $P^i(K^*) = (Mc, \boldsymbol{P}) = \left(\gamma \frac{E^*}{c}, \gamma \frac{E^*}{c^2} \boldsymbol{V}_\mathrm{C}\right)$

Then

$$\left\|P^i(K^*)\right\|^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2 = \left(\gamma \frac{E^*}{c}\right)^2 - \left(\gamma \frac{E^*}{c^2} \boldsymbol{V}_\mathrm{C}\right)^2 = \left(\gamma \frac{E^*}{c}\right)^2 \left(1 - \left(\frac{\boldsymbol{V}_\mathrm{C}}{1}\right)^2\right) = \left(\frac{E^*}{c}\right)^2$$

$$=> \left(\frac{E^*}{c}\right)^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2$$

$$<=> E^2 = {E^*}^2 + c^2 \boldsymbol{P}^2$$

$$<=> \boxed{E = \sqrt{{E^*}^2 + c^2 \boldsymbol{P}^2}}$$

Having also $E^* = \sum_a E_a^* = \sum_a \sqrt{(m_a c^2)^2 + c^2 {\boldsymbol{P_a}^*}^2}$

$$=> \boxed{E = \sqrt{\left(\sum_a \sqrt{(m_a c^2)^2 + c^2 {\boldsymbol{P_a}^*}^2}\right)^2 + c^2 \boldsymbol{P}^2}}$$

Thus the Hamiltonian map $H: (\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{P}) \rightarrow H(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{P}) \equiv E$ is

$$\boxed{\begin{gathered} H(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{P}) = \sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\})^2 + c^2 \boldsymbol{P}^2} \\ with \\ H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}) = \sum_a \sqrt{(m_a c^2)^2 + c^2 {\boldsymbol{P_a}^*}^2} \end{gathered}}$$

With $H^*: (\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{P}) \rightarrow H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{P}) \equiv E^*$

I give below with evident notation 3 kinds of approximation:

- $H_{\{a\}Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{P}) = \sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\})^2 + c^2 \boldsymbol{P}^2}$

with

$$H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}) = \sum_a \sqrt{(m_a c^2)^2 + c^2 {\boldsymbol{P_a}^*}^2} = \sum_a (m_a c^2) \sqrt{1 + \frac{c^2 {\boldsymbol{P_a}^*}^2}{(m_a c^2)^2}}$$

$$\approx \sum_a m_a c^2 \left(1 + \frac{1}{2} \frac{c^2 {\boldsymbol{P_a}^*}^2}{(m_a c^2)^2}\right) = M_\Sigma c^2 + \sum_a \frac{{\boldsymbol{P_a}^*}^2}{2 m_a}$$

$$=> H_{\{a\}Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\{\boldsymbol{P_a}^*\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) = \sqrt{\left(M_\Sigma c^2 + \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)^2 + c^2\boldsymbol{P}^2}$$

$$= \sqrt{(M_\Sigma c^2)^2\left(1 + \frac{1}{M_\Sigma c^2}\sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)^2 + c^2\boldsymbol{P}^2}$$

$$\approx \sqrt{(M_\Sigma c^2)^2\left(1 + \frac{2}{M_\Sigma c^2}\sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\right) + c^2\boldsymbol{P}^2}$$

$$= \sqrt{(M_\Sigma c^2)^2 + c^2\sum_a \left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2} + c^2\boldsymbol{P}^2} = \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}\sqrt{1 + \frac{c^2\sum_a \left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2}}{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}}$$

$$\approx \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}\left(1 + \frac{1}{2}\frac{c^2\sum_a \left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2}}{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}\right) = \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{(M_\Sigma c^2)^2}}}\sum_a \left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$$

$$=> H_{\{a\}Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\{\boldsymbol{P_a}^*\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{(M_\Sigma c^2)^2}}}\sum_a \left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$$

$$with\ M_\Sigma \equiv \sum_a m_a$$

We have also

$$\frac{1}{\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{(M_\Sigma c^2)^2}}} = \frac{1}{\sqrt{1 + \frac{c^2\gamma^2 M^2 \boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2}}} = \frac{1}{\sqrt{\frac{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2 + \frac{c^2 M^2 \boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2}}{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2 + \frac{c^2 M^2 \boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2 + \frac{c^2\left(M_\Sigma + \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a c^2}\right)^2 \boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2}}}$$

$$= \sqrt{\frac{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2 + \frac{c^2 M_\Sigma^2\left(1 + \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a c^2 M_\Sigma}\right)^2 \boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2 + \frac{c^2 M_\Sigma^2 \boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2} + \frac{c^2 M_\Sigma^2 \sum_a \frac{\boldsymbol{P_a}^{*2}}{m_a M_\Sigma c^2}\boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}{1 + \frac{M_\Sigma \sum_a \frac{\boldsymbol{P_a}^{*2}}{m_a}\boldsymbol{V}_{\mathbf{C}}^2}{(M_\Sigma c^2)^2}}}$$

$$= \sqrt{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}\left(1 - \frac{1}{2}\frac{\sum_a \frac{\boldsymbol{P_a}^{*2}}{m_a}}{M_\Sigma c^2}\frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}\right) = \sqrt{1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}\left(1 - \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\frac{\frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}}{M_\Sigma c^2}\right) = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\left(1 - \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\frac{\left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}{M_\Sigma c^2}\right)$$

$$=> H_{\{a\}Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\{\boldsymbol{P_a}^*\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P})$$

$$= \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\left(1 - \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\frac{\left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2}{M_\Sigma c^2}\right)\sum_a \left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$$

$$\approx \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\sum_a \left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$$

$$=> \boxed{H_{\{a\}Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\sum_a \left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)}$$

This result is surprisingly for the second term $\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\sum_a \left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$ because the dilation of time factor $\gamma(\boldsymbol{V}_{\mathbf{C}})$ divides the internal ("kinematic") energy instead of multiplying it as in the relation E= γ E*.

- $H_{Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx (M_\Sigma c^2)\left[1 + \frac{1}{2}\frac{c^2}{(M_\Sigma c^2)^2}\left(\sum_a \left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2} + \boldsymbol{P}^2\right)\right]$

$$= M_\Sigma c^2 + \frac{1}{2}\frac{1}{M_\Sigma}\left(\sum_a \left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2} + \boldsymbol{P}^2\right) = M_\Sigma c^2 + \sum_a \left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right) + \frac{\boldsymbol{P}^2}{2M_\Sigma}$$

$$=> \boxed{H_{Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx M_\Sigma c^2 + \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a} + \frac{\boldsymbol{P}^2}{2M_\Sigma}}$$

- $H_{C,Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) = \sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\})^2 + c^2\boldsymbol{P}^2}$

$$= \sqrt{\left[\sum_a \sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}}\right]^2 + c^2\boldsymbol{P}^2}$$

$$= \left[\sum_a \sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}}\right]\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{\left[\sum_a \sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}}\right]^2}}$$

$$= \left[\sum_a \sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}}\right]\left(1 + \frac{1}{2}\frac{c^2\boldsymbol{P}^2}{\left[\sum_a \sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}}\right]^2}\right)$$

$$=> \boxed{H_{C,Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx \sum_a \sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}} + \frac{1}{2}\frac{c^2\boldsymbol{P}^2}{\sum_a \sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}}}}$$

- **Hamiltonian equations**

We can verify if the form of the Hamiltonian verifies the Hamilton equation:

$$\frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a'}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) = \frac{\partial}{\partial \boldsymbol{P}_{\boldsymbol{a'}}^{*}}\sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\})^2 + c^2\boldsymbol{P}^2} =$$

$$= H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\})\frac{\partial H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\})}{\partial \boldsymbol{P}_{\boldsymbol{a'}}^{*}}\frac{1}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P})}$$

With

$$\frac{\partial H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\})}{\partial \boldsymbol{P}_{\boldsymbol{a'}}^{*}} = \sum_a \frac{\partial}{\partial \boldsymbol{P}_{\boldsymbol{a'}}^{*}}\sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}} = \sum_a c^2\boldsymbol{P_a}^{*}\frac{\partial \boldsymbol{P_a}^{*}}{\partial \boldsymbol{P}_{\boldsymbol{a'}}^{*}}\frac{1}{H_a^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P_a}^{*}\})}$$

$$= \sum_a c^2\delta_{aa}\frac{\boldsymbol{P_a}^{*}}{\sqrt{(m_a c^2)^2 + c^2\boldsymbol{P_a}^{*2}}} = \frac{c^2\boldsymbol{P_{a'}}^{*}}{\sqrt{(m_{a'} c^2)^2 + c^2\boldsymbol{P_{a'}}^{*2}}} = \frac{c^2\boldsymbol{P_{a'}}^{*}}{H_{a'}^*(\boldsymbol{r}_{\boldsymbol{a'}}^{*},\boldsymbol{P_{a'}}^{*})}$$

Then

$$\frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = \frac{c^2\boldsymbol{P}_{\boldsymbol{a}}^{*}}{H_a^*(\boldsymbol{r}_{\boldsymbol{a}}^{*},\boldsymbol{P}_{\boldsymbol{a}}{}^{*})}\frac{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

$$= \frac{c^2\boldsymbol{P}_{\boldsymbol{a}}^{*}.H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}).H_a^*(\boldsymbol{r}_{\boldsymbol{a}}^{*},\boldsymbol{P}_{\boldsymbol{a}}{}^{*})}$$

$$=> \boxed{c^2\frac{\boldsymbol{P}_{\boldsymbol{a}}^{*}}{\gamma.E_a^*} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

But as for the center of mass we can write

$\boldsymbol{P}_{\boldsymbol{a}}^{*} = \frac{E_a^*}{c^2}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^*}$ with $E_a^* = \gamma_a^* E_a^{K_a^*}$

Where $E_a^{K_a^*}$ is the internal energy of the particle "a" in its own center of mass. This internal energy is equal to its mass only when the particle is free (as for the global center of mass).

$$\left(E_a^{K_a^*}\right)_{a\ is\ free} = m_a c^2$$

Then we can write in general

$$c^2\frac{\frac{E_a^*}{c^2}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^*}}{\gamma.E_a^*} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})$$

$$\frac{1}{\gamma}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^*} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})$$

$$=> \boxed{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

This is again coherent with a first Hamiltonian equation.

$$\frac{\partial H}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = \frac{\partial}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}}\sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})^2 + c^2\boldsymbol{P}^2} =$$

$$= \frac{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}\frac{\partial H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = -\frac{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^*}$$

Since $\frac{\partial H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = \frac{\partial\left[\sum_a \boldsymbol{P}_{\boldsymbol{a}}{}^{*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} - L^*\right]}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = -\frac{\partial L^*}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = -\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^*}$

$$\frac{\partial H}{\partial \boldsymbol{r}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = -\frac{E^*}{\gamma E^*}\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^*} = -\frac{1}{\gamma}\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^*} = -\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt}$$

$$=> \boxed{\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

This is again consistent with a second Hamiltonian equation.

We then see that for the particular variables chosen in the Lagrangian analysis $\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt},\boldsymbol{r}_{\boldsymbol{a}}^{*}\right)$ we find what we should expect for a Hamiltonian analysis with the variable $(\boldsymbol{P}_{\boldsymbol{a}}{}^{*},\boldsymbol{r}_{\boldsymbol{a}}^{*})$, that is to say the Hamiltonian equation.

For the center of mass we have obviously

$$\frac{\partial H}{\partial \boldsymbol{P}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = \frac{\partial}{\partial \boldsymbol{P}}\sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})^2 + c^2\boldsymbol{P}^2} = \frac{c^2\boldsymbol{P}}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

$$= \frac{c^2\frac{E}{c^2}\boldsymbol{V}_{\mathbf{C}}}{E} = \boldsymbol{V}_{\mathbf{C}}$$

$$=> \boxed{\boldsymbol{V}_{\mathrm{C}} = \frac{\partial H}{\partial \boldsymbol{P}}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P}_{\boldsymbol{a}}{}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P})}$$

The second equation, as already showed (cf. [2]):

$$\boxed{\frac{d\boldsymbol{P}}{dt} = \frac{\partial \sum_{\boldsymbol{a}} \boldsymbol{P}_{\boldsymbol{a}}{}^* \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \boldsymbol{P}\boldsymbol{V}_{\mathrm{C}} - H'}{\partial \boldsymbol{R}_{\mathrm{C}}} = -\frac{\partial H'}{\partial \boldsymbol{R}_{\mathrm{C}}}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P}_{\boldsymbol{a}}{}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P})}$$

We see that if we want to quantize any system in parallel with its center of mass, we should choose the quantum operator associated to the corresponding canonical couples of classical variables:

- $\{(\boldsymbol{r}_{\boldsymbol{a}}^*, \boldsymbol{P}_{\boldsymbol{a}}{}^*)\}, (\boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P})$ for a system of particles
- $\left\{\left(\varphi^*, \frac{\partial \varphi^*}{\partial \boldsymbol{r}^*}\right)\right\}, (\boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P})$ for a field (scalar for example)
- $etc.$.

2.2. Application: the electromagnetic model of the electron [3], [9], [23]-[25]

Just before and during the construction of the Special Relativity , some theoretical physicists used an electromagnetic model of the electron in order to untangle the ball of wool constituted by different to date of physical theories and experiments about the electrodynamics (and the optic) of moving bodies. The model of electron was used by notably Lorentz and improving by Poincaré (who shares with Einstein the privilege to have realized the last step of the discovery/invention of Special Relativity, both (very different) way of thinking have their own charm) which is interesting to use at least to treat classically the interaction between matter and electromagnetic field without divergence. The latter appears indeed for a material point as showed in [1]. One can always (in a classical universe with matter and electromagnetism living in a static Minkowskian space-time) physically replace a material point by a continuum if one always works for dimensions infinitely larger than the dimension of the continuum. The interest is to have a clear mathematic expression for the mass, even if the model is actually fundamentally wrong (but the ugly last point is here "sufficiently" hidden).

My interest in using this model is to see how a complex system behaves with the particular choice of variables and so thus to see the influence of the dynamics of the center of mass on the internal dynamics, in particular the mass behaviour itself. The model I decided to use is slightly different from the one used by Poincaré since I want to maintain the mass of the continuum without let all the mass to the electromagnetic energy field (as Poincaré & Lorentz & others have done).

I will present the first attempt of the electron model which is unstable and then the one used by Poincaré with his internal "pressure".

The electron model is:

- a continuum spherical surface in its rest frame K* characterized by a surface density of mass σ;
- the speed of all the material points of the continuum are radial (at an instant t)
- and the mass distribution is spherical K*(at an instant t).

We assume that, the internal spherical behaviour is maintained during motion, although according to [9], this model is in fact unstable.

The Lagrangian is

$$L\left(\{\boldsymbol{r_a}(t)\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\},t\right)=\sum_a\left[-m_a.c\frac{ds_a}{dt}-\frac{e_a}{c}.A^i(x_{ai})\frac{dx_{ai}}{dt}\right]$$

$$\Rightarrow L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*(t)\},\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right)=-\frac{\sum_a\left[\frac{m_a.c^2}{\gamma_{a^*}}+e_a.\varphi^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}+\sum_a\frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*).\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}$$

$$=-\frac{\sum_a\left[\frac{m_a.c^2}{\gamma_{a^*}}+e_a.\varphi^*(\boldsymbol{r}_{\boldsymbol{a}}^*,t^*)\right]}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\quad\text{via the isotropy hypothesis}$$

$$=-\frac{\iint\left[\frac{\sigma_m^*.c^2}{\gamma^*}+\sigma_e^*.\varphi^*(\boldsymbol{r}^*,t^*)\right]dS^*}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\quad\text{since we have a continuum}$$

$= -\frac{\left[\frac{\sigma_m^* . c^2}{\gamma^*} + \sigma_e^* . \varphi^*(\boldsymbol{r}^*, t^*)\right] S^*}{\gamma(\boldsymbol{V}_C)}$ via the isotropy of the speed in K*

$= -\frac{\frac{M_\Sigma . c^2}{\gamma^*} + e.\varphi^*(r^*, t^*)}{\gamma(\boldsymbol{V}_C)}$ since the additive mass $M_\Sigma = \sigma_m^* S^*$ and the charge e=$\sigma_e^* S^*$ are relativistic invariants and the distribution of material points is spherical.

We have $e.\varphi^*(r^*, t^*) = E_{em}^*(r^*) = E_{em,eq}^* \frac{r_{eq}^*}{r^*}$ where $E_{em}^*(r^*)$ is the electromagnetic energy and the quantities with index "eq" are the associated quantities for an eventual equilibrium point.

We have also $\gamma^* = \gamma^*\left(\frac{dr^*}{dt^*}\right) = \frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{dr^*}{dt^*}\right)^2}} = \gamma^*\left(\gamma(\boldsymbol{V}_C)\frac{dr^*}{dt}\right) = \frac{1}{\sqrt{1-\frac{\gamma(\boldsymbol{V}_C)^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}} = \gamma^*\left(\boldsymbol{V}_C, \frac{dr^*}{dt}\right)$

=> $\boxed{L'\left(\{\boldsymbol{r}_a^*(t)\}, \left\{\frac{d\boldsymbol{r}_a^*}{dt}\right\}, \boldsymbol{R}_C, \boldsymbol{V}_C, t\right) = L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_C, \boldsymbol{V}_C, t\right) = -\frac{1}{\gamma(\boldsymbol{V}_C)}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_C, \frac{dr^*}{dt}\right)} + E_{em,eq}^* \frac{r_{eq}^*}{r^*}\right)}$

Which gives

$$\boldsymbol{P}_c = \frac{\partial L'}{\partial \boldsymbol{V}_C} = \gamma(\boldsymbol{V}_C)\left(\frac{\sum_a [\gamma_a{}^* . m_a . c^2 + e_a . \varphi^*(\boldsymbol{r}_a^*, t^*)]}{c^2}\right)\boldsymbol{V}_C = \gamma(\boldsymbol{V}_C)\left(\gamma^* . M_\Sigma + \frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*}\right)\boldsymbol{V}_C$$

$$\Rightarrow \boldsymbol{P}_c = \gamma(\boldsymbol{V}_C) M \boldsymbol{V}_C$$

With $M = \gamma^* . M_\Sigma + \frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*}$

And $\gamma^* = \gamma^*\left(\boldsymbol{V}_C, \frac{dr^*}{dt}\right) = \frac{1}{1-\frac{\gamma(\boldsymbol{V}_C)^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}$

We see that the mass is (modulo c²) the sum of the total internal free energy $\gamma^* . M_\Sigma$ with the electromagnetic energy $\frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*}$ (a potential energy).

Moreover, the value of the mass depends of the "external" dynamics of the center of mass.

The relativistic dynamic is:

$$\frac{d}{dt}\left(\gamma(\boldsymbol{V}_C)\frac{E^*}{c^2}\boldsymbol{V}_C\right) = \frac{\partial}{\partial \boldsymbol{R}_C} L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_C, \boldsymbol{V}_C, t\right)$$

$$\frac{d}{dt}\left(\gamma_a^* m_a . \frac{d\boldsymbol{r}_a^*}{dt^*}\right) = \frac{1}{\gamma(\boldsymbol{V}_C)}\frac{\partial}{\partial \boldsymbol{r}_a^*} L^*\left(\{\boldsymbol{r}_a^*\}, \left\{\gamma(\boldsymbol{V}_C)\frac{d\boldsymbol{r}_a^*}{dt}\right\}\right)$$

=> $\frac{d}{dt}\left(\gamma^* M_\Sigma . \frac{dr^*}{dt^*}\right) = \frac{-1}{\gamma(\boldsymbol{V}_C)}\frac{\partial}{\partial r^*}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_C, \frac{dr^*}{dt}\right)} + E_{em,eq}^* \frac{r_{eq}^*}{r^*}\right) = \frac{1}{\gamma(\boldsymbol{V}_C)} E_{em,eq}^* \frac{r_{eq}^*}{r^{*2}}$

=> $\boxed{\frac{d}{dt}\left(\gamma(\boldsymbol{V}_C) . \gamma^* M_\Sigma . \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_C)} E_{em,eq}^* \frac{r_{eq}^*}{r^{*2}}}$

One can see that this model is internally radially unstable since there is only a repulsive term.

In order to improve he model we can add to it a (Poincaré-)truncated cosmological constant [9] which is null everywhere but not into the spherical electron.

The new Lagrangian is ([9] & [3])

$$L'\left(r^*,\frac{dr^*}{dt},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right)=-\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\left(\frac{M_{\Sigma}c^2}{\gamma^*\left(\boldsymbol{V}_{\mathrm{C}},\frac{dr^*}{dt}\right)}+E^*_{em,eq}\frac{r^*_{eq}}{r^*}-\frac{c^4}{8\pi k}\iiint \Lambda_P.\theta(R^*-r^*)\sqrt{-g^*}d^3\boldsymbol{R}^*\right)$$

With

$$\theta(R^*-r^*)\equiv 1\, for R^*\leq r^*$$

$$\equiv 0\, for R^* > r^*$$

But the space-time is Minkowskian and the electron is spherical in K*. Then

$$\boxed{L'\left(r^*,\frac{dr^*}{dt},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right)=-\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\left(\frac{M_{\Sigma}c^2}{\gamma^*\left(\boldsymbol{V}_{\mathrm{C}},\frac{dr^*}{dt}\right)}+E^*_{em,eq}\frac{r^*_{eq}}{r^*}-\frac{c^4}{8\pi k}\frac{4}{3}\pi\Lambda_P.r^{*3}\right)}$$

<u>A false problem</u>

We can remark that this Lagrangian naively suggests that the interaction terms acts instantaneously which would be inconsistent with Relativity. But actually the interaction term comes from fields that act just exactly at points where the material points are located, that is to say on the sphere and not at the center of the sphere.

<u>A digression towards some intriguing Uniform-Energy-Region</u>

The cosmological term in the Lagrangian is not the one used by Einstein since it is not applied to the whole space-time. This is very surprising for me since the general famous theorem (Lovelock) established that the cosmological term à la Einstein (in addition to the Ricci term) is the only one allowed in General Relativity in order to respect the general requirement of this theory: second order equation for dynamics and invariance of physical laws for any transformation of coordinates. A natural question is why the addition of the Poincaré term is authorized in Relativity ? In a more intuitive reasoning (which allows to reveal the solution): saying that a cosmological term applied only to a given fixed region seems to contradict the epistemological views of General Relativity [22] saying in particular that any effect of a phenomenon has to be caused by a direct measurable cause. This direct measurable cause has to be a physical phenomenon, governed by **dynamical** equations, which interact with other fields and matters (that is why reference frame must not be allowed to influence phenomena via inertial forces, the equivalence principle permitting precisely to make the latters dynamic by unifying them with the **dynamical** field of gravity). The solution to my problem is therefore that the boundary of the region, where the Poincaré pressure term is applied, is dynamically coupled with the distribution of the material system localized in the region. This has an interesting consequence: General Relativity allows a priori the existence of an arbitrary number n of deformed closed surfaces surrounding internal regions, of volume $\iiint \theta(\|\boldsymbol{R_n}\|-\|\boldsymbol{r_n}\|)\sqrt{-g}.d^3\boldsymbol{R_n}$, each containing a "Constant-cosmological" term Λ_n with an arbitrary value. This in the condition that all these borders are dynamical coupled with a border variable $\boldsymbol{r_n}$. Explicitly, General Relativity permits an action as

$$S[\{g_{ik}(x,t)\},\{r_n(t)\}] = \frac{-c^3}{16\pi k}\int\iiint(R-2\Lambda_E)\sqrt{-g}d\Omega+\sum_n\frac{c^4}{8\pi k}\int\iiint\Lambda_n.\theta(\|\boldsymbol{R_n}\|-\|\boldsymbol{r_n}\|)\sqrt{-g}.d^3\boldsymbol{R_n}\,dt+S[\{r_n(t),\dots\}]$$

Thus, Lovelock theorem applied, as it should, to a free gravitation field and the other "cosmological" terms are not affected by it since they necessitate the use of other dynamical variables.

Of course, although allowed, the other cosmological terms are not very "natural" because we have to add them arbitrary by hand. However they are not more "unnatural" than the complex topologies already often used and a priori allowed. If one accepts such new terms we must therefore complete the action with another part implying the dynamic of a 2D membrane for which every point behaves as a material point, each providing a "ds" term in the action. Hence, this membrane is sensitive to (as it should) the gravitation field (and a priori only to it) and is deformed by it. We can imagine a space time bathed by these Uniform-Energy-Regions. The problem of this kind of Uniform-Energy-Regions is the instabilities of their shape since they behave internally like a dynamical min-de-sitter (or anti-de-sitter) universe and not like a wiser Einstein-static one. Another problem that comes in mind is the possible appearance of gravitational singularities when 2 free point of the same (infinitely thin surface) surface (or even several surface) meet at the same point during their "free" movement (but this problem can be maybe cured by a quantum "bandage"). In spite of all these oddities, it is important to keep in mind (surely already known, perhaps by Dirac) all the mathematical possibilities permitted by the standard paradigm of physics, which is still today partly constituted by classical General Relativity. After a reading of the Jean Pierre Luminet's book [17], it seems that these speculations look a bit like the concept of gravastars which were conceptually invented in 2001 by Mazur & Mottola: Is "the Uniform-Energy-Regions" the same speculative concept as gravastars ? Is the gravastar the rebirth of the old Lorentz-Poincaré electron in an astrophysical domain? One of the differences would be that the Uniform-Energy-Regions are put by hand as one can put by hand a cosmological constant or the existence of some material points instead of being the result of a dynamical collapse of an existing massive star (with some exotic-innovative behaviour).

<u>Returning to our (simpler) initial problem</u>

$$L'\left(r^*,\frac{dr^*}{dt},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C},t\right)=-\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_\mathrm{C},\frac{dr^*}{dt}\right)}+E^*_{em,eq}\frac{r^*_{eq}}{r^*}-\frac{c^4}{2k}\frac{1}{3}\Lambda_P.r^{*3}\right)$$

Which gives

$$\boldsymbol{P_c}=\frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}}=\gamma(\boldsymbol{V}_\mathrm{C})\left(\frac{\sum_a[\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r}^*_{\boldsymbol{a}},t^*)]}{c^2}\right)\boldsymbol{V}_\mathrm{C}$$
$$=\gamma(\boldsymbol{V}_\mathrm{C})\left(\gamma^*.M_\Sigma+\frac{E^*_{em,eq}}{c^2}\frac{r^*_{eq}}{r^*}-\frac{c^2}{2k}\frac{1}{3}\Lambda_P.r^{*3}\right)\boldsymbol{V}_\mathrm{C}$$

$$=>\boldsymbol{P_c} = \gamma(\boldsymbol{V}_\mathrm{C}) M \boldsymbol{V}_\mathrm{C}$$

With $M = \gamma^*. M_\Sigma + \frac{E^*_{em,eq}}{c^2}\frac{r^*_{eq}}{r^*} - \frac{c^2}{6k}\Lambda_P . {r^*}^3$

And $\gamma^* = \gamma^*\left(\boldsymbol{V}_\mathrm{C}, \frac{dr^*}{dt}\right) = \frac{1}{\sqrt{1-\frac{\gamma(\boldsymbol{V}_\mathrm{C})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}}$

We see that the mass is (modulo c²) the sum of the total internal free energy with the electromagnetic energy (which behaves as a potential energy) and with the pressure-Poincaré energy.

Moreover, the value of the mass depends of the "external" dynamics of the center of mass.

The relativistic dynamic for the internal part is now:

$$\frac{d}{dt}\left(\gamma^*_a m_a . \frac{d\boldsymbol{r}^*_a}{dt^*}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{\partial}{\partial \boldsymbol{r}^*_a} L^*\left(\{\boldsymbol{r}^*_a\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}^*_a}{dt}\right\}\right)$$

$$=>\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma . \frac{dr^*}{dt}\right) = \frac{-1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{\partial}{\partial r^*}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_\mathrm{C}, \frac{dr^*}{dt}\right)} + E^*_{em,eq}\frac{r^*_{eq}}{r^*} - \frac{c^4}{6k}\Lambda_P . {r^*}^3\right)$$

$$= \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} E^*_{em,eq}\frac{r^*_{eq}}{{r^*}^2} + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{c^4}{2k}\Lambda_P . {r^*}^2$$

$$=>\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma . \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} E^*_{em,eq}\frac{r^*_{eq}}{{r^*}^2} + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{c^4}{2k}\Lambda_P . {r^*}^2$$

In addition to the repulsive Coulomb term, the Poincaré term add a pressure force

$$S^*. p = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{c^4}{2k}\Lambda_P . {r^*}^2$$

$$=> p = \frac{\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{c^4}{2k}\Lambda_P . {r^*}^2}{4\pi {r^*}^2} = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{c^4}{8\pi k}\Lambda_P$$

In order to stabilize the sphere, we put $\Lambda_P = -|\Lambda_P|$

Hence we have

$$\boxed{\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma . \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} E^*_{em,eq}\frac{r^*_{eq}}{{r^*}^2} - \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{c^4}{2k}|\Lambda_P| . {r^*}^2}$$

The internal equilibrium is realized when (we put by definition $r^* = r^*_{eq}$):

$$\left(\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})} E^*_{em,eq}\frac{r^*_{eq}}{{r^*}^2} = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{c^4}{2k}|\Lambda_P| . {r^*}^2\right)_{r^* = r^*_{eq}}$$

$$<=> E^*_{em,eq} = \frac{c^4}{2k}|\Lambda_P| . {r^*_{eq}}^3$$

$$<=> \boxed{{r_{eq}^*}^3 = 2\frac{k}{|\Lambda_P|c^4}E_{em,eq}^*}$$

Moreover

$$=> \frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = \frac{1}{\gamma(V_\mathrm{C})}E_{em,eq}^*\frac{r_{eq}^*}{r^{*2}} - \frac{1}{\gamma(V_\mathrm{C})}\left(\frac{c^4}{2k}|\Lambda_P|\right).r^{*2} = \frac{1}{\gamma(V_\mathrm{C})}E_{em,eq}^*\frac{r_{eq}^*}{r^{*2}} - \frac{1}{\gamma(V_\mathrm{C})}\left(\frac{E_{em,eq}^*}{{r_{eq}^*}^3}\right).r^{*2}$$

$$=> \boxed{\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}E_{em,eq}^*\left(\frac{r_{eq}^*}{r^{*2}} - \frac{r^{*2}}{{r_{eq}^*}^3}\right)}$$

$$M = \gamma^*.M_\Sigma + \frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*} - \frac{c^2}{6k}\Lambda_P.r^{*3} = \gamma^*.M_\Sigma + \frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*} + \frac{1}{3}\left(\frac{c^2}{2k}|\Lambda_P|\right).r^{*3}$$
$$= \gamma^*.M_\Sigma + \frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*} + \frac{1}{3}\left(\frac{E_{em,eq}^*}{c^2{r_{eq}^*}^3}\right).r^{*3} = \gamma^*.M_\Sigma + \frac{E_{em,eq}^*}{c^2}\left(\frac{r_{eq}^*}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r_{eq}^*}^3}\right)$$

$$=> \boxed{M = \gamma^*.M_\Sigma + \frac{E_{em,eq}^*}{c^2}\left(\frac{r_{eq}^*}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r_{eq}^*}^3}\right)}$$

The mass is a function $M = M\left(r^*, \frac{dr^*}{dt}, \boldsymbol{V}_\mathrm{C}\right)$

For the equilibrium point we have the well known result (with the famous intriguing 4/3 term).

$$\boxed{M_{eq} = \gamma^*.M_\Sigma + \frac{4}{3}\frac{E_{em,eq}^*}{c^2} = \left(M_\Sigma + \frac{4}{3}\frac{E_{em,eq}^*}{c^2}\right)_{if\ \frac{dr^*}{dt}=0}}$$

Hence, as already many time said, for example in [3] & [9], the a priori astonishing factor mass $\frac{4}{3}$ is due to the necessity of a confining term (Poincaré term) which add an energical contribution.

$$\boxed{M_{eq} = (\gamma^*.M_\Sigma)_{Material\ points} + \left(\frac{E_{em,eq}^*}{c^2}\right)_{Electromagnetic\ field} + \left(\frac{1}{3}\frac{E_{em,eq}^*}{c^2}\right)_{Poincaré\ confinment}}$$

<u>Another look at the internal equation of motion of the sphere</u>

We start from the equation

$$\gamma(\boldsymbol{V}_\mathrm{C})\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = E_{em,eq}^*\left(\frac{r_{eq}^*}{r^{*2}} - \frac{r^{*2}}{{r_{eq}^*}^3}\right)$$

The quantity $\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt}$ is physically clear as it is the Lagrangian well-defined internal momentum of the sphere:

$$\gamma(\boldsymbol{V}_\mathrm{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt} = \gamma^* M_\Sigma.\frac{dr^*}{dt^*} = P^* = \frac{\partial}{\partial\frac{dr^*}{dt}}L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}, t\right)$$

We can re-express it as

$$P^* = M_K^{K^*}.\frac{dr^*}{dt}$$

Where $M_K^{K^*} \equiv \gamma(\boldsymbol{V}_\mathbf{C}).\gamma^* M_\Sigma$ is the inertia of r^* "$seen\ from\ K$ " (=with its time t)

The equation can be re-written

$$\gamma(\boldsymbol{V}_\mathbf{C})\frac{d}{dt}\left(M_K^{K^*}.\frac{dr^*}{dt}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> \gamma(\boldsymbol{V}_\mathbf{C})\frac{dM_K^{K^*}}{dt}.\frac{dr^*}{dt} + \gamma(\boldsymbol{V}_\mathbf{C})M_K^{K^*}.\frac{d}{dt}\left(\frac{dr^*}{dt}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> M_K^{K^*}.\frac{d}{dt}\left(\frac{dr^*}{dt}\right) = \frac{E^*_{em,eq}}{\gamma(\boldsymbol{V}_\mathbf{C})}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{dM_K^{K^*}}{dt}.\frac{dr^*}{dt}$$

$$M_K^{K^*}\frac{d}{dt}\left(\frac{dr^*}{dt}\right) = \vartheta.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \alpha.\frac{dr^*}{dt}$$

With:

- $M_K^{K^*} \equiv \gamma(\boldsymbol{V}_\mathbf{C}).\gamma^* M_\Sigma$
- $\alpha = \alpha\left(\boldsymbol{V}_\mathbf{C}, \boldsymbol{a}_\mathbf{C}, \frac{dr^*}{dt}, \frac{d^2r^*}{dt^2}\right) = \frac{dM_K^{K^*}}{dt}$
- $\vartheta = \vartheta(\boldsymbol{V}_\mathbf{C}) = \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2}$

We want express the coefficient of the viscous term α in term of Energy and internal energy (or mass).

$$E = \gamma(\boldsymbol{V}_\mathbf{C})E^* = \gamma(\boldsymbol{V}_\mathbf{C})Mc^2 = \gamma(\boldsymbol{V}_\mathbf{C})\left(\gamma^*.M_\Sigma c^2 + E^*_{em,eq}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right)\right)$$

$$= M_K^{K^*}c^2 + \gamma(\boldsymbol{V}_\mathbf{C})E^*_{em,eq}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right)$$

$$=> \frac{dE}{dt} = \frac{dM_K^{K^*}}{dt}.c^2 + \frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right) + \gamma(\boldsymbol{V}_\mathbf{C})E^*_{em,eq}\frac{d}{dt}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right)$$

$$= \frac{dM_K^{K^*}}{dt}.c^2 + \frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right) - \gamma(\boldsymbol{V}_\mathbf{C})E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)\frac{dr^*}{dt}$$

$$=> \frac{dM_K^{K^*}}{dt} = \frac{1}{c^2}\frac{dE}{dt} - \frac{1}{c^2}\frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right) + \frac{1}{c^2}\gamma(\boldsymbol{V}_\mathbf{C})E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)\frac{dr^*}{dt}$$

$$= \frac{1}{c^2}\frac{dE}{dt} - \frac{1}{c^2}\frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt}[Mc^2 - \gamma^*.M_\Sigma c^2] + \frac{1}{c^2}\gamma(\boldsymbol{V}_\mathbf{C})E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)\frac{dr^*}{dt}$$

Then

$$M_K^{K^*}.\frac{d}{dt}\left(\frac{dr^*}{dt}\right)=\frac{E^*_{em,eq}}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)$$
$$-\left[\frac{1}{c^2}\frac{dE}{dt}-\frac{1}{c^2}\frac{d\gamma(\boldsymbol{V}_{\mathbf{C}})}{dt}[Mc^2-\gamma^*.M_\Sigma c^2]+\frac{1}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)\frac{dr^*}{dt}\right].\frac{dr^*}{dt}$$

$$<=>M_K^{K^*}.\frac{d}{dt}\left(\frac{dr^*}{dt}\right)=$$
$$=E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)\left(\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}-\frac{1}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})\left(\frac{dr^*}{dt}\right)^{\mathbf{2}}\right)-\left[\frac{1}{c^2}\frac{dE}{dt}-\frac{1}{c^2}\frac{d\gamma(\boldsymbol{V}_{\mathbf{C}})}{dt}[Mc^2-\gamma^*.M_\Sigma c^2]\right].\frac{dr^*}{dt}$$

Using

- $\gamma^*=\gamma^*\left(\boldsymbol{V}_{\mathbf{C}},\frac{dr^*}{dt}\right)=\frac{1}{\sqrt{1-\frac{\gamma(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}}$
  $$=>\gamma^{*2}=\frac{1}{1-\frac{\gamma(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}$$
- $E=\gamma E^*$
  $=>\frac{dE}{dt}=\frac{d\gamma(\boldsymbol{V}_{\mathbf{C}})}{dt}E^*+\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{dE^*}{dt}$
  $$<=>\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\frac{dE}{dt}=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\frac{d\gamma(\boldsymbol{V}_{\mathbf{C}})}{dt}E^*+\frac{dE^*}{dt}$$
  $$<=>\frac{d\gamma(\boldsymbol{V}_{\mathbf{C}})}{dt}=\frac{1}{E^*}\frac{dE}{dt}-\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{1}{E^*}\frac{dE^*}{dt}$$

Thus

$$M_K^{K^*}.\frac{d}{dt}\left(\frac{dr^*}{dt}\right)=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^{*2}}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)$$
$$-\left[\frac{1}{c^2}\frac{dE}{dt}-\frac{1}{c^2}\left(\frac{1}{E^*}\frac{dE}{dt}-\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{1}{E^*}\frac{dE^*}{dt}\right)[Mc^2-\gamma^*.M_\Sigma c^2]\right].\frac{dr^*}{dt}$$
$$=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^{*2}}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)$$
$$-\frac{1}{c^2}\left[\frac{dE}{dt}-\left(\frac{1}{E^*}\frac{dE}{dt}[Mc^2-\gamma^*.M_\Sigma c^2]-\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{1}{E^*}\frac{dE^*}{dt}[Mc^2-\gamma^*.M_\Sigma c^2]\right)\right].\frac{dr^*}{dt}$$
$$=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^{*2}}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)$$
$$-\frac{1}{c^2}\left[\left(1-\frac{1}{Mc^2}[Mc^2-\gamma^*.M_\Sigma c^2]\right)\frac{dE}{dt}+\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{1}{Mc^2}\frac{dMc^2}{dt}[Mc^2-\gamma^*.M_\Sigma c^2]\right].\frac{dr^*}{dt}$$
$$=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^{*2}}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)-\left(\frac{\gamma^*.M_\Sigma}{Mc^2}\frac{dE}{dt}+\gamma(\boldsymbol{V}_{\mathbf{C}})\frac{dM}{dt}\left(1-\frac{\gamma^*.M_\Sigma}{M}\right)\right).\frac{dr^*}{dt}$$

=>

$$M_K^{K^*}\frac{d}{dt}\left(\frac{dr^*}{dt}\right) = \vartheta . E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \alpha . \frac{dr^*}{dt}$$

- $M_K^{K^*} \equiv \gamma(\boldsymbol{V}_\mathbf{C}).\gamma^* M_\Sigma$
- $\alpha = \alpha\left(\boldsymbol{V}_\mathbf{C}, \boldsymbol{a}_\mathbf{C}, \frac{dr^*}{dt}, \frac{d^2r^*}{dt^2}\right)$

$$= \frac{\gamma^* . M_\Sigma}{Mc^2}\frac{dE}{dt} + \gamma(\boldsymbol{V}_\mathbf{C})\frac{dM}{dt}\left(1 - \frac{\gamma^* . M_\Sigma}{M}\right) = \left[\gamma(\boldsymbol{V}_\mathbf{C})\frac{dM}{dt}\left(1 - \frac{\gamma^* . M_\Sigma}{M}\right)\right]_{if\ E=cte}$$

$$= \frac{\gamma^* . M_\Sigma}{E}\frac{dE}{dt} + \gamma(\boldsymbol{V}_\mathbf{C})\frac{1}{c^2}\frac{dE^*}{dt}\left(1 - \frac{\gamma^* . M_\Sigma c^2}{E}\right) = \left[\gamma(\boldsymbol{V}_\mathbf{C})\frac{1}{c^2}\frac{dE^*}{dt}\left(1 - \frac{\gamma^* . M_\Sigma c^2}{E}\right)\right]_{if\ E=cte}$$

- $\vartheta = \vartheta\left(\boldsymbol{V}_\mathbf{C}, \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})\gamma^{*2}}$

Or in a "more internal" point of view

$$\gamma^* M_\Sigma \frac{d}{dt}\left(\frac{dr^*}{dt}\right) = \vartheta . E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \alpha . \frac{dr^*}{dt}$$

- $\alpha = \alpha\left(\boldsymbol{V}_\mathbf{C}, \boldsymbol{a}_\mathbf{C}, \frac{dr^*}{dt}, \frac{d^2r^*}{dt^2}\right) = \left[\frac{dM}{dt}\left(1 - \frac{\gamma^* . M_\Sigma}{M}\right)\right]_{if\ E=cte}$

$$= \left[\frac{1}{c^2}\frac{dE^*}{dt}\left(1 - \frac{\gamma^* . M_\Sigma c^2}{E}\right)\right]_{if\ E=cte}$$

- $\vartheta = \vartheta\left(\boldsymbol{V}_\mathbf{C}, \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^{*2}}$

With this effective Newtonian form one can interpret more intuitively the internal relativistic equation of the sphere:

- The latter has an apparent mass $\gamma^* M_\Sigma$ (=internal kinetic energy of the material system)
- a factor ϑ affecting the repulsive Coulombian force and the attractive Poincaré-pressure force
- a viscous term -$\alpha . \frac{dr^*}{dt}$ due to the exchange between internal energy (the mass) and the total energy since it is proportional to the rate of increase of internal energy $\frac{dE^*}{dt}$.

It is very surprising that <u>if internal energy increases, the viscous term increases (!)</u> ($\frac{\gamma^* . M_\Sigma c^2}{E} < 1$). This is the total opposite of what I expected since one might think that the representative of this internal dynamics $\frac{dr^*}{dt}$ must intuitively "accelerate" ($\equiv$ "r^* accelerate relative to t"). But we must remember that $\frac{dr^*}{dt}$ is actually not the representative of the true internal dynamics which is played by $\frac{dr^*}{dt^*} = \gamma(\boldsymbol{V}_\mathbf{C})\frac{dr^*}{dt}$. Moreover if E^*increases, then $\gamma(\boldsymbol{V}_\mathbf{C})$decreases ($E = \gamma(\boldsymbol{V}_\mathbf{C})E^* = cte$), and, in this case, $\vartheta''$and thus the Coulombian force increases. It is then finally not obvious that the accumulation of internal energy have to "accelerate" the quantity $\frac{dr^*}{dt}$.

We can still notice that, if E^* increases, $\gamma(\boldsymbol{V}_\mathbf{C})$ decreases ($E = \gamma(\boldsymbol{V}_\mathbf{C})E^* = cte$) and then

$\frac{dr^*}{dt} = \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})}\frac{dr^*}{dt^*}$ seems to decrease (for a given $\frac{dr^*}{dt^*}$). But the growth of E^* can increase $\frac{dr^*}{dt^*}$ which counterbalance the time dilation effect. So the reasoning is finally consistent.

Instability of the expression of the viscous force with respect to the studied quantity

Despite the apparent paradoxical behaviour of the viscous force above, I will now show the radical modification of the viscous force if we change slightly the studied quantity:

- $\gamma^* M_\Sigma.\frac{dr^*}{dt^*} = \gamma(\boldsymbol{V}_\mathbf{C})M_\Sigma.\frac{dr^*}{dt}$
- or $M_\Sigma.\frac{dr^*}{dt}$
- or $\gamma^*\ M_\Sigma.\frac{dr^*}{dt}$

Now, we study the equation associated to $\gamma^*\ M_\Sigma.\frac{dr^*}{dt}$

In we start again from

$$\gamma(\boldsymbol{V}_\mathbf{C})\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathbf{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

(which is equivalent to $\frac{d}{dt^*}\left(\gamma^* M_\Sigma.\frac{dr^*}{dt^*}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$)

$$<=> \gamma(\boldsymbol{V}_\mathbf{C})\frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt}.\gamma^* M_\Sigma.\frac{dr^*}{dt} + \gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^* M_\Sigma.\frac{d}{dt}\left(\frac{dr^*}{dt}\right) + \gamma(\boldsymbol{V}_\mathbf{C})^2\frac{d\gamma^*}{dt}M_\Sigma.\frac{dr^*}{dt} = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> \gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^* M_\Sigma.\frac{d^2r^*}{dt^2} + \gamma(\boldsymbol{V}_\mathbf{C})^2\frac{d\gamma^*}{dt}M_\Sigma.\frac{dr^*}{dt} = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt}\gamma(\boldsymbol{V}_\mathbf{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt}$$

$$<=> \gamma(\boldsymbol{V}_\mathbf{C})^2\frac{d}{dt}\left(\gamma^* M_\Sigma\frac{dr^*}{dt}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt}\gamma(\boldsymbol{V}_\mathbf{C}).\gamma^* M_\Sigma.\frac{dr^*}{dt}$$

But $\frac{d\gamma(\boldsymbol{V}_\mathbf{C})}{dt} = \frac{1}{E^*}\frac{dE}{dt} - \gamma(\boldsymbol{V}_\mathbf{C})\frac{1}{E^*}\frac{dE^*}{dt}$

Thus

$$\frac{d}{dt}\left(\gamma^* M_\Sigma\frac{dr^*}{dt}\right) = \vartheta.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \alpha.\frac{dr^*}{dt}$$

- $\alpha = \alpha\left(\boldsymbol{V}_\mathbf{C}, \boldsymbol{a}_\mathbf{C}, \frac{dr^*}{dt}\right) = \frac{1}{E^*}\frac{dE}{dt} - \gamma(\boldsymbol{V}_\mathbf{C})\frac{1}{E^*}\frac{dE^*}{dt}$

$$= \left(-\frac{\gamma^* M_\Sigma}{E^*}\frac{dE^*}{dt}\right)_{if\ E=cte} = \left(-\frac{\gamma^* M_\Sigma}{M}\frac{dM}{dt}\right)_{if\ E=c}$$

- $\vartheta = \vartheta(\boldsymbol{V}_\mathbf{C}) = \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2}$

One can remark the change of the sign of the viscous force. Here, an increase of the internal energy is traduced by an increase of the quantity $\gamma^* M_\Sigma\frac{dr^*}{dt}$.

In this sense the latter quantity maintain characteristics more in accordance with our intuition. But of course, whatever the quantity used, the dynamics of the different point of view express a total physical equivalence.

A summary:

If we studied:

- $P^* = \gamma^* M_\Sigma . \frac{dr^*}{dt^*} = \gamma(\boldsymbol{V}_\mathbf{C}) M_\Sigma . \frac{dr^*}{dt}$ then :
  - the equation of motion is $\gamma(\boldsymbol{V}_\mathbf{C}) \frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathbf{C})\gamma^* M_\Sigma . \frac{dr^*}{dt}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$
  - with no viscous force: $F_v = 0$
- $M_\Sigma \frac{dr^*}{dt}$ then :
  - the equation of motion is $\gamma^* \frac{d}{dt}\left(M_\Sigma \frac{dr^*}{dt}\right) = \vartheta . E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \alpha . \frac{dr^*}{dt}$
  - with viscous force $F_v = -\alpha . \frac{dr^*}{dt} = -\frac{dM}{dt}\left(1 - \frac{\gamma^* . M_\Sigma}{M}\right) . \frac{dr^*}{dt}$
- $\gamma^* M_\Sigma \frac{dr^*}{dt}$ then :
  - the equation of motion is $\frac{d}{dt}\left(\gamma^* M_\Sigma \frac{dr^*}{dt}\right) = \vartheta . E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \alpha . \frac{dr^*}{dt}$
  - with viscous force $F_v = -\alpha . \frac{dr^*}{dt} = +\left(\frac{\gamma^* M_\Sigma}{M}\frac{dM}{dt}\right)\frac{dr^*}{dt}$

Consistency check (at least for myself) with another way of computation :

Since

$$\frac{d^2}{dt^2} = -\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C} . \boldsymbol{a}_\mathrm{C}}{c^2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{d}{dt^*}\left(\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{d}{dt^*}\right)$$
$$= \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left[\frac{d\gamma(\boldsymbol{V}_\mathrm{C})^{-1}}{dt^*}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$
$$= \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left[\frac{dt}{dt^*}\frac{d\gamma(\boldsymbol{V}_\mathrm{C})^{-1}}{dt}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$
$$= \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left[\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\left(1 - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2}\right)^{1/2}}{dt}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$
$$= \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left[\gamma(\boldsymbol{V}_\mathrm{C})\frac{1}{2}\left(-2\frac{\boldsymbol{V}_\mathrm{C}}{c^2}\right)\boldsymbol{a}_\mathrm{C}\left(1 - \frac{{\boldsymbol{V}_\mathrm{C}}^2}{c^2}\right)^{-1/2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$
$$= \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left[-\gamma(\boldsymbol{V}_\mathrm{C})^2\frac{\boldsymbol{V}_\mathrm{C} . \boldsymbol{a}_\mathrm{C}}{c^2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$
$$= -\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C} . \boldsymbol{a}_\mathrm{C}}{c^2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)$$

$$=> \gamma(\boldsymbol{V}_\mathrm{C})^2\frac{d}{dt}\left(\gamma^* M_\Sigma \frac{dr^*}{dt}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{\boldsymbol{V}_\mathrm{C} . \boldsymbol{a}_\mathrm{C}}{c^2}\gamma(\boldsymbol{V}_\mathrm{C})^4 . \gamma^* M_\Sigma . \frac{dr^*}{dt}$$

$$\gamma(\boldsymbol{V}_\mathrm{C})^2\frac{d\gamma^*}{dt}M_\Sigma\frac{dr^*}{dt} + \gamma(\boldsymbol{V}_\mathrm{C})^2\gamma^* M_\Sigma\frac{d}{dt}\left(\frac{dr^*}{dt}\right) = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{\boldsymbol{V}_\mathrm{C} . \boldsymbol{a}_\mathrm{C}}{c^2}\gamma(\boldsymbol{V}_\mathrm{C})^4 . \gamma^* M_\Sigma . \frac{dr^*}{dt}$$

$$\gamma(\boldsymbol{V}_\mathrm{C})^2\frac{d\gamma^*}{dt}M_\Sigma\frac{dr^*}{dt} + \gamma(\boldsymbol{V}_\mathrm{C})^2\gamma^* M_\Sigma\left[-\gamma(\boldsymbol{V}_\mathrm{C})\frac{\boldsymbol{V}_\mathrm{C} . \boldsymbol{a}_\mathrm{C}}{c^2}\left(\frac{dr^*}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2}\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)\right] = E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{\boldsymbol{V}_\mathrm{C} . \boldsymbol{a}_\mathrm{C}}{c^2}\gamma(\boldsymbol{V}_\mathrm{C})^4 . \gamma^* M_\Sigma . \frac{dr^*}{dt}$$

$$\gamma(\boldsymbol{V}_\mathbf{C})^2\frac{d\gamma^*}{dt}M_\Sigma\frac{dr^*}{dt}-\gamma(\boldsymbol{V}_\mathbf{C})\gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^*M_\Sigma\frac{\boldsymbol{V}_\mathbf{C}.\boldsymbol{a}_\mathbf{C}}{c^2}\left(\frac{dr^*}{dt^*}\right)+\frac{\gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^*M_\Sigma}{\gamma(\boldsymbol{V}_\mathbf{C})^2}\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)=E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)-\frac{\boldsymbol{V}_\mathbf{C}.\boldsymbol{a}_\mathbf{C}}{c^2}\gamma(\boldsymbol{V}_\mathbf{C})^4.\gamma^*M_\Sigma.\frac{dr^*}{dt}$$

$$\gamma(\boldsymbol{V}_\mathbf{C})^2\frac{d\gamma^*}{dt}M_\Sigma\frac{dr^*}{dt}-\gamma(\boldsymbol{V}_\mathbf{C})^3\gamma^*M_\Sigma\frac{\boldsymbol{V}_\mathbf{C}.\boldsymbol{a}_\mathbf{C}}{c^2}\left(\frac{dr^*}{dt^*}\right)+\gamma^*M_\Sigma\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)=E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)-\frac{\boldsymbol{V}_\mathbf{C}.\boldsymbol{a}_\mathbf{C}}{c^2}\gamma(\boldsymbol{V}_\mathbf{C})^4.\gamma^*M_\Sigma.\frac{dr^*}{dt}$$

$$\gamma(\boldsymbol{V}_\mathbf{C})^2\frac{d\gamma^*}{dt}M_\Sigma\frac{dr^*}{dt}+\gamma^*M_\Sigma\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)=E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$\frac{d}{dt^*}\left(\gamma^*M_\Sigma.\frac{dr^*}{dt^*}\right)=E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

As it should.

<u>The internal equation of motion of the sphere in second order approximation</u>

We clarify the following factor

$$\gamma^*=\gamma^*\left(\boldsymbol{V}_\mathbf{C},\frac{dr^*}{dt}\right)=\frac{1}{\sqrt{1-\frac{\gamma(\boldsymbol{V}_\mathbf{C})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}}=\frac{1}{\sqrt{\frac{1-\frac{(\boldsymbol{V}_\mathbf{C})^2}{c^2}-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2}{1-\frac{(\boldsymbol{V}_\mathbf{C})^2}{c^2}}}}=\sqrt{\frac{1-\frac{(\boldsymbol{V}_\mathbf{C})^2}{c^2}}{1-\frac{(\boldsymbol{V}_\mathbf{C})^2}{c^2}-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2}}$$

$$\approx\left(1-\frac{1}{2}\frac{(\boldsymbol{V}_\mathbf{C})^2}{c^2}\right)\left(1+\frac{1}{2}\frac{(\boldsymbol{V}_\mathbf{C})^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)$$

$$=\left(1+\frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2-\frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}\left(1+\frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\right)$$

$$=1+\frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2-\frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}-\frac{1}{4}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}-\frac{1}{4}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2$$

$$=>\gamma^*=1+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2-\frac{1}{4}\frac{(\boldsymbol{V}_\mathbf{C})^4}{c^4}-\frac{1}{4}\frac{(\boldsymbol{V}_\mathbf{C})^2}{c^2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2$$

$$=>\boxed{\gamma^*=1+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2+\theta\left(\frac{v^4}{c^4}\right)}$$

Then, using

$$\frac{dM}{dt}=\frac{1}{c^2}\frac{dMc^2}{dt}=\frac{1}{c^2}\frac{dE^*}{dt}=\frac{1}{c^2}\frac{d\frac{E}{\gamma(\boldsymbol{V}_\mathbf{C})}}{dt}=\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})}\frac{1}{c^2}\frac{dE}{dt}+\frac{E}{c^2}\frac{d\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})}}{dt}$$

$$=\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})}\frac{1}{c^2}\frac{dE}{dt}+\frac{E}{c^2}\frac{d\sqrt{1-\left(\frac{\boldsymbol{V}_\mathbf{C}}{c}\right)^2}}{dt}$$

$$= \frac{1}{\gamma(\boldsymbol{V_C})}\frac{1}{c^2}\frac{dE}{dt} + \frac{\gamma(\boldsymbol{V_C})}{2}\frac{E}{c^2}\frac{d\left(1-\left(\frac{\boldsymbol{V_C}}{c}\right)^2\right)}{dt} = \frac{1}{\gamma(\boldsymbol{V_C})}\frac{1}{c^2}\frac{dE}{dt} - \frac{\gamma(\boldsymbol{V_C})}{2}\frac{E}{c^2}2\frac{\boldsymbol{V_C a_C}}{c^2}$$

$$= \frac{1}{\gamma(\boldsymbol{V_C})}\frac{1}{c^2}\frac{dE}{dt} - \gamma(\boldsymbol{V_C})^2 M\frac{\boldsymbol{V_C a_C}}{c^2}$$

$$=> \boxed{\frac{dM}{dt} = \frac{1}{\gamma(\boldsymbol{V_C})}\frac{1}{c^2}\frac{dE}{dt} - M.\gamma(\boldsymbol{V_C})^2\frac{\boldsymbol{V_C a_C}}{c^2} = \left[-\gamma(\boldsymbol{V_C})^2 M\frac{\boldsymbol{V_C a_C}}{c^2}\right]_{if\ E=cte}}$$

We have

$$\gamma^* M_\Sigma \frac{d}{dt}\left(\frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V_C})^2\gamma^{*2}}.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{r^{*\,3}_{eq}}\right) - \frac{dM}{dt}\left(1 - \frac{\gamma^*.M_\Sigma}{M}\right).\frac{dr^*}{dt}$$

$$<=> M_\Sigma \frac{d}{dt}\left(\frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V_C})^2\gamma^{*3}}.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{r^{*\,3}_{eq}}\right) - \frac{dM}{dt}\left(\frac{1}{\gamma^*} - \frac{M_\Sigma}{M}\right).\frac{dr^*}{dt}$$

$$<=> M_\Sigma \frac{d^2r^*}{dt^2} = \frac{1}{\gamma(\boldsymbol{V_C})^2\left(1+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)^3}.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{r^{*\,3}_{eq}}\right) + \gamma(\boldsymbol{V_C})^2 M\frac{\boldsymbol{V_C a_C}}{c^2}\left(1 - \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2 - \frac{M_\Sigma}{M}\right).\frac{dr^*}{dt}$$

$$= \frac{1 - \frac{3}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2}{\gamma(\boldsymbol{V_C})^2}.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{r^{*\,3}_{eq}}\right) + M\gamma(\boldsymbol{V_C})^2\frac{\boldsymbol{V_C a_C}}{c^2}\left(1 - \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2 - \frac{M_\Sigma}{M}\right).\frac{dr^*}{dt}$$

Then

$$M_\Sigma \frac{d^2r^*}{dt^2} = \vartheta.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{r^{*\,3}_{eq}}\right) - \alpha.\frac{dr^*}{dt}$$

- $\vartheta = \vartheta\left(\boldsymbol{V_C}, \frac{dr^*}{dt}\right) = \frac{1-\frac{3}{c^2}\left(\frac{dr^*}{dt}\right)^2}{\gamma(\boldsymbol{V_C})^2}$
- $\alpha = \alpha\left(\boldsymbol{V_C}, \boldsymbol{a_C}, \frac{dr^*}{dt}\right) = \left[-M_\Sigma\gamma(\boldsymbol{V_C})^2\left(1 - \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2 - \frac{M_\Sigma}{M}\right)\frac{\boldsymbol{V_C a_C}}{c^2}\right]_{if\ E=cte}$

As noted above we find (surprisingly) that the viscous term shows a “capture” of the kinetic energy of the center of mass by the internal system when the center of mass is accelerated.

But as explain there is an ambiguity between different variables that can express the internal dynamics and the one we have chosen to study $M_\Sigma\frac{dr^*}{dt}$ is different from the classical $\gamma^* M_\Sigma.\frac{dr^*}{dt^*}$.

<u>The Newtonian oscillator in the field of the center of mass</u>

We see that in general there is a coupling between the external dynamic and the internal dynamic. But this coupling is clearly due to the relativistic regime: outside this regime, the external dynamic does not affect the internal dynamic (since K* is a local Galilean frame, there are no inertial forces).

The case of a non-relativistic internal dynamic ($\frac{1}{c}\frac{dr^*}{dt} \approx 0$) gives:

- $\vartheta = \vartheta\left(\boldsymbol{V}_\mathbf{C}, \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2} = 1 - \frac{\boldsymbol{V}_\mathbf{C}^2}{c^2}$
- $\alpha = \alpha\left(\boldsymbol{V}_\mathbf{C}, \frac{dr^*}{dt}\right) \approx -M_\Sigma \gamma(\boldsymbol{V}_\mathbf{C})^2 \left(1 - \frac{M_\Sigma}{M}\right) \frac{\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}}{c^2} = -\frac{M_\Sigma\left(1-\frac{M_\Sigma}{M}\right)}{c^2} \frac{\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}}{1-\frac{\boldsymbol{V}_\mathbf{C}^2}{c^2}}$

$$=> \alpha \frac{dr^*}{dt} = -\frac{M_\Sigma\left(1-\frac{M_\Sigma}{M}\right)}{c^2} \frac{\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}}{1-\frac{\boldsymbol{V}_\mathbf{C}^2}{c^2}} \frac{dr^*}{dt} = K.\frac{1}{c}\frac{dr^*}{dt} \approx 0$$

$$=> M_\Sigma \frac{d^2 r^*}{dt^2} \approx \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

To simplify the dynamics, I will assume that the system is close to the equilibrium point. Then we can Taylor the function $f(r^*) \equiv \frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}$ near this point.

$$f(r^*) \approx f(r^*_{eq}) + \frac{df}{dr^*}(r^*_{eq}).(r^* - r^*_{eq}) = \left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)(r^*_{eq}) + \left(-2\frac{r^*_{eq}}{r^{*3}} - 2\frac{r^*}{{r^*_{eq}}^3}\right)(r^*_{eq}).(r^* - r^*_{eq})$$

$$= (0) - \frac{4}{{r^*_{eq}}^2}.(r^* - r^*_{eq})$$

$$=> M_\Sigma \frac{d^2 r^*}{dt^2} \approx -\frac{4}{{r^*_{eq}}^2}\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2} E^*_{em,eq}(r^* - r^*_{eq})$$

If the speed of the center of mass varies sufficiently slowly (adiabatically), we have as desired the case of an effective oscillator around a center of mass velocity $\boldsymbol{V}_\mathbf{C}^2$:

$$M_\Sigma \frac{d^2 r^*}{dt^2} \approx -k_{\boldsymbol{V}_\mathbf{C}}.(r^* - r^*_{eq})$$

With $k_{\boldsymbol{V}_\mathbf{C}} \equiv \frac{4E^*_{em,eq}}{{r^*_{eq}}^2}\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2} = \frac{k_\mathbf{0}}{\gamma(\boldsymbol{V}_\mathbf{C})^2}$

$$\omega_{\boldsymbol{V}_\mathbf{C}} = \sqrt{\frac{k_{\boldsymbol{V}_\mathbf{C}}}{M_\Sigma}} = \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})}\sqrt{\frac{k_\mathbf{0}}{M_\Sigma}} = \frac{\omega_\mathbf{0}}{\gamma(\boldsymbol{V}_\mathbf{C})}$$

Moreover, taking into account ${r^*_{eq}}^3 = 2\frac{k}{|\Lambda_P|c^4}E^*_{em,eq}$, the pulsation of the oscillator is then:

$$\omega_{\boldsymbol{V}_\mathbf{C}} = \sqrt{\frac{k_{\boldsymbol{V}_\mathbf{C}}}{M_\Sigma}} \approx \frac{2}{r^*_{eq}}\sqrt{E^*_{em,eq}}\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})} = \frac{2\sqrt{E^*_{em,eq}}}{\left(2\frac{k}{|\Lambda_P|c^4}E^*_{em,eq}\right)^{1/3}}\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})} = 2^{2/3}\left(\frac{c^4|\Lambda_P|}{k}\right)^{1/3} {E^*_{em,eq}}^{1/6}\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})}$$

$$\omega_{\boldsymbol{V}_\mathbf{C}} = \frac{\omega_\mathbf{0}}{\gamma(\boldsymbol{V}_\mathbf{C})} \approx \left(1 - \frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}\right)\omega_\mathbf{0} = \omega_\mathbf{0} + \Delta\omega_{\boldsymbol{V}_\mathbf{C}}$$

- $\omega_\mathbf{0} \equiv \sqrt{\frac{k_\mathbf{0}}{M_\Sigma}} = 2\sqrt{\frac{E^*_{em,eq}}{M_\Sigma . {r^*_{eq}}^2}} \approx 2^{2/3}\left(\frac{c^4|\Lambda_P|}{k}\right)^{1/3} {E^*_{em,eq}}^{1/6}$
- $\Delta\omega_{\boldsymbol{V}_\mathbf{C}} \approx -\frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}\omega_\mathbf{0}$

Hence,

- if the "internal" system has Newtonian dynamics ;
- if the velocity of the center of mass is not negligible relative to the "Einstein constant" c ([5']);
- and if the speed of the center of mass varies sufficiently slowly with respect to the internal dynamics,

Then the "internal" oscillator sees its frequency $\omega_{\boldsymbol{V}_\mathbf{C}}$ decreasing (red-shift) to the value:

$$-\Delta\omega_{\boldsymbol{V}_\mathbf{C}} \approx \frac{1}{2}\frac{{\boldsymbol{V}_\mathbf{C}}^2}{c^2}\omega_\mathbf{0}$$

A complex system whose center of mass moves at a sufficiently high speed affects the internal dynamics of the system.

<u>The dynamic effect is actually a kinematic one</u>

This, a priori dynamics effect, is actually rather a kinematic one, Einstein's law of time dilation: Indeed, the latter said

$$dt = \gamma(\boldsymbol{V}_\mathbf{C})dt^*$$

Which is traduced in term of decreases of frequencies $\left(\propto \frac{1}{dt}\right)$ since

$$\frac{1}{dt} \approx \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})}\frac{1}{dt^*}$$

Another way to see it is to consider the relation given above

$$\frac{d^2}{dt^2} = -\gamma(\boldsymbol{V}_\mathbf{C})\frac{\boldsymbol{V}_\mathbf{C} . \boldsymbol{a}_\mathbf{C}}{c^2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)$$

In the Newtonian limit, for the same reasons than above, we can neglect the viscous term :

$$M_\Sigma\frac{d^2 r^*}{dt^2} \approx -k_{\boldsymbol{V}_\mathbf{C}} . \left(r^* - r^*_{eq}\right) = -\frac{k_\mathbf{0}}{\gamma(\boldsymbol{V}_\mathbf{C})^2} . \left(r^* - r^*_{eq}\right)$$

$$<=> M_\Sigma \frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)r^* \approx -\frac{k_\mathbf{0}}{\gamma(\boldsymbol{V}_\mathbf{C})^2} . \left(r^* - r^*_{eq}\right)$$

$$<=> \boxed{M_\Sigma \frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right) \approx -k_\mathbf{0} . \left(r^* - r^*_{eq}\right)}$$

$$=> \omega_\mathbf{0} = \sqrt{\frac{k_\mathbf{0}}{M_\Sigma}} \text{ is indeed the "true" internal frequency}$$

=>What we see with the variable $\frac{dr^*}{dt}$ is indeed the time dilated internal dynamics of the "true" one classically given by $\frac{dr^*}{dt^*}$

This is consistent with the intuition coming from the simple relation

$$\frac{dr^*}{dt} = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\frac{dr^*}{dt^*}$$

It is tempting to anticipate a quantum treatment (as I tried to do in an older version of this article) by quantizing the system associated with the dynamics variable pair $\left(r^*, M_\Sigma \frac{dr^*}{dt}\right)$ which behaves as an oscillator with pulsation $\omega_{V_{\mathrm{C}}} = \sqrt{\frac{k_{V_{\mathrm{C}}}}{M_\Sigma}} = \frac{1}{\gamma(V_{\mathrm{C}})}\sqrt{\frac{k_0}{M_\Sigma}} = \frac{\omega_0}{\gamma(V_{\mathrm{C}})}$ .This eventual quantum treatment would produce quantum characteristics (quantum energy & zero point energy) red-shifted (a kind of tiny renormalization) by thermal energy (for a macroscopic number of electrons thanks to the energy equipartition theorem). But this reasoning is finally suspect to me as I showed above during the Hamiltonian analysis that the canonical variable that I have identified is whereas $\left(r^*, \gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^* M_\Sigma \frac{dr^*}{dt}\right)$.

<u>The mass for the Newtonian oscillator</u>

In this situation the mass is now

$$M = \gamma^*. M_\Sigma + \frac{E^*_{em,eq}}{c^2}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right) \approx \left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right). M_\Sigma + \frac{E^*_{em,eq}}{c^2} g(r^*)$$

With the function $g(r^*) \equiv \frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}$

Taking account that for the equilibrium point $\left(\frac{dg}{dr^*}\right)_{r^*_{eq}} = 0$, since the mass is an internal energy near the equilibrium point, we have

$$g(r^*) = g(r^*_{eq}) + \frac{1}{2}\left(\frac{d^2 g}{dr^{*2}}\right)_{r^*_{eq}} (r^* - r^*_{eq})^2 = \frac{4}{3} + \frac{1}{2}\left(\frac{d}{dr^*}\left[\frac{d}{dr^*}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right)\right]\right)_{r^*_{eq}} (r^* - r^*_{eq})^2$$

$$= \frac{4}{3} + \frac{1}{2}\left(\frac{d}{dr^*}\left(-\frac{r^*_{eq}}{r^{*2}} + \frac{r^{*2}}{{r^*_{eq}}^3}\right)\right)_{r^*_{eq}} (r^* - r^*_{eq})^2 = \frac{4}{3} + \frac{1}{2}\left(2\frac{r^*_{eq}}{r^{*3}} + 2\frac{r^*}{{r^*_{eq}}^3}\right)_{r^*_{eq}} (r^* - r^*_{eq})^2$$

$$= \frac{4}{3} + \frac{1}{2}\left(2\frac{1}{{r^*_{eq}}^2} + 2\frac{1}{{r^*_{eq}}^2}\right)(r^* - r^*_{eq})^2 = \frac{4}{3} + \frac{2}{{r^*_{eq}}^2}(r^* - r^*_{eq})^2$$

$$=> M \approx \left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right). M_\Sigma + \frac{E^*_{em,eq}}{c^2}\left[\frac{4}{3} + \frac{2}{{r^*_{eq}}^2}(r^* - r^*_{eq})^2\right]$$

$$= \left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right). M_\Sigma + \frac{E^*_{em,eq}}{c^2}\left[\frac{4}{3} + \frac{2}{{r^*_{eq}}^2}(r^* - r^*_{eq})^2\right]$$

$$=> M \approx M_{eq} + \frac{1}{2} M_{\Sigma} \frac{1}{c^2} \left(\frac{dr^*}{dt}\right)^2 + \frac{E^*_{em,eq}}{c^2} 2 \frac{\left(r^* - r^*_{eq}\right)^2}{{r^*_{eq}}^2}$$

With

$$M_{eq} \equiv M_{\Sigma} + \frac{4}{3} \frac{E^*_{em,eq}}{c^2}$$

The mass is a function $M = M\left(r^*, \frac{dr^*}{dt}, \boldsymbol{V}_{\mathbf{C}}\right)$.

<u>The dynamic of the center of mass</u>

$$\frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}}) M \boldsymbol{V}_{\mathbf{C}}) = M \frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}}) + (\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}}) \frac{dM}{dt}$$

But we know that

$$\frac{dM}{dt} = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})} \frac{1}{c^2} \frac{dE}{dt} - \gamma(\boldsymbol{V}_{\mathbf{C}})^2 M \frac{\boldsymbol{V}_{\mathbf{C}} \boldsymbol{a}_{\mathbf{C}}}{c^2}$$

$$But \frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}}) M \boldsymbol{V}_{\mathbf{C}}) = \frac{\partial}{\partial \boldsymbol{R}_{\mathbf{C}}} L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}, t\right)$$

$$=> M \frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}}) + (\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}}) \frac{dM}{dt} = \frac{\partial}{\partial \boldsymbol{R}_{\mathbf{C}}} L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}, t\right)$$

Then

$$M \frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathbf{C}})\boldsymbol{V}_{\mathbf{C}}) = \frac{\partial}{\partial \boldsymbol{R}_{\mathbf{C}}} L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}, t\right) - \alpha . \boldsymbol{V}_{\mathbf{C}}$$

with

- $\alpha = \gamma(\boldsymbol{V}_{\mathbf{C}}) \frac{dM}{dt} = \frac{dM}{dt^*}$
- $\frac{dM}{dt} = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} \frac{1}{c^2} \frac{dE}{dt} - \gamma(\boldsymbol{V}_{\mathbf{C}})^2 M \frac{\boldsymbol{V}_{\mathbf{C}} \boldsymbol{a}_{\mathbf{C}}}{c^2} = \left[-\gamma(\boldsymbol{V}_{\mathbf{C}})^2 M \frac{\boldsymbol{V}_{\mathbf{C}} \boldsymbol{a}_{\mathbf{C}}}{c^2}\right]_{if\ E=cte}$
- $M = \gamma^* . M_{\Sigma} + \frac{E^*_{em,eq}}{c^2} \left(\frac{r^*_{eq}}{r^*} + \frac{1}{3} \frac{{r^*}^3}{{r^*_{eq}}^3}\right)$

  $\approx \left(M_{eq} + \frac{1}{2} M_{\Sigma} \frac{1}{c^2} \left(\frac{dr^*}{dt}\right)^2 + \frac{E^*_{em,eq}}{c^2} 2 \frac{\left(r^* - r^*_{eq}\right)^2}{{r^*_{eq}}^2}\right)_{for\ a\ Newtonian\ internal\ oscillator}$

The internal dynamics influence the dynamics of the center of mass. This coupling is not due to an eventual relativistic behaviour of the internal dynamics but especially to the relativistic behaviour of the center of mass itself. Indeed, we see that in the Newtonian limit ($\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2} = 0$), there is no longer a viscous term where the coupling appears (except if energy is exchange $\frac{dE}{dt} \neq 0$ between the whole system and the exterior). This coupling is of course due to an exchange between the internal energy M and that of the center of mass (if E=cte). This variation of the internal energy modifies the inertia and then acts on the speed for a given momentum.

2.3. Hamiltonian analysis: Hamilton-Jacobi equation(an attempt) for a material system free

As for [1] and [2], we start from the norm equation:

$$\left(\frac{E^*}{c}\right)^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2$$

We have to express the different quantities in term of the action. For that, I search the expression of the action as the function of coordinates : that is to say the action resulting from the injection of the equation of motion in its variation. I need the 2 expressions below in term of coordinate:

- By Mixing internal and external degree of freedom $(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t)$
- And only using internal degree of freedom $(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*)$

- $S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) \equiv \{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}, t]\}_{real\ trajectory} = \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \frac{L^*}{\gamma}.dt = \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} L'.dt$

$=> \delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t)$

$$= \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \left(\sum_a \frac{\partial L'}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} \delta \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \frac{\partial L'}{\partial \boldsymbol{R}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}} + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{V}_{\mathrm{C}} + \frac{\partial L'}{\partial t}\delta t\right) dt$$

$$= \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \left(\sum_a \frac{\partial L'}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \frac{d}{dt}\left(\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\delta \boldsymbol{r}_{\boldsymbol{a}}^* - \delta \boldsymbol{r}_{\boldsymbol{a}}^* \frac{d}{dt}\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\right) + \frac{\partial L'}{\partial \boldsymbol{R}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}} \right.$$
$$\left. + \left(\frac{d}{dt}\left(\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}}\right) - \delta \boldsymbol{R}_{\mathrm{C}}\left(\frac{d}{dt}\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\right)\right) + \frac{\partial L'}{\partial t}\delta t\right) dt$$

$$= \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \left\{\sum_a \left(\frac{\partial L'}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} - \frac{d}{dt}\frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\right)\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \left(\frac{\partial L'}{\partial \boldsymbol{R}_{\mathrm{C}}} - \frac{d}{dt}\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\right)\delta \boldsymbol{R}_{\mathrm{C}} + \frac{\partial L'}{\partial t}\delta t \right.$$
$$\left. + \frac{d}{dt}\left(\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}}\right)\right\} dt$$

$$=> \delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) = \int_{t_1}^{t,\{\delta r_a^*\},\delta R_{\mathrm{C}}} d\left(\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}}\right)$$

Since

- $\left(\frac{\partial L}{\partial r_a^*} - \frac{d}{dt}\frac{\partial L\prime}{\partial \frac{d r_a^*}{dt}} = 0\right)_{for\ a\ real\ trajectory}$
- $\left(\frac{\partial L}{\partial \boldsymbol{R}_{\mathrm{C}}} - \frac{d}{dt}\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} = 0\right)_{for\ a\ real\ trajectory}$

Then we have the following result

$$=> dS(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) = \sum_a \left(\frac{\partial S}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} d\boldsymbol{r}_{\boldsymbol{a}}^*\right) + \frac{\partial S}{\partial \boldsymbol{R}_{\mathrm{c}}} d\boldsymbol{R}_{\mathrm{c}} + \frac{\partial S}{\partial t} dt$$

$$= \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} d\boldsymbol{r}_{\boldsymbol{a}}^* + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{c}}} d\boldsymbol{R}_{\mathrm{c}} + \frac{\partial S}{\partial t} dt$$

$$= \sum_a \boldsymbol{P}_{\boldsymbol{a}}^* d\boldsymbol{r}_{\boldsymbol{a}}^* + \boldsymbol{P} d\boldsymbol{R}_{\mathrm{c}} + \frac{\partial S}{\partial t} dt$$

Then

- $\boldsymbol{P}_{\boldsymbol{a}}^* \equiv \frac{\partial L'}{\partial \frac{dr_{\boldsymbol{a}}^*}{dt}} = \frac{\partial S}{\partial r_{\boldsymbol{a}}^*}$
- $\boldsymbol{P} \equiv \frac{\partial L'}{\partial V_{\mathrm{c}}} = \frac{\partial S}{\partial R_{\mathrm{c}}}$
- $L' \equiv \frac{dS}{dt}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t)$
- $H(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, \boldsymbol{V}_{\boldsymbol{c}}) \equiv \sum_a \boldsymbol{P}_{\boldsymbol{a}}^* \frac{dr_{\boldsymbol{a}}^*}{dt} + \boldsymbol{P}\boldsymbol{V}_{\boldsymbol{c}} - L' = \frac{\partial S}{\partial t}$
- $S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) \equiv \{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}(t)]\}_{real\ trajectory}$

- $S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*) = \{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*), t^*(t)\}]\}_{real\ trajectory} = \int_{t_1^*}^{t^*, \{\boldsymbol{r}_{\boldsymbol{a}}^*\}} L^* dt^*$

$$\delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*) = \{\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, t^*(t)]\}_{real\ trajectory} = \int_{t_1^*}^{t^*, \{\boldsymbol{r}_{\boldsymbol{a}}^*\}} \delta(L^* dt^*)$$

$$= \int_{t_1^*}^{t^*, \{\boldsymbol{r}_{\boldsymbol{a}}^*\}} \delta_{t^*}(L^* dt^*) + \delta_{\{\boldsymbol{r}_{\boldsymbol{a}}^*\}}(L^* dt^*) + \delta_{\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}}(L^* dt^*) + \delta_t(L^* dt^*)$$

$$= \int_{t_1^*}^{t^*, \{\boldsymbol{r}_{\boldsymbol{a}}^*\}} \left\{ L^* \delta_{t^*}(dt^*) + dt^* \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} \delta \boldsymbol{r}_{\boldsymbol{a}}^*\right) dt^* + \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} \delta \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right) dt^* \right\}$$

But dt^* is variable:

$$\delta \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} = \delta\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right) = \frac{\delta(d\boldsymbol{r}_{\boldsymbol{a}}^*)}{dt^*} + d\boldsymbol{r}_{\boldsymbol{a}}^* \delta\left(\frac{1}{dt^*}\right) = \frac{d\delta\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} + d\boldsymbol{r}_{\boldsymbol{a}}^*\left(\frac{-\delta dt^*}{{dt^*}^2}\right) = \frac{d\delta\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} - d\boldsymbol{r}_{\boldsymbol{a}}^*\left(\frac{d\delta t^*}{{dt^*}^2}\right)$$

$$= \frac{d\delta\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} - \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\left(\frac{d\delta t^*}{dt^*}\right)$$

$$\delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*) = \int_{t_1^*}^{t^*, \{r_a^*\}} \left\{ d(L^* \delta t^*) - \delta t^* dL^* + dt^* \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} \delta \boldsymbol{r}_{\boldsymbol{a}}^*\right) dt^* \right.$$
$$\left. + \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} \left(\frac{d\delta\boldsymbol{r}_{\boldsymbol{a}}^*}{ddt^*} - \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\left(\frac{d\delta t^*}{dt^*}\right)\right)\right) dt^* \right\}$$

$$= \int_{t_1^*}^{t^*,\{\boldsymbol{r}_a^*\}} \left\{ \frac{d(L^*\delta t^*)}{dt^*} - \delta t^* \frac{dL^*}{dt^*} + \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_a^*} \delta \boldsymbol{r}_a^* \right) + \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\delta \boldsymbol{r}_a^*}{dt^*} - \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \left(\frac{d\delta t^*}{dt^*} \right) \right\} dt^*$$

$$= \int_{t_1^*}^{t^*,\{\boldsymbol{r}_a^*\}} \left\{ \frac{d(L^*\delta t^*)}{dt^*} - \delta t^* \frac{dL^*}{dt^*} + \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_a^*} \delta \boldsymbol{r}_a^* \right) \right.$$

$$+ \left(\frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \delta \boldsymbol{r}_a^* \right) - \sum_a \delta \boldsymbol{r}_a^* \frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \right) \right)$$

$$\left. - \left(\frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \delta t^* \right) - \delta t^* \frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \right) \right) \right\} dt^*$$

$$= \int_{t_1^*}^{t^*,\{\boldsymbol{r}_a^*\}} \left\{ \delta t^* \left[-\frac{dL^*}{dt^*} + \frac{\partial L^*}{\partial t^*} + \frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \right) \right] + \sum_a \left(\frac{\partial L^*}{\partial \boldsymbol{r}_a^*} - \frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \right) \right) \delta \boldsymbol{r}_a^* \right.$$

$$\left. + \frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \delta \boldsymbol{r}_a^* - \delta t^* \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^* \right) \right) \right\} dt^*$$

- $\left(\frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \right) - \frac{dL^*}{dt^*} = -\frac{\partial L^*}{\partial t^*} \right)_{for\ a\ real\ trajectory}$
- $\left(\frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \right) = \frac{\partial L^*}{\partial \boldsymbol{r}_a^*} \right)_{for\ a\ real\ trajectory}$

Then we have the following result

$$=> dS(\{\boldsymbol{r}_a^*\}, t^*) = \sum_a \frac{\partial S}{\partial \boldsymbol{r}_a^*} d\boldsymbol{r}_a^* + \frac{\partial S}{\partial t^*} dt^*$$

$$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} d\boldsymbol{r}_a^* - \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^* \right) dt^*$$

$$= \sum_a \boldsymbol{P}_a^* d\boldsymbol{r}_a^* - E^* dt^*$$

Then

- $\boldsymbol{P}_a^* \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} = \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} = \frac{\partial S}{\partial \boldsymbol{r}_a^*}$
- $E^* \equiv \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^* = \frac{\partial S}{\partial t^*}$
- $L^* \equiv \frac{dS}{dt^*}(\{\boldsymbol{r}_a^*\}, t^*, t)$
- $H^*(\{\boldsymbol{r}_a^*\}, \{\boldsymbol{P}_a^*\}) \equiv \sum_a \boldsymbol{P}_a^* \frac{d\boldsymbol{r}_a^*}{dt} - L^* = \frac{\partial S}{\partial t^*}$
- $S(\{\boldsymbol{r}_a^*\}, t^*) \equiv \{S[\{\boldsymbol{r}_a^*(t^*)\}, t^*]\}_{real\ trajectory}$

$$\left(\frac{E^*}{c}\right)^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2$$

We have the first equation

$$\boxed{\frac{1}{c^2}\left(\frac{\partial S}{\partial t^*}\right)^2_{t^*,\{\boldsymbol{r}_a^*\}} = \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2_{\{\boldsymbol{r}_a^*\},\boldsymbol{R}_c,t} - \left(\frac{\partial S}{\partial \boldsymbol{R}_c}\right)^2_{\{\boldsymbol{r}_a^*\},\boldsymbol{R}_c,t}}$$

The expression uses the same quantity S but expressed as 2 functions of different variables.

We can also express the equation in term of internal position

$$E^* = \sum_a E_a^*$$

With $\left(\frac{m_a c^2}{c}\right)^2 = \left(\frac{E_a^*}{c}\right)^2 - \boldsymbol{P}_a^{*\,2}$

$$=> E_a^* = \sqrt{(m_a c^2)^2 + c^2 \boldsymbol{P}_a^{*\,2}}$$

$$=> E^* = \sum_a \sqrt{(m_a c^2)^2 + c^2 \boldsymbol{P}_a^{*\,2}}$$

$$=> \boxed{\left(\sum_a \sqrt{\left(\frac{m_a c^2}{c}\right)^2 + \left(\frac{\partial S}{\partial \boldsymbol{r}_a^*}\right)^2_{t^*,\{\boldsymbol{r}_a^*\}}}\right)^2 = \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2_{\{\boldsymbol{r}_a^*\},\boldsymbol{R}_c,t} - \left(\frac{\partial S}{\partial \boldsymbol{R}_c}\right)^2_{\{\boldsymbol{r}_a^*\},\boldsymbol{R}_c,t}}$$

$$<=> \boxed{\sum_a \sqrt{\left(\frac{m_a c^2}{c}\right)^2 + \left(\frac{\partial S}{\partial \boldsymbol{r}_a^*}\right)^2_{t^*,\{\boldsymbol{r}_a^*\}}} = \sqrt{\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2_{\{\boldsymbol{r}_a^*\},\boldsymbol{R}_c,t} - \left(\frac{\partial S}{\partial \boldsymbol{R}_c}\right)^2_{\{\boldsymbol{r}_a^*\},\boldsymbol{R}_c,t}}}$$

This equation is pretty complicated. To develop again the analysis, a quantum version (à la Schrödinger) using the action as the phase of a wave function would be interesting to obtain (with the internal degree of freedom and the center of mass as variables). Here we see that is seems not very straightforward (or not possible?).

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