The CADO Reference Frame for an Accelerating Observer

Author: Michael Leon Fontenot

Abstract:

The CADO reference frame[1] is defined for an observer who accelerates in any manner whatsoever. Specifically, the observer's acceleration *a(t)*, where *t* is any instant in the observer's life, can be whatever the accelerating observer wants it to be, without restriction.

Contents

The CADO Frame, for the Standard Twin Paradox Scenario

Although the CADO frame is applicable to *any* acceleration profile, the concepts and terminology needed to describe the CADO reference frame are most quickly and easily understood if they are initially couched in the context of the standard well-known twin "paradox" scenario.
First, consider the even simpler scenario where two perpetually-inertial observers are moving at some fixed velocity *v* relative to one another, and when they momentarily are co-located, they just happen to be exactly the same age then. For example, it could just happen that they are both born at that instant of their co-location, even though their mothers could have had a relative velocity of *v* at that instant. Since each of those newborns is a perpetually-inertial observer, they are each entitled to use the Lorentz equations to determine, at any instant of their own life, the current age of the other.[2] And each of them is entitled to use the well-known time-dilation result of special relativity[3] to determine how fast or how slowly the other is currently ageing, relative to their own ageing.

In the standard twin paradox, the "home twin" is perpetually inertial by assumption, and thus is entitled to use either the Lorentz equations or the time-dilation result (or both) to determine the current age of the "traveling twin". To allow more brevity and less clutter in the writing which follows, the home-twin will always be referred to as a "she", and the traveling twin will always be referred to as a "he".

The traveling twin must accelerate, in order to accomplish his turnaround, so he is *not* a perpetually-inertial observer, and his reference frame during his trip *cannot* be an inertial frame. Specifically, he is *not* allowed, during his *entire* trip, to use the time-dilation result to determine the current age of his twin. And, depending on exactly how his reference frame is defined, he *might* or *might not* be allowed to use the Lorentz equations at each instant of his life during his trip.

So what *is* the reference frame of the traveling twin? There are five requirements that any such frame must have.

1. It must be such that the traveler is perpetually located at its spatial origin.
2. It must specify how the traveler, at each instant of his life, is to determine the current age and the current position of each and every object (or person) in the (assumed flat) universe.
3. It must be internally consistent.
4. It must not contradict special relativity.
5. It must be such that the traveler and the home-twin agree with one another about the correspondence between their ages, when they are reunited.

More than one reference frame for an accelerating observer have been defined, and there is not yet a consensus about which one is most appropriate. This article describes one such reference frame: the CADO frame.

The CADO frame was originally inspired by an example (Example 49) in Taylor's and Wheeler's classic book.[4] The results of their example are consistent with those obtained from the common gravitational time dilation explanation, but do not depend on the use of any fictitious gravitational fields. Their basic approach is clearly applicable to scenarios with finite accelerations, although they didn't pursue that generalization. The CADO frame accomplishes that generalization.

Even though the frame of the traveling twin, since he accelerates during some portion his trip, cannot be an inertial frame, there is, at each instant *t*of the traveler's life, a unique inertial frame which is momentarily stationary with respect to the traveler at that instant, with a spatial axis pointing in the same direction as the home-twin's spatial axis, and such that the traveler is located at the spatial origin of that frame at that instant. Furthermore, for uniqueness, we require that the time coordinate of that inertial frame be equal to the traveler's age, at that instant. That unique inertial frame is called the "momentarily stationary inertial reference frame, at the instant *t* in the traveler's life", abbreviated as the MSIRF*(t)*. In general, MSIRF*(t)* will correspond to a *different* inertial frame from one instant in the traveler's life to the next. It is only during unaccelerated segments of the traveler's life that the MSIRF*(t)* will consist of the *same* inertial frame for the entire segment.

Given this (generally infinite) collection of inertial frames, the CADO frame is defined to be the single unique frame having the property that its conclusions about the current age and location of all objects or persons in the (assumed flat) universe, at any instant *t* of the traveler's life, is the same as the corresponding conclusions of the MSIRF*(t)*. I.e., at each instant of his life, the traveler adopts the viewpoint (about the simultaneity and location of distant objects) of the inertial frame with which he is momentarily stationary at that instant. The acronym "CADO" originates from the phrase "the current age of a distant object".

The CADO Equation

Given the above definition of the CADO frame, it is possible to derive a very simple, and very useful equation, called "the CADO equation",[5] which allows the traveler to determine, at each instant *t* in his life, the current age of any given distant perpetually-inertial object or person (the "home-twin" in the twin paradox scenario).

First, it is important to understand that, for any given instant *t* in the traveler's life, the home-twin and the traveler will generally *disagree* with one another about how old the home-twin is at that instant of the traveler's life. There are two quantities in the CADO equation which represent each of the twins' conclusions about the home-twin's age when the traveler's age is *t*. The quantity *CADO_T* denotes the *traveler's conclusion* about the home-twin's age, when the traveler's age is t, whereas the quantity $CADO\_H$ denotes the *home-twin's conclusion* about the home-twin's age, when the traveler's age is t.

The CADO equation can be written (most simply) as

$$CADO\_T = CADO\_H - v * L$$

where

v is their current relative speed, according to the home-twin, at the given instant t in the traveler's life, with v taken as positive when the twins are moving apart,

L is the distance from the home-twin to the traveler, at the given instant t in the traveler's life, according to the home twin,

and

the asterisk denotes multiplication.

Strictly speaking, the quantity $L(t)$ is the position of the traveler, relative to the home-twin, according to the home-twin, when the traveler's age is t. The distinction will be clarified later (in Section 5), but for now, it's simplest to just think of it as a distance (a number either positive or zero).

The above equation gives the relationship between those four quantities ($CADO\_T$, $CADO\_H$, v, and L), at the given instant t of the traveler's life. I.e., although it is not shown explicitly, each of the four quantities in the equation are functions of t.

What makes the CADO equation especially useful is that it allows the quantity $CADO\_T$, which is a quantity which is otherwise relatively difficult to determine, to be easily calculated from the other three quantities ($CADO\_H$, L, and v), which are each very easy to determine.

In order to make the equation strictly correct, a factor of $c*c$ dividing the last term is required, where c is the constant speed of any light pulse, as determined by any perpetually-inertial observer. If the time and spatial units are chosen so that c has unity value, the factor in that case is required only for dimensional correctness. In this article, units of years and lightyears will be used exclusively (but often abbreviated as y and ly), and the factor of $c*c$ will be suppressed entirely, purely for simplicity and brevity.

It can be immediately seen from the CADO equation that if at any instant , either v or L is zero, then CADO_T is equal to CADO_H. I.e., if, at any instant in the traveler's life, he is stationary with respect to his twin, then he will agree with her about their respective ages, regardless of how far apart they are. Or, if at any instant they are co-located, they will agree about their respective ages, regardless of whether or not they have any relative motion at that instant. And it is equally clear from the CADO equation that at any instant t when v and L are non-zero, the two twins will *not* agree about their respective ages. (This last statement is true for the one-dimensional motion we have been considering so far, but the statement must be modified for motion in two or three spatial dimensions. The higher-dimensional case will be addressed in a later section.)

I originally derived the CADO equation, many years ago, using only the Minkowski diagram for the twin "paradox" scenario. I show explicitly show how to do that derivation near the the end of Section 11, on "The Graphical Interpretation of the CADO Frame".

Idealized Instantaneous Velocity Changes

In the idealized, limiting case of the instantaneous turnaround usually assumed in the twin paradox scenario, the quantities *CADO_H* and *L* that are needed in the CADO equation are very easy to obtain, and *v* is given in the statement of the scenario.

For example, suppose that immediately after the twins are born, the traveling twin moves away from the home-twin at a constant relative velocity of 0.866 lightyears/year for 20 years of his life. That complicated-looking value of the velocity was chosen for this example because it produces the very nice value of 2 for the gamma factor (the time-dilation factor):

gamma = 1 / sqrt(1 - *v* * *v*) ,

where "sqrt( )" denotes the square-root operation, and where, again for simplicity, the factor *c*c* that should actually be dividing the *v*v* term has been omitted.

The traveler then instantaneously reverses course, and spends the next 20 years of his life returning to his home-twin. The magnitude of his velocity is still 0.866 ly/y, but since he is now moving *toward* his twin, by convention his velocity is now negative, -0.866 ly/y. Since *gamma* depends only on the magnitude of the velocity, *gamma* is still equal to 2.

So, the traveler is 20 years old at his turnaround, and 40 years old when he is reunited with his twin. Since the home-twin is perpetually inertial, she is entitled to use the time dilation result for his entire trip. Since *gamma* = 2 for the entire trip, she concludes that the traveler ages half as fast as she herself does, so she concludes that she is 40 years old when he turns around, and 80 years old when they are reunited. (Of course, when they are reunited, they will each *know* both of their ages). So, just from the time-dilation result, we've been able to quickly determine that

CADO_H(20) = 40 years old.

Now, from the definition of the CADO frame, the MSIRF*(t)* for all *t* from 0 years up to, but not including, 20 years, is the *same* inertial frame ... it's the one which is moving at a velocity relative to the home-twin of 0.866 ly/y, and in which the traveler is located at the spatial origin. During that entire segment, 0 <= *t* < 20, the traveler (by definition) agrees with that single MSIRF about the age of any distant inertial object or person, and thus he also agrees with that MSIRF about how fast or how slowly any distant person is ageing, compared to his own ageing. So, during that outbound leg (but not including the instant at *t* = 20), the traveler is entitled to use the time-dilation result, and he concludes that the home-twin is ageing half as fast as he himself is. So he concludes that, right at the end of his constant-velocity outbound leg (but before he does his instantaneous turnaround), that the home-twin is 10 years old. Therefore we've been able to determine that

CADO_T(immediately before turnaround) = 10 years old.

The fact that the traveler is entitled to use the time-dilation result, during his entire unaccelerated outbound segment, is also true of *any* unaccelerated segment, of finite duration, in his life. During any unaccelerated finite segment of his life, he is a *full-fledged* inertial observer during that entire segment, and he is entitled to use the Lorentz equations to determine simultaneity at a distance, and he is entitled to use the time-dilation and length-contraction results that follow from the Lorentz equations.

So, for the entire outbound leg, we didn't need to use the CADO equation at all ... the time-dilation result was all that we needed. But we *do* need the CADO equation in order to determine what happens during the turnaround, right at the instant *t* = 20 years. How do we do that?

To make use of the CADO equation during the turnaround, we need to know the values of the three quantities on the right-hand-side of the CADO equation (*CADO_H*, v, and L), immediately before and immediately after the instantaneous turnaround. *CADO_H* and L are quantities that are computed in the home-twin's inertial frame, and they are always continuous ... they never change discontinuously, even when v changes discontinuously. So *CADO_H* and L don't change during the turnaround, but v does change.

We can denote the instant in the traveler's life, immediately *before* the turnaround, as t = 20-, and the instant immediately *after* the turnaround as t = 20+. So, we have

v(20-) = 0.866 ly/y,

and

v(20+) = -0.866 ly/y.

We also already know that

CADO_H(20-) = *CADO_H*(20+) = *CADO_H*(20) = 40 years.

So all we still need to determine is L(20). How do we do that? We know that, in the home-twin's frame, the velocity of the traveler is 0.866 ly/y during the outbound frame, and we know that that outbound leg lasts for 40 years of the home-twin's life, so she will conclude that the traveler's distance from her at the turnaround is

L = 0.866 * 40 = 34.64 ly.

Since, in the CADO equation, all of the quantities need to be specified as functions of the variable t (the traveler's age), we therefore have

L(20-) = L(20+) = L(20) = 34.64 ly.

So, we've got all the quantities we need, to evaluate *CADO_T*(20-) and *CADO_T*(20+) using the CADO equation. We actually were already able to determine *CADO_T*(20-) using only the time-dilation result for the outbound leg ... we got the value 10 years. But it is instructive to use the CADO equation for the instants immediately before and immediately after the instantaneous turnaround, just to understand why the CADO frame concludes that the home-twin's age abruptly changes during the instantaneous turnaround. Immediately before the turnaround, we get

CADO_T(20-) = *CADO_H*(20-) - v(20-) * L(20-) = 40 - 0.866 * 34.64,

so

CADO_T(20-) = 40 - 30 = 10 years.

And, immediately after the turnaround, we get

CADO_T(20+) = *CADO_H*(20+) - v(20+) * L(20+) = 40 + 0.866 * 34.64,

so

CADO_T(20+) = 40 + 30 = 70 years.

So, the CADO equation says that, according to the traveler, the home-twin instantaneously get 60 years older during his instantaneous turnaround. And the CADO equation makes it clear *why* the traveler's abrupt velocity change causes (according to the traveler) the abrupt change in the home-twin's age: by definition, at any instant t of the traveler's life, he adopts as his own the conclusions of his MSIRF, at that instant, about simultaneity. The MSIRF at the instant immediately *before* the turnaround, *MSIRF*(20-), and the MSIRF at the instant immediately *after* the turnaround, *MSIRF*(20+), have very different conclusions about the current age and current position of the home-twin.

The change in the home-twin's age, before and after the instantaneous velocity change, is

delta_CADO_T(20) = *CADO_T*(20+) - *CADO_T*(20-),

and since nothing on the right-hand-side of the CADO equation changes during the instantaneous turnaround except the velocity, we get the very simple equation

delta_CADO_T(20) = - *L*(20) * (*v*(20+) - *v*(20-))

or

delta_CADO_T(20) = - *L* * delta_v(20).

So, getting the change in the home-twin's age during an instantaneous velocity change is very simple: you just multiply the negative of their separation by the change in the velocity.

Note that in this case (for the turnaround that occurs in the standard twin paradox scenario), the change in the velocity is *negative*:

delta_v(20) = *v*(20+) - *v*(20-) = (-0.866) - (0.866) = -1.732,

and so the change in the home-twin's age is

delta_CADO_T(20) = -34.64 * (-1.732) = 60 years.

But note that, for other scenarios, the traveler *could* change his velocity from (say) -0.866 ly/y to +0.866 ly/y (corresponding to an acceleration *away from* the home twin), and in that case, his velocity change would be positive (+1.732), and so the home-twin's age change would be -60 years .... i.e., she would suddenly get 60 years *younger* (according to the traveler).

The fact, that the traveler concludes that the home-twin's age changes abruptly whenever he abruptly changes his velocity, certainly has no impact on the home-twin's own perception of the progression of her own age. Lots of additional accelerating observers would generally come to very different conclusions about the way her age changes while they accelerate in various ways, and it is really of no consequence to her *what* they conclude. But no one's conclusions are any more correct than any one else's conclusions. They are *all* correct ... in special relativity, different observers generally just have to agree to disagree.

To complete our application of the CADO frame to the standard twin paradox, we've still got to analyze the inbound leg. The analysis is essentially the same as for the outbound leg. Since the traveler is unaccelerated during the entire inbound leg, the CADO frame says that the traveler is a full-fledged inertial observer during that entire 20-year segment of his life. So he uses the time-dilation result, and concludes that the home-twin ages 10 years during the inbound leg. So, when they are reunited, she is 80 years old, and he is 40 years old. The home-twin and the traveler agree, about the correspondence between their two ages, when they are reunited (as of course they must), even though they generally disagreed about that correspondence, during the trip.

Instead of using the time-dilation result to determine CADO_T when the twins are reunited (as we did above), we can also easily get the answer from the CADO equation: since L is obviously zero when they are reunited, the CADO equation says that CADO_T = CADO_H there (and so the twins agree about their ages there).

Given the above results, it is easy (and very useful) to sketch an "age-correspondence graph" ... a plot of the home-twin's age (according to the traveler) as a function of the traveler's age. I.e, we want a graph, with the home-twin's age plotted vertically, and the traveler's age plotted horizontally. (The following description is most easily understood if the reader roughly sketches the graph as the description proceeds). What does that graph look like?

On the outbound leg, the traveler says that the home-twin's age increases half as fast as his own age. So the curve starts from the origin, and increases linearly along a straight line of slope 1/2, until his (the traveler's) age is 20, and her (the home-twin's) age is 10. At that point, the curve jumps vertically to 70 for her age (with no increase in his age). Finally, the curve increases linearly from there, along a straight line of slope 1/2, until she reaches 80 years old, and he reaches 40 years old. After that, as long as they remain together, they will age at the same rate, but she will always be 40 years older than he is.

The home-twin can do her own age-correspondence graph, again with her age plotted vertically, and his age plotted horizontally. I.e., both graphs show her age as a function of his age; the only difference is that the two graphs show the conclusions of two different observers.

Her graph will be quite different from his graph: hers will consist of a single, straight line of slope 2, because the time-dilation result tells her that, during his entire trip, *he* ages half as fast as *she* does, which means that *she* ages twice as fast as *he* does. But the two different graphs *do* start at the same point (the origin), and they *do* end at the same point (the point where she is 80, and he is 40). But in between those two points, the curves are very different.

In the standard paradox scenario (with a single instantaneous velocity change, and a reunion at the end of the trip), it is actually possible to avoid having to use the CADO equation to determine how the home-twin's age changes during the turnaround. That change can simply be *inferred* by determining the sum of the amount of her ageing (according to him) during the two unaccelerated segments of his life (10 + 10 = 20 years), and then using the fact that her age at the end of the trip *must* be 80 years. So we have to come up with an additional 60 years somewhere, and the turnaround is the *only* place that extra time could have occurred.

But for more complicated scenarios, where the traveler can instantaneously change his velocity *multiple* times during the trip (both positively and negatively), and in cases where there is never any reunion of the twins), then the CADO equation is indispensable in determining how much the distant perpetually-inertial person instantaneously ages (positively or negatively) during the traveler's instantaneous velocity changes. And even in the standard paradox scenario, the use of the CADO equation at the turnaround makes it clear *why* the home-twin's age (according to the traveler) instantaneously increases during the instantaneous turnaround. And the CADO equation also makes it clear why the traveler's *initial* instantaneous velocity change (when he begins his trip), and his *final* instantaneous velocity change (when they are reunited), does *not* cause any instantaneous change in her age (because *L* is zero then).

Finite Accelerations

In all of the above, the non-inertial behavior by the traveler consisted only of instantaneous velocity changes. But the CADO frame, and the CADO equation, is not restricted to these idealized, limiting cases ... the traveler can accelerate in *any* manner that he chooses. I.e., he can choose any function *a(t)* for his acceleration, for *t* ranging over his entire life.

For any choice of the acceleration profile *a(t)*, the CADO equation remains exactly the same as given above. The only difference is that the quantities *v(t)*, *CADO_H(t)*, and *L(t)*, on the right-hand-side of the CADO equation, are no longer quite as simple to determine. For completely general acceleration profiles *a(t)*, all three quantities will generally require numerical integration for their determination. For the (very useful and important) cases that consist of a sequence of segments of the traveler's life in which his acceleration is constant within each segment (and possibly including segments of zero acceleration ... coasting), each of the three quantities needed for evaluation of the CADO equation can be determined analytically. But in any case, once those three quantities have been determined (for any given age of the traveler), the quantity *CADO_T* can be determined from the same CADO equation, with (as always) only a single multiplication and a single addition or subtraction.

The way the three quantities *v*, *CADO_H*, and *L* can be determined, for each instant of the traveler's life, will first be very briefly verbally described. Since all three quantities correspond to the conclusions of a perpetually-inertial observer (the "home-twin"), their determination is fairly widely known. For example, Taylor and Wheeler[6] use basically the same approach in their Example 51 of how far a traveler can go, by constantly accelerating at 1g in a straight line.

The acceleration, *a(t)*, at any given instant *t* of the traveler's life, is the acceleration that would be measured on an accelerometer carried by the traveler (taken as positive when directed away from the home-twin, and negative when directed toward her). This acceleration is *not* the acceleration that would be measured by the home-twin. At each instant *t* , it is the acceleration that would be measured by the MSIRF(*t*) , i.e., by the traveler's MSIRF at that instant. The particular MSIRF doing the measurement is generally *different* from one instant to the next. The entire acceleration profile, for the whole range of *t* corresponding to the traveler's life, is *not* what would be measured by any one single inertial frame.

Once we know the function *a(t)* for the entire trip of the traveler, we can compute the "rapidity" *eta(t)*. The rapidity is a one-to-one nonlinear function of the velocity *v*, having the needed property that it is linearly additive across inertial reference frames (the velocity *v* itself is *not* linearly additive across inertial frames). Specifically, if we know what the infinitesimal changes in the rapidity is, according to each MSIRF in any finite segment of the traveler's life, we can just add up all those infinitesimal changes to get the total change in the rapidity over that whole finite segment. So, we can get the rapidity *eta(t)* for the entire trip simply by integrating the acceleration *a(t)* with respect to *t*, over the range of *t* corresponding to the entire trip. (In the above description of the calculations required, and in subsequent descriptions, there are actually some factors of *c* that, strictly speaking, should be present, but we will always choose our units such that $c = 1$, and so those factors of *c* are needed only for dimensional correctness. In the interest of simplicity of description, those factors will be omitted here. They can always be inserted wherever needed, if required.)

For completely general acceleration profiles *a(t)*, the integral to get *eta(t)* must be calculated numerically. But in the very important (and very useful) special cases where *a(t)* is some sequence of segments in which the acceleration is constant (positive, negative, or zero) within each segment, the change in *eta(t)* over any given one of those segments is just equal to *A* times the duration of that segment, where *A* is the value of the constant acceleration in that segment. So *eta(t)* is very easy to determine for those cases.

Once we know *eta(t)*, we can compute *v(t)*, because *v* is the hyperbolic tangent of *eta*. And once we know *v(t)*, we can compute *gamma(t)*.

CADO_H(t) can then be computed as the integral, with respect to *t*, of *gamma(t)*. In the general case, that integration will also have to be carried out numerically. But in the cases where *a(t)* is piecewise-constant, the change in *CADO_H(t)*, within each segment, is just the total change in the hyperbolic sine of *eta(t)* within that segment, divided by the constant acceleration *A* in that segment.

Finally, *L(t)* can be computed as the integral, with respect to *t*, of *v(t)* * *gamma(t)*, which again requires numerical integration in the general case. In the piecewise-constant cases, the evaluation is handled just like the CADO_H evaluation, except that the hyperbolic cosine is used instead of the hyperbolic sine.

The above calculations, in the case of the piecewise-constant accelerations, are very easy and quick to carry out on a computer, and can even be done (although with considerably more effort) with a good hand-calculator, if absolutely necessary.

Here are the explicit equations that are verbally described above:

The basic CADO equation, in its more general form, and with the traveler's age "t" shown explicitly, is

CADO_T(t, beta(t)) = CADO_H(t) - beta(t) * L(t) / c,

where beta is the dimensionless quantity

beta(t) = v(t) / c.

The velocity parameter beta(t) is related to the rapidity parameter theta(t) via

beta(t) = tanh [theta(t)],

or with the inverse

theta(t) = arctanh [beta(t)].

The rapidity, in the most general case of a completely arbitrary acceleration profile, is proportional to the integral of the acceleration a(t):

theta(t) = theta(t_0) + (1/c) * int_from_t_0_to_t (a(s) ds).

The quantity CADO_H(t) is the integral of the gamma parameter, gamma(t):

CADO_H(t) = CADO_H(t_0) + int_from_t_0_to_t (gamma(s) ds),

where gamma(t) in terms of beta(t) is

gamma(t) = 1 / sqrt (1 - beta(t) * beta(t)).

The distance L(t) is proportional to the integral of beta(t) * gamma(t):

L(t) = L(t_0) + c * int_from_t_0_to_t (beta(s) * gamma(s) ds).

In the very important special case where the acceleration is piece-wise constant, the above quantities become (during any interval starting at time t_1 in which the acceleration "A" is constant)

theta(t) = theta(t_1) + (1/c) * A * (t - t_1)

CADO_H(t) = CADO_H(t_1) + c * (1 / A) * { sinh [theta(t)] - sinh [theta(t_1)] }

L(t) = L(t_1) + (c * c) * (1 / A) * { cosh [theta(t)] - cosh [theta(t_1)] }.

The hyperbolic trig functions in the above can be calculated with a good "scientific" calculator via

arctanh (beta) = (1/2) * ln ([1 + beta] / [1 - beta]) tanh (theta) = [exp (theta) - exp (-theta)] / [exp (theta) + exp (-theta)]

sinh (theta) = (1/2) * [exp (theta) - exp (-theta)]

cosh (theta) = (1/2) * [exp (theta) + exp (-theta)],

where "ln" in the first equation is the natural logarithm (base "e").

Current Position of a Distant Perpetually-Inertial Object or Person

The foregoing descriptions have described the CADO equation, and have given a brief description of how the three required quantities on the right-hand-side of that equation can be determined, both for the idealized cases of instantaneous velocity changes, and for completely general finite accelerations, and also for the especially useful cases of piecewise-constant accelerations. Those results satisfy the requirement that a reference frame for an accelerating observer must specify how the observer is to determine, at each instant of his life, the current *age* of any given distant object or person. But a reference frame must also specify how the observer can determine the current *position* of that distant person, at each instant of his life. That turns out to be very easy, for the CADO frame.

Each of the accelerating observer's MSIRFs (one for each instant *t* of his life) will conclude that the current position of the distant person, at the instant *t* when the inertial frame of that MSIRF is momentarily stationary with respect to the accelerating observer, is

L_T(t) = - L(t) / gamma(t),

where *L(t)* is the position of the accelerating traveler, according to the distant inertial person, when the accelerating observer's age is *t*.

The minus-sign above requires some elaboration. *L(t)* was defined earlier as the *distance* to the traveler, when his age is *t*, according to the home-twin. That was done because that terminology makes the CADO equation more intuitive, and easier to initially understand. But that terminology isn't completely precise. The term "distance" normally is understood to be a positive quantity, whereas a "position" (in one-dimensional space) can be either positive of negative. In usual descriptions of the standard twin paradox scenario, for simplicity the issues of how spatial axes are chosen are usually not discussed, and it is just tacitly assumed that the position of the traveler can just be specified by giving a (positive) distance to the traveler. But, in more flexible scenarios, if the traveler returns, but continues to travel on past the home-twin, then his position, relative to the home-twin, will be *negative*. So, to be precise, the quantity *L(t)* in the CADO equation is actually defined as the *position* of the traveler when his age is *t*, relative to the home-twin, according to the home-twin. If the traveler's outbound velocity is positive, then his position during the trip will be positive, and thus his position is always the same as his distance from her, provided that he doesn't go on past her when he returns. For the case where *L(t)* is positive, then the position of the home-twin, relative to the traveler, when the traveler's age is *t*, will be *negative* (because the position of any person P, relative to any person Q, is *always* the negative of the position of person Q, relative to person P). That's why the minus-sign is present in the above equation for *L_T(t)*. The term "distance" (relative to the spatial origin) corresponds to the *absolute value* of some given position.

Since *gamma(t)* can change very quickly, for small increases in *t*, it's clear from the above equation that *L_T(t)* can also change very quickly. For idealized instantaneous velocity changes, the position of the home-twin, according to the traveler, will instantaneously change, and so he will conclude that her distance from him has instantaneously increased or decreased.

Some Additional CADO Equation Results for Instantaneous Velocity Changes

It is easy to see from the *delta_CADO_T* equation that, for instantaneous velocity changes, it is possible for the current age of the distant perpetually-inertial person, in years, to instantaneously vary over an open time-interval, in years, which is (almost) numerically equal to twice their current separation, in lightyears. For example, if their separation *L* at the instant *t* in the traveler's life when the velocity change occurs (according to the distant person) is 40 lightyears, and if the velocity immediately before and immediately after the velocity change is (almost) +1 ly/y and -1 ly/y, respectively, then

delta_CADO_T(t) = - *L(t)* * (*v(t+)* - *v(t-)*) = (-40) * (-2) = 80 years (almost).

The above example corresponds to the case where the traveler is moving *away from* the distant person at a velocity arbitrarily close to the velocity of light, and then instantaneously reverses course, and moves *toward* her, again at a velocity arbitrarily close to the velocity of light. In that case, the distant person instantaneously gets *older* by an amount arbitrarily close to 80 years.

In the opposite extreme case, where the traveler is initially moving *toward* the distant person at almost the velocity of light, and then instantaneously reverses course, and moves *away from* her, again at almost the velocity of light, then the delta*(CADO_T)* equation gives -80 years ... i.e., she instantaneously gets *younger* by (almost) 80 years.

Of course, depending on the current age of the distant person immediately before the instantaneous velocity change, the age change of +80 years might well exceed her indisputable age at death. In that case, the traveler is really determining how old she currently is, assuming that she is still alive. Similarly, the age change of -80 years might well precede her birth, which really just tells the traveler how much *her mother's* current age has decreased during his velocity change. Of course, the CADO equation is *actually* telling the traveler what the current date and time is, in the inertial frame in which the distant person (and her predecessors) are perpetually inertial. Stated another way, the CADO equation just determines *simultaneity*, according to an accelerating observer. Couching the CADO equation in terms of the age of a particular distant person is just a way to make it more intuitively meaningful, and less abstract.

Some CADO Equation Results for Finite Accelerations

One might reasonably suspect that the results for the idealized cases of instantaneous velocity changes are of no value in understanding what happens for actual realizable accelerations, where velocities don't change instantaneously. But examples obtained by evaluating the CADO equation for finite accelerations show that, provided the separation is sufficiently great, the age changes of the distant perpetually-inertial person are qualitatively quite similar to the idealized results, even for perfectly reasonable 1 g accelerations. (It just happens that a 1 g acceleration is very close to an acceleration of 1 ly/y. More precisely, 1 ly/y is approximately equal to 0.970 g, and 1 g is approximately equal to 1.03 ly/y.) For 1 g accelerations, the age changes of the distant person aren't *discontinuous*, but her age changes (both positive and negative) can be very large, for relatively small increases in the age of the traveler.

For example, suppose that he (the traveler) and she (the home-twin) happen to be separated by 39.97 lightyears, when he is 26 years old, and she is 47.93 years old (all according to *her*). His velocity at that instant happens to be +0.7739 ly/y (he is moving away from her), and *gamma* therefore equals about 1.58.

The CADO equation says that, at that instant (when his age t is 26), that her current age (according to him) is

CADO_T(26) = *CADO_H*(26) - *v*(26) * *L*(26) = 47.93 - (0.7739) * 39.97 = 17.00 years old.

Then, he accelerates at -1 g for two years (of his life). I.e., he points his rocket ship toward her, and fires his rocket engine for 2 years.

During the first half of that acceleration, he is slowing down, but is still getting farther away from her. Half way through the acceleration (when he is 27), he momentarily comes to a standstill, and their separation is 40.53 lightyears then. She is then 49.12 years old (according to her). As can easily be seen from the CADO equation, they will *always* agree about their corresponding ages whenever their relative velocity *v* is zero. So he also concludes that she is 49.12 years old at that instant.

During the second half of his -1 g acceleration, he is moving back toward her, and speeding up. At the end of that acceleration, he is 28 years old, she is 50.31 years old, their separation is again 39.97 lightyears, and his velocity is -0.7739 ly/y, all according to *her*. The CADO equation then says that she is 81.24 years old (according to *him*).

So he concludes that, during that entire -1 g acceleration, she gets 64.24 years older, whereas he only got 2 years older. During the -1 g acceleration, she doesn't *instantaneously* get older, but she *does* age *much* faster than he does. So it is qualitatively fairly similar to what happens for the idealized instantaneous velocity-change case.

We can also use the CADO equation to determine the values of the various quantities for as many intermediate times during that -1 g acceleration as we want. Then, we can plot the age-correspondence graph ... the plot of the home-twin's age (according to the traveler) as a function of the traveler's age. I.e, we want a graph that has the home-twin's age plotted vertically, and the traveler's age plotted horizontally. We did this earlier for the case of the instantaneous turnaround in the standard twin paradox. (Again, the reader is encouraged to roughly sketch the graph as the following description proceeds). What does the curve look like for this -1 g acceleration?

In this case, the curve starts out, when the magnitude of the velocity is fairly high (0.7739 ly/y), with a positive slope of about 17. I.e., she is ageing then about 17 times faster than he is. (He is 26 then, and she is 17). As the velocity *v* decreases, the slope of the curve gets steeper, reaching a maximum of about 41 at the instant when *v* is momentarily zero. I.e., at that instant, she is ageing about 41 times faster than he is. (He is 27 then, and she is 49.12). Then, as his velocity increases again (negatively, toward her) during the second half of the -1 g acceleration, the slope of the curve again decreases, until it gets to about 17 at the end of the acceleration, when his velocity reaches -0.7739. I.e., at the end, she is again ageing about 17 times faster than he is. (He is 28 then, and she is 81.24). Overall, during the entire two-year -1 g acceleration, she aged about 32 times faster than he did (64.24 / 2). So the curve has a steep, thin "S" shape. Actually, it's more accurate to say that the shape of the curve is similar to the shape of an integral sign, because the slope of the curve never changes sign during the constant acceleration.

At the end of the -1 g acceleration, suppose that he turns his spaceship around (pointing it *away from* her), and accelerates at +1 g for the next two years of his life. During the first half of that acceleration, he is slowing down, but is still getting closer to her. Half way through the acceleration (when he is 29), he momentarily comes to a standstill, and their separation is 39.41 lightyears then. She is then 51.49 years old (according to her). Again, they will *always* agree about their corresponding ages whenever their relative velocity *v* is zero. So, since *v* is zero at that instant, he agrees that she is 51.49 years old then.

During the second half of his +1 g acceleration, he is moving away from her, and speeding up. At the end of that acceleration, he is 30 years old, she is 52.68, their separation is again 39.97 lightyears, and his velocity is +0.7739 ly/y, all according to *her*. The CADO equation then says that she is 21.75 years old, according to *him*.

So he concludes that, during that entire +1 g acceleration, she gets 59.49 years *younger*, whereas he got 2 years older. During the +1 g acceleration, she doesn't *instantaneously* get younger, but her age *does* decrease *much* faster than his age increases. So it is qualitatively fairly similar to what happens for an idealized instantaneous velocity-change.

If we compute more intermediate data during that +1 g acceleration, we can continue the age-correspondence graph that we drew above for the preceding -1 g acceleration. We get a curve similar to what we got before, but this time the slopes are negative, and the curve is like a "steep thin *backward* S shape". Again, a more accurate description of the shape is to say that it is similar to a "backwards" integral sign, i.e., an integral sign that has been reversed left-to-right (or rotated 180 degrees about its vertical axis).

In this case, the curve starts out, when the magnitude of the velocity is fairly high (-0.7739 ly/y), with a negative slope of about -16. I.e., she is getting younger then about 16 times faster than he is getting older. (He is 28 then, and she is 81.24). As the magnitude of the velocity *v* decreases, the slope of the curve gets steeper, reaching a maximum of about -39 at the instant when *v* is momentarily zero. I.e., at that instant, she is getting younger about 39 times faster than he is getting older. (He is 29 then, and she is 51.49). Then, as his velocity increases again during the second half of the +1 g acceleration, the curve again gets less steep, until the slope gets to about -16 at the end of the acceleration, when his velocity reaches +0.7739. I.e., at the end, she is again getting younger about 16 times faster than he is getting older. (He is 30 then, and she is 21.75). Overall, during the entire +1 g acceleration, she got younger about 30 times faster than he got older (-59.49 / 2).

A useful rule of thumb, provided their separation is sufficiently great, is that for a +-1g acceleration, the maximum rate of change in the age of the distant perpetually-inertial person, relative to the traveler's rate of ageing, will be approximately numerically equal to their separation, in lightyears. When the acceleration is directed *toward* the distant person, she will be getting *older* at that relative rate. When the acceleration is directed *away from* the distant person, she will be getting *younger* at that relative rate, as the traveler gets older. And in either case, that maximum relative rate will occur when accelerating through zero relative speed.

Here are the explicit equations:

The derivative of CADO_T can be shown to be

d(CADO_T)/dt = (1 / gamma) - (1 / (c*c) * (a * L / [gamma * gamma]),

where all of the above quantities are understood to be functions of the traveler's age "t". Now consider the special case where the acceleration a(t) is piecewise-constant and equal to "A" in the interval of interest. Then if the above equation is differentiated wrt t, and the result set equal to zero, the result is that beta is zero. So the home twin's rate of ageing (in magnitude), relative to the traveler's rate of ageing, occurs when the traveler is accelerating through zero relative speed. So we then have

max { abs [d(CADO_T)/dt] } = abs [1 - (1 / {c*c} * A * L)].

The above equation says that, using units of years and lightyears (so that c = 1), and when the acceleration is plus or minus 1 ly/y/y, and when abs(L) >> 1 ly, then the home twin's maximum (in magnitude) rate of ageing is greater than the traveler's rate of ageing by a factor approximately equal to their distance apart, as measured in the home twin's frame, in lightyears.

Another interesting result of the CADO frame, is that, if an observer, at some instant of his life, begins some constant acceleration (either positive *or* negative) that lasts for the rest of his (assumed very long) life, then the distant perpetually-inertial person's age (according to the accelerating traveler) will approach a finite limit. And if their separation, at that beginning instant, has a certain critical value, the distant person's age will not change at all, from that initial value, at all later times.

Here are the details:

If the traveler accelerates forever after some initial time t_1 in his life, with a constant acceleration A ly/y/y, then CADO_T(t) will approach some finite limit as t goes to infinity. There is a critical distance L_c such that, if L is equal to L_c when the constant acceleration begins, then CADO_T will never change after the constant acceleration begins.

If the initial distance L(t_1) is greater than L_c, then CADO_T(t) will decrease if the acceleration A is positive, and will approach the finite limit from above. (The acceleration is positive when it is in the direction of positive velocity. When L is positive, the velocity v will be positive when the home twin and the traveler are moving apart.)

If the initial distance L(t_1) is less than L_c, then CADO_T(t) will increase if the acceleration A is positive, and will approach the finite limit from below.

The critical distance is

$$L\_c = \{ (c*c) * \text{gamma} (t\_1) \} / A,$$

where "A" is the constant acceleration in ly/y/y.

The finite limit approached by CADO_T(t), as t goes to infinity, is

$$\lim \text{CADO\_T}(t) = \text{CADO\_T}(t\_1) + \text{sign}(A) * \{(c/A)*\text{sqrt}([1-\text{beta}(t\_1)]/[1+\text{beta}(t\_1)]) - L(t\_1)/c\}.$$

Velocities, According to the Accelerating Observer

The velocity v, that appears in the CADO equation, is the velocity of the observer, relative to the perpetually-inertial distant person, according to that distant person. Each of the traveler's MSIRFs agrees with the distant inertial person's conclusions about that relative velocity. Similarly, the velocity of any light pulse, c, is the velocity of that light pulse according to the distant inertial person, and all inertial observers agree about that.

But an accelerating observer will generally *disagree* with the distant person about their relative velocity[7]. And he will generally disagree with her about the velocity of any given light pulse.

The velocity of the accelerating observer, relative to the distant inertial person, according to the accelerating observer, is

$$v\_T = v - (L * v * a) / gamma,$$

where L is the position of the accelerating observer, relative to the distant inertial person, according to the distant person. The quantity a is the observer's acceleration (as measured on the accelerating observer's accelerometer), in ly/y. a is positive when in the direction of positive v. All of the quantities in the equation are for some arbitrary, but given, instant of the accelerating observer's life.

Since the acceleration a can be arbitrarily large, and either positive or negative, it's clear that the accelerating observer can conclude that the magnitude of their relative velocity is much larger, or much smaller, than the distant inertial person (and the MSIRF) says it is. In particular, it can be larger than c. And the accelerating observer can conclude that the direction of their relative velocity is the *opposite* of what the distant person, and the MSIRF, say it is. Also, since the acceleration a can change essentially instantaneously, v_T can change essentially instantaneously.

According to the accelerating observer, the velocity of some given light pulse is

$$c_T = c + a * R_T / c,$$

where c is the velocity of light, according to any inertial observer, a is the acceleration in ly/yr, and R_T is the position to the light pulse, relative to the accelerating observer, according to the accelerating observer. R_T, a, c, and c_T are positive when in the accelerating observer's positive spatial direction. The quantity c, which usually denotes a positive constant, is here a *signed* quantity (a one-dimensional vector), positive in the distant person's positive spatial direction, but negative in her opposite spatial direction.

Note that, according to the accelerating observer, the velocity of a light pulse depends on how far away it is from him. And a light pulse, as it passes him, always has the velocity c, regardless of his acceleration (because R_T is zero then).

Note also that, since his acceleration a can be arbitrarily large in magnitude, and either positive or negative, he can conclude that the velocity of a distant light pulse is much larger or much smaller than inertial observers say it is. And he can conclude that the pulse is moving in the opposite direction than the inertial observers say it is. Also, since the acceleration a can change essentially instantaneously, c_T can change essentially instantaneously.

The Non-Invertibility of the CADO Frame

It is important to stress that all of the age correspondences to be discussed in this section (as in most of the sections as well) are *according to the traveler*. The home-twin will generally come to very different conclusions about the correspondences between their ages.

In the example of the previous section, of a -1 g acceleration lasting for two years of the traveler's life, followed immediately by a +1 g acceleration lasting for another two years of his life, we got an age-correspondence graph that shows how the distant perpetually-inertial person's age changes (according to the traveler) during those four years of the traveler's life. That continuous curve looks a bit like a very high, but very narrow mountain peak, rising from 17 years old for her age when he is 26, to a peak of 81.24 years old for her age when he is 28, then back down to 21.75 years old for her age when he is 30.

For each age of the traveler during that segment between when he is 26 years old and when he is 30 years old, there is some specific value for her current age then. For example, the question "How old is *she*, when *he* is t years old?", where t is some age between 26 and 30, *always* has an answer, and it *never* has more than one answer. As a specific example, when he is 26.8 years old, she is 48.92 years old.

But if you ask "How old is *he*, when *she* is 48.9 years old?", you *don't* get only one answer. During his two year -1 g acceleration, he was 26.8 years old when she was about 48.9 years old. But during his +1 g acceleration during his two years immediately after his -1 g acceleration, he was about 29.07 years old when she was 48.9 years old. And in between his two ages of 26.8 and 29.07, she reached an age *much* greater than 48.9 years old ... she was 81.24 years old when he was 28 years old.

So during that four-year segment of his life, she was 48.9 years old twice: once when he was 26.8 years old, and once again when he was 29.07 years old. And it is possible that he could have other ages, outside that four year segment of his life, when she is 48.9 years old.

The above is a specific numerical example, but it's easy to see from the age-correspondence graph that for *any* given age for her, between the bottom and the summit of that mountain-like curve, there will be *two* ages for him, not just one. And there could be more possible ages for him, when she has that given age, for regions of the age-correspondence curve outside of the region that we chose to investigate. (Again, it is important to stress that all of the above discussion refers to the traveler's conclusions about their age correspondences. The home-twin will never conclude that the traveler has multiple ages at any instant in her life.)

So it's clear that the CADO frame isn't "invertible". I.e., it is *not* true that for any choice of the age of some given distant perpetually-inertial person, that there is a unique corresponding age for the traveler (according to the traveler). By contrast, the inertial frame for a perpetually-inertial observer *is* invertible, because in that case, the age-correspondence graph is just some one-to-one (invertible) curve.

The fact that the CADO frame isn't invertible, means that the CADO frame *cannot* be used as one of the possible *charts*[8] that general relativity "knits together" in order to cover the entire universe. Such charts must be invertible: they must provide a one-to-one mapping between the spacetime points within the domain of coverage of the chart, and the coordinate values of the chart. But there is no need to impose that requirement in the definition of a frame for an accelerating observer. All that matters to an accelerating observer is that he be able to determine the current age and current position of any given distant perpetually-inertial object or person in the (assumed everywhere flat) universe, in a way that is internally self-consistent, and in a way that is consistent with special relativity. The fact that some distant inertial person, at some instant in her own life, might realize that the traveler can legitimately conclude that he has more than one single age then, is certainly of no fundamental importance to either the traveler *or* to the distant inertial person. From the inertial person's *own* perspective, the traveler *does* have a unique age at each instant of the inertial person's life. And the fact that some accelerating observer somewhere happens to conclude that some given perpetually-inertial person is rapidly getting younger, is certainly of no fundamental importance to that given inertial person.

The chart consisting of the Rindler coordinates[9] is quite similar to the CADO frame. The primary difference between the two, is that the Rindler chart is restricted to a neighborhood of the accelerating observer (which is necessary to make the chart be a one-to-one mapping, as required in general relativity). In contrast, the CADO frame applies to all of (the assumed flat) universe. That is possible for the CADO frame, because the CADO frame isn't intended to be, nor has it any need to be, a chart.

Empirical Determination of the Current Age of a Distant Person

If the traveler were perpetually inertial, he could determine (at each instant of his life) the current age of the distant perpetually-inertial person, by using the Lorentz equations. (Or, he could get the same answer by using the CADO equation, of course). If he uses the Lorentz equations to get the answer, it can seem like an abstract operation, without any intuitive meaning. But he can also get the same answer in a very intuitive, and meaningful way.

Suppose this perpetually-inertial traveler arranges for the distant person to periodically broadcast a TV image of herself, holding a sign that states her current age. When the traveler receives one of those images, he knows that the age being reported on the sign is *not* her current age at the instant that he receives that image, because the image doesn't travel infinitely fast ... the age on the sign tells him what her age *was*, at the instant when she transmitted that image. She is obviously older when he receives that image.

It is possible for the traveler, by using only elementary observations and elementary calculations, to determine how much she has aged while that image was in transit, and thus to determine what her actual current age was at the instant that he received that image.[10] If he does that correctly, he will get exactly the same result that the Lorentz equations would have given him (and the same result that the CADO equation would have given him).

Here are the details about how an inertial observer, using only elementary observations and elementary calculations, can determine the current age of a distant inertial person.

An inertial observer (he) can determine the current age of a distant inertial person (she) using only "first-principles" observations and calculations, with no synchronized clocks, and without any knowledge of special relativity other than the fact that an electromagnetic signal will always travel at a constant and specific speed of "c" relative to that inertial observer.

What ARE these "first-principles" observations and calculations? I'm going to be using that term a lot, so it's important to know what I mean by the term. "First-principles" things are things that are elementary, in the sense that they are equally applicable to inertial (non-accelerating) observers in both Newtonian physics AND in special relativity. Examples are basic Euclidean geometry, trigonometry, and algebra. And something else that will be ESPECIALLY important is this: suppose there is an inertial distant person whose velocity relative to the inertial observer is constant at "v" ly/y. Then the inertial observer can ALWAYS use the fact that changes delta_d in the distance of the distant inertial person from him, between any two instants t_1 and t_2 in his life, is ALWAYS equal to

delta_d = v * (t_2 - t_1).

So how does the inertial observer (he) empirically determine the current age of the distant person (her)?

First of all, he needs to know what his velocity is, relative to the distant person (negative when they are converging, and positive for when they are diverging). The speed "v" is given in units of lightyears per year (i.e., as a fraction of the speed of light). In the most general case, he may have accelerated in various ways in the past, and may have lost track of how far away she is from him, or what his current velocity is with respect to her (or, equivalently, what her velocity is with respect to him). So he needs to determine these things from first principles.

The first job is to determine her velocity relative to him. He does that by broadcasting a message to her at the instant t1_b1, asking her to immediately transmit a reply message to him. Then, a short time later, at the instant t1_b2, he broadcasts another message to her with the request to immediately transmit a second reply message to him. When he receives her first reply to him, he labels that instant in his life as t3_r1. And when, a short time later, he receives her second reply to him, he labels that instant in his life as t3_r2. Let the quantity delta_b equal the time lapse between his two broadcast messages to her, and delta_r equal the time lapse between the two replies he received from her. So we have

delta_b = t1_b2 - t1_b1

and

delta_r = t3_r2 - t3_r1.

Denote the ratio of the two delta quantities as

R = delta_r / delta_b.

He can determine her velocity v relative to him as a function of the ratio R from first principles, knowing only that the velocity of light is always the constant c (whose value is equal to 1 in units of lightyears/year). To see how he does that, it's necessary to describe the Minkowski diagram corresponding to this scenario. Draw the diagram yourself as I describe it ... that's the ONLY way it will be understandable. If you have two plastic right triangles (a 30-60-90 one, and a 45-45-90 one), that will make it easier to accurately draw the sloped worldlines that I'm about to describe.

The horizontal axis of the Minkowski diagram shows all of his ages t ... so label that axis t. The vertical axis (labeled x) gives the distance of all objects from him. His distance from himself is always zero, so the horizontal axis corresponds to HIS worldline. As he ages, he progresses along that horizontal line.

Vertically above the point t = t1_b1 on the horizontal axis, we need to arbitrarily pick a point P to represent her unknown distance from him then, even though he doesn't actually know her distance yet. What we DO know is that her worldline passing through her position there has to have a slope equal to her unknown velocity v. (That follows from my above explanation of the meaning of the term "first-principles"). Just pick any velocity so that we can see how that velocity is related to the various important times on the horizontal axis. A convenient choice is the velocity that gives an upward-sloping line with an angle of 30 degrees wrt the horizontal. So the slope v is then

v = tan(30 degrees) = 0.57735.

So, through the point P, draw a long straight line sloping upward toward the right at an angle of 30 degrees to the horizontal. That is then HER worldline on the diagram. As she ages, she progresses along that line.

We can also plot the worldline of his first request message to her. It starts at the point t1_b1 on the horizontal axis, and slopes upward toward the right at an angle of 45 degrees wrt the horizontal (because 45 degrees is the angle whose tangent is 1, which is the velocity of the light pulse, in units of ly/y).

That worldline of his first request message to her intersects her worldline at the point Q. Label the point on the horizontal axis vertically below the point Q as his age t2.

The point Q is the event where she receives his first request message. She immediately sends her first reply message to him, and that message's worldline starts at Q, and then slopes downward to the right at angle of -45 degrees. It intersects the horizontal axis at the time t3_r1. The time lapse between t3_r1 and t2 is the same as the time lapse between t2 and t1_b1.

The worldlines of the two messages form a right isosceles triangle whose hypotenuse lies on the horizontal axis, and whose 90-degree corner is at point Q. A similar (but larger) right isosceles triangle can be drawn for the second set of messages. That triangle starts at the point t1_b2 on the horizontal axis, and ends at the point t3_r2 on the horizontal axis. Its vertex is at the point S on her worldline.

On the diagram, there is a very important triangle whose base is horizontal, starting at point Q on the left and extending to the right until it intersects the worldline of the second reply message, at point U. The top corner of the triangle is the point S on her worldline. The angle of the left corner of that triangle is 30 degrees, and the angle of the right corner is 45 degrees.

There is another important triangle, contained within the above triangle, that shares the right side of the above triangle, and also lies along the right portion of the base of the above triangle. That second triangle's third side on the left is the upper portion of the second request message's worldline. The angle of the left corner of the second triangle is 45 degrees, so the second triangle is a right isosceles triangle. It will help if you redraw those two triangles as a separate enlarged drawing well below the Minkowski diagram, because it plays a very important role in this derivation. Also use the same labels for the important points on those two triangles that have already been defined.

The base of the larger triangle has length delta_r. (Show that on the enlarged drawing). Also on the enlarged drawing, denote the length of the base of the smaller triangle as Z. And the remaining segment (on the left end) of the base of the large triangle is delta_b.

Draw a vertical line from point S down to the base of the two triangles. Call that the intersection of that vertical line with the base of the two triangles point W. Note that W evenly divides the base of the smaller triangle. The small triangle has thus been divided into two right 45-45-90 triangles, whose perpendicular sides all have length Z/2.

The left corner of the large triangle has the angle 30 degrees. So we now have that

tan (30) = v = (Z/2) / (delta_b + Z/2)

or

tan (30) = v = Z / (2 * delta_b + Z).

But Z = delta_r - delta_b, so

v = (delta_r - delta_b) / (2 * delta_b + delta_r - delta_b)

or

v = (delta_r - delta_b) / (delta_r + delta_b).

Dividing top and bottom of the right-hand-side by delta_b, and recalling that

R = delta_r/delta_b,

gives

v = (R - 1) / (R + 1).

That's the result we need ... it allows him to calculate her velocity relative to him in terms of two quantities that he can experimentally determine. He has been able to determine that empirically, from first principles. (He was able to do that, because we are assuming that he is an inertial observer).

Now that he knows her velocity v relative to him, he can empirically determine her distance from him, again using only first-principles observations and calculations, together with the knowledge that messages (light signals) always travel at the constant velocity c (which has the magnitude 1 when units of years and lightyears are used). We are again assuming that he is an inertial observer.

To understand how he can empirically determine her distance from him, we first need to draw a Minkowski diagram for a trivially simple case, the case where their relative velocity is zero. As with the empirical determination of velocity, we draw a Minkowski diagram similar to what we drew there. His world line is again the horizontal axis. Her world line is also horizontal, but is vertically above the horizontal axis at the unknown distance d_a. Mark a point on the horizontal axis to represent his age at some instant in his life. Choose the location of that point on the horizontal axis to be near (and to the right of) the origin of the diagram, roughly a sixth or so of the way to the right end of the axis. Label that point t1, corresponding to some arbitrary instant in his life. Suppose he sends a message to her at that instant, telling her to immediately send him a reply when she receives his message. The world line of that request message starts at t1, and then slopes upward to the right, at an angle of 45 degrees (because its slope must be equal to 1, the velocity of light). That line terminates on her worldline. Label that point on her worldline as point P. Draw a vertical line downward from point P to the horizontal axis, and label the time there as t2. The time t2 is his age when she receives his request message. She then sends her reply message to him. The worldline of that reply message starts at P, and then slopes downward to the right at an angle of -45 degrees, until it intersects the horizontal axis at t3, his age when he receives her reply. Clearly, the time lapse between t3 and t2 is the same a the time lapse between t2 and t1, and so

(t3 - t1) = 2 * (t2 - t1).

During the elapsed time (t2-t1), the request message traveled the distance d_a. Likewise, during the elapsed time (t3-t2), the reply message also traveled the distance d_a. So during the total elapsed time (t3 - t1), the round-trip light pulse has traveled a total distance 2 * d_a. So for this simple case with v = 0, we get

d_a = (t3 - t1) / 2.

Now, we need to do the case where v is not zero. We'll do the case where v is some positive number (so that she is moving AWAY from him). Like we did in the empirical determination of v, we'll choose v = 0.57735 ly/y, giving a slope of her world line that angles upward to the right, at a 30 degree angle.

We need to do a new Minkowski diagram. Start out by exactly copying the previous diagram, except don't draw in her horizontal worldline that we drew before. But DO copy the triangle we got before for the worldlines of the two messages, along with the labels t1, t2, and t3 on the horizontal axis. Also draw the vertical line rising from t2 to the upper edge of the triangle at the point P. Then draw in her new worldline, passing through the point P, with slope v (and angle 30 degrees). Draw her worldline so that its left end starts above the horizontal axis, vertically above a time value on the horizontal axis that is a little less than t1. And draw her worldline so that its right end is vertically above a time value a little greater than t3. (Her complete worldline of course covers her whole life, but we only are interested in the above segment of it). Also draw a short horizontal line from point P, extending a short distance out to the right, and draw a vertical line rising upward from t3 and extending to where it intersects the short horizontal line.

That diagram is all we need to get the equation we want, that gives the distance to her, when he receives her reply message. First, note that by design, her distance from him when she receives his request is STILL what it was in the first (v = 0) example: it is STILL true that

d_a = (t3 - t1) / 2.

And note that during the time which elapses between her receipt of his request at t2 and his receipt of her reply at t3, the increase delta_d in her distance from him is

delta_d = v * (t3 - t2).

(That follows from the definition of slope).

Since (t3 -t2) = 2 * (t3 - t1), we get

delta_d = v * (t3 - t1) / 2.

So her distance d_b from him at t3 is then

d_b = d_a + delta_d

d_b = (t3 - t1) / 2 + v * (t3 - t1) / 2

d_b = (t3 - t1) * (1 + v) / 2.

That is the equation we need. It tells him her distance from him at the instant that he receives her reply message. And he has been able to determine that empirically, from first principles. Remember that we were assuming that he is an inertial observer.

Note that the "1" being added to v in the above equation is more generally written as "c", the velocity of light. But in units of years and lightyears, the magnitude of c is equal to 1. So in its general form, the equation would be written

d_b = (t3 - t1) * (c + v) / 2.

So he now knows what her current distance from him is. Knowing that one distance at that one time, together with the speed "v", he can then easily calculate her distance from him at all other times in his life (as long as the speed "v" doesn't change).

Now, it's time to move on to the most important question: How can he empirically determine (using only first-principle observations and calculations) her current age at any instant of his life? We are again assuming that he is an inertial observer.

To follow the material below, it is probably indispensable that you draw a Minkowski diagram as we go along. The one we need is very similar to the one that was described in the determination of the relative velocity "v", but some of the labels will be different. And as before, it will be easier to draw an accurate diagram if you have a 30-60-90 right triangle (for drawing the 30-degree angle of the perpetually-inertial distant person's (HER) worldline), and also a 45-45-90 right triangle for drawing the worldlines of the electromagnetic messages. Start by drawing a horizontal line for the "t" axis of the diagram, about half-way down the page, and covering most of the width of the paper. That's HIS worldline: he progresses to the right along that line as he ages. So any point along that line represents some instant "t" in his life.

Next, draw a vertical line for his "x" axis, so that vertical height above the horizontal axis represents the distance of all objects or persons from him (according to HIM, not HER).

Now, starting at a point near the left margin of the paper, at a height above the horizontal axis fairly close the horizontal axis (maybe about a sixth of the distance to the top of the diagram), start a line sloping upward to the right at an angle of 30 degrees to the horizontal, and extending almost to the right edge of the page. The 30 degree angle is the angle whose tangent is their relative velocity. It corresponds to a velocity of 0.57735 ly/y, but that's just a convenient choice, because we're using a 30-60-90 right triangle to draw the diagram ... the actual velocity will be determined empirically, in the way that was described earlier in this section. That upwardly sloping line is HER worldline.

Slightly to the right of the left end of her worldline, put a short "tic mark" perpendicular to her worldline, and slightly above that "tic", write her age tau_b_1. (Write it with the same slope (30 degrees) that her worldline has). That point represents how old she is when he decides to send her an electromagnetic message, asking her to immediately send a reply message to him when she receives his message, giving her age then. Draw a vertical line through that point, extending down to the horizontal axis. Label that point on the horizontal axis t_b_1 ... that's HIS age when he sends her the message. And label the height of that vertical line d_b_1 ... that's the distance between them (according to him) when he sends his message to her. We don't know what that distance actually is yet, but we will be able to calculate it later. WE are intentionally NOT making any assumptions yet about what the time t_b_1 or the distance d_b_1 is. Specifically, we are NOT assuming that he has directly come from her, as in the standard twin "paradox" ... he may not even have ever been co-located wit her, and he may not be her twin.

Next, through the point t_b_1 on the horizontal axis, draw a line sloping upward to the right, and at an angle of 45 degrees wrt the horizontal. Extend it until it intersects her worldline. That's the worldline of his request message to her. At that intersection on her worldline, put a "tic" and label it with her age tau_0. That's her age when she receives his message. And draw a vertical line between that point and the horizontal axis. Label that point on the horizontal axis t_0 ... that's his age when she receives his message.

Next, from the point labeled tau_0, draw a line extending downward to the right, with the angle -45 degrees wrt the horizontal, until it intersects the horizontal axis. That's the worldline of her reply message to him. Put a vertical "tic" where that line intersects the horizontal axis, and label that point t_r_1, his age when he receives her reply message. Draw a vertical line upward from that point, until it intersects her worldline. Put a "tic" there, perpendicular to her worldline, and label that point tau_r_1, her age when he receives her message, according to him. Label the length of that vertical line between t_r_1 and tau_r_1 as d_r_1. Note that the point t_0 is half way between the points t_b_1 and t_r_1. And note that the vertical line rising from t_0 divides the large 45-90-45 triangle into two adjacent 45-90-45 triangles.

Now, we repeat the above construction of the 45-90-45 triangle (which connected the points t_b_1, tau_0, and t_r_1, formed by the first request and reply message), but this time it's for a second request message that he decides to send her at his age t_b_2. Position the point t_b_2 a small distance to the right of t_b_1 (about a tenth of the length of the horizontal axis, and well to the left of the point t_0). We get a second (much larger) 45-90-45 triangle, labeled with the point t_b_2 on the bottom left, and the point tau_1 at the top, and the point t_r_2 at the bottom right. Label as t_1 the point on the horizontal axis vertically below the point tau_1. And draw a vertical line up from t_r_2, intersecting her worldline at the point tau_r_2. Label the height of that line d_r_2.

Note that in all of the above construction, we have used NO knowledge of special relativity, other than the fact that in any inertial reference frame, light always travels at the same constant specific speed "c".

So now, we need to show exactly how he can use the above diagrams to tell him what her current age was when he was t_b_1 years old.

First of all, he needs to determine what their relative velocity "v" is. It was shown earlier in this section how he can do that. So he has the quantity "v". Keep a list of the quantities that he has determined, as we go along.

Next, he needs to determine what her distance is from him (according to him), at some instant "t_d" in his life. It was shown how he can do that in an earlier part of this section. Call that distance "d_t". So our list now includes v, t_d, and d_t. Once he knows t_d and d_t, he can compute their separation at any instant of his life ... in particular, right back to the instant t_b_1. The equation he uses to do that is

d = d_t + v * (t - t_d).

So he now knows what the previously unknown distance d_b_1 was, on the current Minkowski diagram:

d_b_1 = d_t + v * (t_b_1 - t_d).

So our list now is v, t_d, d_t, d_b_1.

Next, when he receives her reply message to his his first request message, he records his age t_r_1 then, and he also records what she tells him her age tau_0 was when she received his request message. Also, from the diagram, he knows that his age when her age was tau_0 is half way between his ages t_b_1 and t_r_1, so

t_0 = (t_b_1 + t_r_1) / 2.

Our list is now v, t_d, d_t, d_b_1, t_r_1, tau_0, t_0.

The only remaining thing he needs to know is what the time-dilation parameter gamma is. That is the important result in special relativity, which says that an inertial observer (he) will always conclude that a distant person (she), who is moving at a speed "v" relative to him, will age slower than he is ageing, by the ratio gamma (where gamma is greater than 1). I.e., if t_a is his age at some instant, and t_b is his age at a latter instant, and if the corresponding ages for her (according to him) are tau_a and tau_b, then

(1 / gamma) = (tau_b - tau_a) / (t_b - t_a).

Special relativity gives an equation for gamma, as a function of the velocity "v". Our traveler doesn't know any special relativity, so he doesn't know that equation. But he can determine the value of gamma, for any specific velocity, from the Minkowski diagram that he has been able to draw using only first principles.

I've already described how he can determine that when he was age t_0, she was age tau_0. From the second request and reply message (which produced the much larger triangle on the diagram than the first request and reply message did), he can likewise determine that when he was age t_1, she was age tau_1. So he can compute

(tau_1 - tau_0)

and

(t_1 - t_0)

from first principles.

So he can now compute the ratio

Q = (tau_1 - tau_0) / (t_1 - t_0)

from first principles. And he can determine, by sending one or more ADDITIONAL request messages, that Q is ALWAYS the same constant, as long as their relative velocity remains unchanged. WE know that the constant Q he has determined is (1/gamma).

So he now knows the value of the constant Q, and he knows how to use it to relate the rate of her ageing to the rate of his ageing, for ANY length interval of his ageing. And so he knows the value of (1/Q), which is equal to gamma, even though he's never heard the name gamma. To simplify the discussion from here on out, let's say that someone tells him that his constant Q is equal to the reciprocal of the famous parameter gamma. So from here on, we'll say he knows the value of gamma and how to use it, even though he doesn't know the famous equation that relates it to the relative velocity v.

So our list is now complete: it's v, t_d, d_t, d_b_1, t_r_1, tau_0, t_0, and gamma. The quantities on that list are all known by him.

Recall that our overall objective was to show how he could determine tau_b_1, her current age (according to him), when he was t_b_1 years old, using only elementary observations, and first-principles calculations. Since he now knows gamma for the current relative velocity, he proceeds as follows:

He knows her age tau_0 when he was t_0, because she told him her age tau_0 when she sent him her first reply message. So he knows that

(tau_0 - tau_b_1) / (t_0 - t_b_1) = 1/gamma.

Therefore he can compute

tau_b_1 = tau_0 - (t_0 - t_b_1) / gamma.

So he has been able to compute her age tau_b_1 when he is age t_b_1, purely from first principles. I.e., he has been able to EVENTUALLY determine what her current age was, when he BEGAN to do those observations and first-principles calculations. THAT is the result we've been pursuing, with ALL the above work!

Remember, in all of the above, we've been assuming that he is an inertial observer.

Now, what about an observer who is NOT always inertial?

The traveler who is not *perpetually* inertial, can in principle carry out the same type of elementary observations and elementary calculations, that the perpetually-inertial traveler carries out above. And he will find that, during any segment of his life in which his acceleration is zero, his conclusions from those elementary calculations will *always* agree with a co-located perpetually-inertial observer's calculations. During that entire unaccelerated interval, he has just a single MSIRF ... the co-located perpetually-inertial observer is just an observer permanently at rest at the spatial origin of that MSIRF.

The fact that the traveler's calculations, during that unaccelerated segment, always agree with the calculations of that co-located perpetually-inertial observer, remains true no matter how short that segment is. It even remains true when the traveler is stationary with respect to that inertial observer for only a single instant. So, the definition of the CADO frame (that the accelerating observer always agrees with his MSIRF, about the current age (and current position) of a distant object or person) is not just an arbitrary, abstract definition ... it defines a reference frame that is consistent with the traveler's own (potential) elementary observations and (potential) elementary calculations.

Of course, those elementary measurements, and elementary calculations, would actually require a finite amount of time to actually carry out ... for some situations, they could take a very long time. So how is the above conclusion arrived at?

The argument is basically a *counter-factual* and *causality* argument.[11] At any instant of his life, the traveler *can*, if he so chooses, decide to stop accelerating, and continue to move thereafter at a constant velocity equal to what his velocity was at the instant that he stopped accelerating. He may not choose to ever do that, but he *can* if he wants. *If* he does that, he can make the *same* kind of observations and calculations that a perpetually-inertial observer who is (temporarily) co-located with him during that segment can make, and they will always arrive at exactly the same answer. And recall that, if he *does* choose to stop accelerating, and make the elementary observations and elementary first-principles calculations that were described in detail for inertial observers, he will be able to calculate the distant person's current age *all the way back* to the instant when he stopped accelerating.

So, the definition of the CADO frame (that the accelerating observer always agrees with his MSIRF, about the current age (and current position) of a distant object or person) is not just an arbitrary, abstract definition ... it defines a reference frame that is consistent with the traveler's own (potential) elementary observations and (potential) elementary first-principles calculations. Those observations and calculations are characterized as being *potential* observations and calculations, because he *may* choose to stop accelerating long enough to make them, or he may *not*. But by causality (i.e, by the fact that the future *cannot* affect the past), his conclusions about the distant person's current age, at some given instant of his life, *cannot* depend on how he may choose to accelerate in the future, *after* that instant. So his conclusion about the current age of the distant person at some instant when he is accelerating doesn't depend on whether or not he continues to accelerate after that instant.

The CADO frame is a reference frame that has a tangible meaningfulness to the accelerating observer. According to him, the current age of the distant person, as given by the CADO equation, is completely meaningful and "real" for him, because it agrees with his own (potential) observations and (potential) elementary first-principles calculations.

Graphical Interpretation of the CADO Frame

We can create a graph (a Minkowski diagram) that shows (two-dimensional) spacetime from the home-twin's perspective. The usual convention is to plot the home-twin's time coordinate (which we will denote by the variable name *tau*) vertically, and her spatial coordinate *X* horizontally. However, that choice is arbitrary, and it will be more convenient here to plot *tau* horizontally, and *X* vertically.

It will probably help if you sketch the Minkowski diagram as we go along.

In the home-twin's inertial frame, she is always located at the spatial origin (i.e., her position is always at $X = 0$). And we can choose the time coordinate *tau* of that frame to directly correspond to her age. Then, the positive horizontal axis of the diagram corresponds to her world line: as her age increases, her spacetime point moves to the right along that positive horizontal axis. Any point along that positive horizontal axis corresponds to the home-twin at some particular age. We can put "tic-marks" along that axis, showing how her age progresses. If we put a tic-mark for every one of her birthdays, those tic-marks will be equally spaced along the positive horizontal axis.

Now, if the traveler's acceleration *a(t)* is given (so that *a(t)* gives the acceleration on his accelerometer at each instant *t* of his life), then his location *X*, according to the home-twin, can be determined at each instant *tau(t)* of the home-twin's life. This is just the quantity *L*, whose determination was given earlier, specified as a function of *tau(t)*. So we can plot the curve corresponding to the traveler's location, according to the home-twin, at each instant *tau* of her life. That curve is *his* world line, plotted on *her* Minkowski diagram.

At any point on his world line, the tangent to that line has a slope that is numerically equal to his velocity *v*, relative to her, according to her. So the slope of that curve must always be less than +1, and greater than -1, since according to her, the magnitude of his velocity can never exceed, or even exactly reach, the speed of light. Other than that restriction, his world line curve can have any shape, except that it will always be continuous, even for instantaneous velocity changes. The curve *can* have "kinks" in it (where the slope changes instantaneously), in the idealized limiting cases with instantaneous velocity changes.

There is one further restriction for the world line of the traveler. Since we have (for simplicity) written the CADO equation is terms of a (positive) distance L, rather than as the position X of the traveler, we need to always start the trip with a positive *v*, so that the world line of the traveler stays above the horizontal axis, keeping *X* positive. We can later have segments with negative *v*, but not lasting so long that *X* becomes negative. In practice, this restriction doesn't adversely affect our ability to handle all the interesting scenarios that can arise in variants of the twin "paradox". It just means that (in the interest of simplifying the CADO equation) we are choosing to ignore scenarios where the traveler passes back by the home twin, and continues on in the opposite direction. If such a scenario ever became important to investigate (which is unlikely), we could do it, but at the expense of complicating the simple conventions that we have established.

Now, back to the Minkowski diagram. Just as we did for *her* world line (the horizontal axis), we can put tic-marks along *his* world line, showing how *his* age (given by the variable *t*) progresses. If we put a tic-mark for every one of his birthdays, those tic-marks will *not* in general be equally spaced along his world line (when drawn on *her* Minkowski diagram): his tic-mark spacings will generally vary, but will be more widely spaced than her tic-marks (except for any finite segments of his life when their relative velocity is zero, in which case the spacing of the traveler's tic marks will be the same as the spacing of the home twin's tic marks during that segment).

At any point on his world line, we can determine his "line of simultaneity" that passes through that point. His line of simultaneity corresponds to "now", according to him. I.e., according to him, the current position of every object or person in the (flat) universe, at that particular instant *t* in his life, corresponds to some unique point on that line.

His line of simultaneity, through any given point on his world line, has a slope of 1/*v*. This can be visualized as follows: if the angle that the tangent to his world line (whose slope is *v*) makes with the *horizontal* axis is *alpha*, then the angle that the line of simultaneity makes with the *vertical* axis is also *alpha*. For example, if the velocity *v* is a small positive number (much less than 1), then *alpha* will be a small angle above the horizontal axis (much less than 45 degrees), and the tangent to his world line will be rotated only slightly *counter-clockwise* with respect to the horizontal axis. His line of simultaneity will be rotated by that same small angle, *clockwise* with respect to the vertical axis. For a velocity *v* near +1, the angle of the tangent to the world line will be just slightly less than 45 degrees CCW with respect to the horizontal axis. The line of simultaneity will be rotated by that same angle (almost 45 degrees), CW with respect to the vertical axis. If we draw a 45-degree line (slope +1) through the given point on his world line, then as *v* gets closer and closer to +1, the tangent line rotates more and more CCW toward that 45-degree line (from below), and his line of simultaneity rotates more and more CW toward it (from above), in such a way that the two lines are always arranged symmetrically with respect to that 45-degree line.

The above description is for the case where the position of the traveler is above the horizontal axis ($X >= 0$, which we always assume), and that the velocity v is positive (the twins are moving apart). When v is negative (the twins are moving toward one another), the rotations are in the opposite direction, and the 45-degree reference line is rotated 45 degrees CW with respect to the horizontal axis (its slope is -1).

So, for any given point on his world line (which is parameterized by the variable t (his age), we can immediately draw his line of simultaneity through that point. And we can then directly see where that line of simultaneity intersects the horizontal axis. Since the horizontal axis is the world line of the home-twin, that point of intersection directly gives us her current age, according to the traveler, when he is age t. That quantity is denoted as CADO_T(t) in the CADO equation. That result, obtained graphically, is exactly what we can get (much more quickly and easily) from the CADO equation.

The above geometrical construction can be used to easily derive the CADO equation. That's how I first derived it, many years ago. Note that the *vertical* line through the point t on the traveler's world line is the line of simultaneity *for the home twin*. The intersection of that vertical line with the horizontal axis gives her age when he is age t, *according to her.* That quantity is just CADO_H(t) in the CADO equation. The CADO equation is then simple to derive, just using the right triangle formed by the two lines of simultaneity and the segment of the horizontal axis between the points CADO_T and CADO_H. For example, in the case where the twins are moving *apart* at the instant t, v is positive, and so the slope $1/v$ of the traveler's line of simultaneity is positive (sloping upward to the right). So the intersection of that line with the horizontal axis (which is CADO_T(t)) lies to the left of the intersection of the home twin's line of simultaneity with the horizontal axis (which is at *tau* = CADO_H(t)). The slope of the hypotenuse is by definition

delta(X) / delta(*tau*),

as we move from the lower end of the hypotenuse to the upper end of the hypotenuse. The change in X is just the distance L, which is always positive. The change in *tau* is (CADO_H - CADO_T), which is positive in this case. So the slope of the hypotenuse of the right triangle is

L(t) / (CADO_H(t) - CADO_T(t)).

But the hypotenuse of the right triangle coincides with the traveler's line of simultaneity, which must always have a slope of 1/v(t). So we must have

1/v(t) = L(t) / (CADO_H(t) - CADO_T(t)).

Solving the above equation for CADO_T(t) gives the CADO equation:

CADO_T(t) = CADO_H(t) - v(t) * L(t).

If we had initially chosen the case where the twins are moving *toward* each other at the instant t, (so that v(t) is negative), the slope of the traveler's line of simultaneity would have been negative, and CADO_T would have been greater than CADO_H. But the final derived result (the CADO equation itself) would have turned out exactly the same. (You might want to test your understanding of the above material by explicitly doing the derivation for that case.)

The CADO Equation for Two or Three Spatial Dimensions

In all of the above sections, the motion of the accelerating traveler has been restricted to be in one spatial dimension. But the CADO equation can be generalized to handle two-dimensional or three-dimensional motion. In fact, the equation remains exactly the same, except that the quantities v and L become the vector quantities $\mathbf{v}$ and $\mathbf{L}$, and the product of $\mathbf{v}$ and $\mathbf{L}$ is the dot product of the two vectors. (The dot product of any two vectors is just the ordinary product of the lengths of the two vectors, times the cosine of the angle between them). $\mathbf{v}$ is the vector velocity of the traveler, relative to the home-twin, according to the home-twin. $\mathbf{L}$ is the vector position of the traveler, relative to the home-twin, according to the home-twin. The CADO equation, with vector quantities, would usually be written

$$CADO\_T = CADO\_H - \mathbf{v} \cdot \mathbf{L}$$

Note that whenever the traveler's motion is transverse with respect to the home-twin, $\mathbf{v}$ and $\mathbf{L}$ are perpendicular, and their dot product will be zero. So the CADO equation says that CADO_T will then equal CADO_H. I.e., the traveler and the home-twin will agree about the correspondence between their ages. This is true regardless of whether the perpendicular motion is permanent, or of short duration, or even momentary. And it is true regardless of whether the traveler's speed (i.e., the length of $\mathbf{v}$) is constant, or varying in an arbitrary manner. So, if the traveler is zipping around on a circle (or, in three dimensions, zipping around on the surface of a sphere), the traveler and the home-twin will agree about their respective ages during the entirety of that motion.

Comparison of the CADO Frame with Some Other Frames

Other reference frames have been defined for the traveling twin (or for any observer who sometimes accelerates), which (like the CADO frame) do not involve the equivalence principle, and which do not involve any fictitious gravitational fields. Any such reference frame *must* be consistent with the indisputable fact that the two twins (in the standard twin paradox scenario) will agree about which of them is younger, when they are again reunited; the twin who accelerated while they were apart will always be the younger.

There is currently no universal consensus about which of these different reference frames is the most appropriate. Two such alternatives (in addition to the CADO frame) are the "Radar" frame,[12] and the "Minguzzi" frame.[13]

The above alternatives give different answers (different than the answers that the CADO frame gives, and different from each other) to the following two important questions.

The first question is, "If an observer accelerates during some segment of his life, can he be considered to be an inertial observer during the other segments of his life in which he is *not* accelerating?". Specifically, *when*, during unaccelerated segments of his life, is he entitled to use the Lorentz equations to determine simultaneity and position at a distance, just as a perpetually-inertial observer is always entitled to do? And, similarly, *when*, during those unaccelerated segments, is he entitled to use the well-known time-dilation result?

The CADO frame says that for an unaccelerated segment of the observer's life, of any length whatsoever (no matter how short), the observer is a *full-fledged* inertial observer during that *entire* segment, regardless of how he has accelerated *before* that segment, or how he may choose to accelerate *after* that segment. (Taylor's and Wheeler's Example 49 is consistent with that answer).

The Radar frame says that for some finite time *before* his acceleration begins, and for some finite time *after* his acceleration has ended, the observer is *not* an inertial observer. The Radar frame is thus *non-causal*, in the sense that the observer's conclusions about simultaneity, during an unaccelerated segment of his life, depend on whether or not he will *choose* to accelerate *in the future*.

The Minguzzi frame says that the observer is an inertial observer at any time *before* the acceleration begins (provided that he has undergone no recent previous segment of acceleration), but that he is *not* an inertial observer for some finite time *after* his acceleration ends. So the Minguzzi frame (like the CADO frame), is causal.

For both the Radar and the Minguzzi frames, the *duration* of those unaccelerated, but non-inertial, portions of the observer's life depends upon how far away the perpetually-inertial object or person is. For perpetually-inertial objects or persons that are sufficiently far away, there will be *no* portions of the unaccelerated segments of the observer's life in which he entitled to be considered to be an inertial observer. During such unaccelerated, yet non-inertial, segments of his life, the traveler may be co-located with perpetually-inertial observers for many years of his life, but he will not be allowed (by the Radar and Minguzzi frames) to agree with them, during all those years, about the current age of distant objects or persons. And he will have to ignore the results he gets from any of his observations and elementary calculations of the type described in the section above on empirical determination of the distant object's current age.

The second important question (for which the different alternative frames give different answers) is, "How does the current age, of some given distant perpetually-inertial object or person, *change* during periods of acceleration by the observer?". The CADO frame says that the current age of the distant perpetually-inertial person (the "home-twin") can change extremely rapidly (either positively or negatively) during rapid velocity changes by the observer. In fact, in the idealized, limiting case of the traveler's instantaneous turnaround in the standard twin paradox, the CADO frame says that the home-twin's age will instantaneously increase during that instantaneous turnaround. (Again, Taylor's and Wheeler's Example 49 is consistent with that answer). Neither the Radar nor the Minguzzi frames ever give such abrupt age changes of the distant perpetually-inertial person, and her age changes are never negative.

The answers to the two questions are not independent. *Any* frame for the traveling twin, which gives no abrupt age change for the home-twin, during the traveler's abrupt turnaround, *cannot* say that the traveler is entitled to use the time-dilation result during the entirety of the unaccelerated segments of the traveler's trip. Otherwise, the two twins would not agree, about the correspondence between their ages, when they are reunited.

The CADO Frame When the Distant Person Is Also Accelerating

If, in addition to the traveler's acceleration, the home-twin also decides to accelerate, the standard CADO equation isn't applicable, except in the very special case where the home twin is initially unaccelerated, and then decides to make her *first* instantaneous velocity change. The standard CADO equation is then applicable and very useful, but *only* for that first instantaneous velocity change. A generalized version of the CADO equation is *sometimes* applicable and useful for multiple instantaneous velocity changes, when the accelerations consist *only* of a sequence of instantaneous velocity changes. The generalized CADO equation is also sometimes applicable when there are *both* finite accelerations *and* instantaneous velocity changes occurring, but it is usually not practically useful in those cases. The generalized CADO equation is explained later in this section.

Although the standard CADO equation and/or the generalized CADO equation can't always be used in the general case, there *is* a way to *always* determine, at each instant t of the traveler's life, what the current age T of the distant accelerating person is, according to the traveler, for *any* choices whatsoever of their two acceleration profiles.

Denote his (the traveler's) acceleration as $a(t)$, as usual, and let her (the distant person's) acceleration be $b(T)$, where each of those accelerations are as measured by an accelerometer stationary with respect to the given observer. We then need to choose some single arbitrary (but given) *inertial* reference frame, and we will then construct the Minkowski diagram for that inertial frame. For twin-type scenarios, where the traveler and the "home-twin" are initially co-located and mutually stationary before the beginning of the traveler's trip, an obvious choice is the inertial frame whose spatial origin is permanently occupied by a third person, say, the mother of the twins. Both twins will accelerate (in spite of our continuing use of the term "home-twin"), but their mother is permanently inertial. Let the time coordinate in the mother's inertial frame be denoted by *tau*, and let her spatial coordinate be denoted by X.

We then construct a Minkowski diagram as before, with *X* as the vertical axis, and *tau* for the horizontal axis. We can then plot the traveler's world line, as before. And as before, we can put tic-marks on that world line, which give the traveler's age *t* for any point on that line. For any given point on that line, we can determine the traveler's line of simultaneity through that point. But we are no longer particularly interested in where that line of simultaneity intersects the horizontal axis *tau*. This time, we want to know where that line of simultaneity intersects the "home-twin's" world line. We can plot the home-twin's world line on that same Minkowski diagram, using the same process that we used to plot the traveler's world line. And we can again put tic-marks on her world line, which give her age *T* at any point on her world line. The value of *T* at the point where the traveler's *t* line of simultaneity intersects her world line gives us the answer we've been seeking: the current age *T* of the (accelerating) "home-twin", according to the traveler, when the traveler's age is *t*. And, by carrying out that process for many different ages *t* of the traveler, we can construct the age-correspondence graph, that shows how the "home-twin's" current age varies, according to the traveler, as the traveler's age increases.

For each age *t* of the traveler, the above described procedure requires an iterative numerical process, in order to "home-in" on the point of intersection. The tic marks on the "home-twin's" world line initially respond to her birthdays. Determining what her exact age is at the intersection requires some iterating, because the spacing between her adjacent birthday tics is generally not constant.

For the special case of instantaneous velocity changes by both twins, a generalized CADO equation can sometimes be usefully employed, but other times it is not applicable at all. When the traveling twin instantaneously changes his velocity, that causes an instantaneous change in the slope of his line of simultaneity, and that results in a new point of intersection of his line of simultaneity with the home twin's world line. That new point of intersection does *not* depend on whether she is instantaneously changing her velocity then or not. If she *is* instantaneously changing her velocity at the point of intersection, then the only effect will be that the *slope* of her world line will change at that point ... i.e., her instantaneous velocity change only causes a "kink" in her world line at that point. That affects the slope of her world line as it progresses beyond that point (and therefore how she will age after that point), but that has no effect on her age at that point.

At the beginning of the typical scenario, the twins have just been born, and are located at the origin of the Minkowski diagram. As the home twin ages, but doesn't accelerate, her world line will move to the right, along the horizontal axis of the diagram. If at some instant in her life she decides to instantaneously change her velocity from zero to some positive or negative value (which must be of magnitude less than 1 ly/y), her world line will instantaneously change its slope from zero to some positive or negative slope of magnitude less than 1, and then as she further ages, her world line will diverge from the horizontal axis in a straight line. So there is a "kink" in her world line at the instant when she decides to instantaneously change her velocity. And if she decides later in her life to instantaneously change her velocity again, her world line will change its slope again, and will get another kink in it.

There is a very concise and general rule to determine if the standard CADO equation can be used when both twins can be instantaneously changing their velocities at various instants. To allow the rule to apply to the most general sorts of scenarios, in what follows, the terms "traveling twin" and "home twin" will be replaced instead by the terms "the observer" and "the observed". The "observer" is the twin who's opinion about the other twin's current age is desired. The "observed" is the twin who's current age is desired, at the instant *t* in the "observer's" life. Most often, the "observer" will be the "traveling" twin, and the "observed" will be "home" twin. The current age of the "observed" twin, according to the "observer" twin, at the instant *t* in "observer" twin's life, is the quantity CADO_T(*t*) in the standard CADO equation. And the current age of the "observed" twin, according to the "observed" twin, at the instant *t* in "observer" twin's life, is the quantity CADO_H(*t*) in the standard CADO equation. But in some cases, it may be desired in a fixed scenario to switch the roles of the twins (in so far as which twin's opinion is desired), and that is why the more general terms "observer" and "observed" have been introduced.

Since both twins may decide to sometimes instantaneously change their velocity, both of their world lines may diverge from the horizontal axis of the Minkowski diagram. And since neither twin is perpetually inertial any more, we need to create another person whose world line corresponds to the horizontal axis of the Minkowski diagram. That person is the mother of the twins. And the vertical axis of the Minkowski diagram is the mother's spatial axis. *The velocities of the twins are then always specified wrt their mother.*

There is a very concise and general *rule* to decide if the standard CADO equation can be used by one accelerating twin when the other twin can also be accelerating at the same or at some other time (with all accelerations being restricted to instantaneous velocity changes).

Here is the explanation of the procedure:

We draw a Minkowski diagram whose horizontal axis is the twins' mother's age. (Their mother never accelerates.) Then, we draw the world lines of the two twins on that diagram. In the following rule, I will call the twin whose opinion we want "the observer", and the other twin "the observed".

Then the rule is this:

We *can* use the standard CADO equation to determine the observed's current age, according to the observer, for that portion of the the observer's life that lies on the diagram to the *left* of any kinks in the observed's world line. That "standard-CADO-correct" region *includes* the vertical line that passes through the *leftmost* kink in the observed's world line.

It should be noted that the above rule gives a *sufficient* condition to be able to use the generalized CADO equation. It is not a *necessary* condition to be able to use the generalized CADO equation. It *is* possible to give a necessary and sufficient condition that allows us to know whether the generalized CADO equation can be used *or not*. *If* there is an instant t in the observer's life when the intersection of his line of simultaneity with the observed's world line forms the hypotenuse of a right triangle whose base is a horizontal section of the observed's world line, and whose vertical side is formed by the vertical line through the point t on the observer's world line, *then* the observer can use the generalized CADO equation at that instant t in his life. That very long and complicated-sounding condition is actually simple to visualize on a sketch of the Minkowski diagram: what the condition defines is a right triangle which is similar to the right triangle you get in the Minkowski diagram when the home twin is perpetually inertial, and from which the standard CADO equation is derived. The only difference in the new triangle is that its base can be above or below the horizontal axis. The new triangle defines a "generalized" CADO equation which looks just like the standard CADO equation, except that the variable name used for the first term on the right-hand-side of the equation should be changed from "CADO_H" to "CADO_M", because it now represents the age of the home twin, according to the *mother*, *not* according to the (sometimes accelerating) *home* twin. (That first term on the right-hand-side of the CADO equation *always* needs to refer to the opinion of a perpetually-inertial person, whose world line is the horizontal axis of the Minkowski diagram.) And also, the *definition* of the quantity L on the right-hand-side of the generalized CADO equation (which is the length of the vertical side of the right triangle) must be changed from "the distance from the home twin to the traveler, according to the home twin" to "the distance from the home twin to the traveler, according to the twins' *mother*". Finally, we also need to change the definition of the quantity v on the right-hand-side of the generalized CADO equation: it now means the velocity of the traveler, relative to the *mother*, according to the *mother*. You can easily spot potential regions where the generalized CADO equation might be usable by looking for portions of the home twin's world line that are *horizontal*. If those horizontal portions are long enough to include the entire base of the right triangle, then the generalized CADO equation *can* be used. If they are *not*, then the generalized CADO equation *cannot* be used. To be clear, if those horizontal portions are long enough to include the entire base of the right triangle, the generalized CADO equation can be used *even if* the home twin's world line changes its slope at the intersection of the base of the triangle with the vertical side of the triangle.

It was stated at the beginning of this section that "The generalized CADO equation is also sometimes applicable when there is a mix of finite accelerations and instantaneous velocity changes, but it is usually not practically useful in those cases". The reason that it is usually not practically useful in those cases is that finite accelerations require that a very accurate (and to scale) Minkowski diagram be produced, not just the rough sketch which is adequate for scenarios with only instantaneous velocity changes. Once the work to produce that accurate Minkowski diagram has been done, the completely general method described earlier in this section provides all the required results, and the generalized CADO equation becomes unnecessary and superfluous. The only situation where the generalized CADO equation can be useful is for instantaneous velocity changes that happen *before* any finite accelerations have occurred, and when there is no interest in what happens later during the finite accelerations.

A CADO Cartoon

Shortly after I first came up with the CADO equation (several decades ago), and after I started to realize some of its bizarre implications, I created a cartoon (only in my mind) that captures (in only a slightly exaggerated way) the essence of what makes those implications so shocking.

Imagine that a spaceship left Earth many years ago (maybe 20 years ago or so, in ship time), and that the spaceship (at some local date-and-time on the ship) is currently very far away from Earth (maybe 50 lightyears or so, as measured in the Earth frame). The passengers on that ship still remember well their previous lives on Earth, and they still often think about the people they cared about then (and still very much care about). They naturally would wonder if their loved-ones are still alive, and if they are OK. The passengers would probably often try to imagine, if they can figure out their loved-ones' current ages, what they might be currently doing, "right now".

In my imagined cartoon, the ship is having its annual New Year's Eve party. One of the passengers asks the captain, "What is the date right now, back on Earth?" The captain, with his hand on a HUGE throttle, answers, "What date would you LIKE it to be?".

One of these days, I'm going to make myself a tee-shirt with that cartoon on it.

References

1. Fontenot, Michael L., "Accelerated Observers in Special Relativity", Physics Essays, December 1999, pp. 629-648.
2. Einstein, A., *Relativity*, Crown Publishers, 1961, p. 30.
3. Einstein 1961, p. 35.
4. Taylor, E. F., and Wheeler, J. A., *Spacetime Physics*, W. H. Freeman and Company, 1966, pp. 94-95.
5. Fontenot 1999, p. 637.
6. Taylor and Wheeler, 1966, p.97.
7. Fontenot, Michael L., "Erratum and Addendum: Accelerated Observers in Special Relativity", Physics Essays, September 2002, pp. 357-358.
8. Wald, Robert M., *General Relativity*, The University of Chicago Press, 1984, p. 12.
9. Rindler, Wolfgang, *Relativity: Special, General, and Cosmological*, Oxford University Press, 2001.
10. Fontenot 1999, pp. 632-633.
11. Fontenot 1999, p. 645.
12. Dolby, Carl E. and Gull, Stephen F, "On Radar Time and the Twin 'Paradox' ", Am.J.Phys., 69 (2001) pp. 1257-1261.

13. Minguzzi, E., "Differential Aging from Acceleration, an Explicit Formula", Am.J.Phys. 73 (2005) pp. 876-880.

Revised/Enhanced: 9-25-17. (Specified the elementary observations and elementary calculations that allow determination of the CADO empirically).

Revised/Enhanced: 10-13-17. (Added (in Section 4) specific equations for finite accelerations, and for the special case of piecewise-constant accelerations. Also provided (in Section 7) equations for the maximum magnitude of the home twin's rate of ageing, and for the finite limiting value of the home twin's age when the traveler accelerates forever into the future.)

Revised 12-31-17. (Changed the notation slightly for the "delta CADO equation").

Revised 8-31-18. Revised Section 11. Showed how the CADO equation can be easily derived from the Minkowski diagram, near the end of the Section 11 on "Graphical Interpretation of the CADO Frame".

Revised 9-2-18. Made another slight change in the notation for the "delta_CADO equation".

Revised 9-5-18. In Section 2, entitled "The CADO Equation", I added a paragraph that explains why the CADO equation is especially useful.

Revised 9-11-18. In section 14, entitled "The CADO Frame When the Distant Person Is Also Accelerating", I added a paragraph at the end that shows why the CADO equation works when the accelerations of the twins consist of instantaneous velocity changes.

Revised 9-14-18. Revised Section 14, to give a rule for determining when the CADO equation can be used, for scenarios in which both twins sometimes instantaneously change their velocity.

Revised 9-16-18. Revised Section 14, to give a necessary and sufficient condition for the CADO equation to be usable, for scenarios in which both twins sometimes instantaneously change their velocity.

Revised 9-18-18. Revised Section 14. Added several new sentences, and a clarification to the end of the above revision. Also changed the definition of the quantities L and v in the generalized CADO equation.

Revised 9-28-18. Revised Section 14, to add information about the applicability of the CADO equation when there are finite accelerations as well as instantaneous velocity changes.

Revised 8-1-19. Revised Section 10, near the beginning, to show how an inertial traveler can empirically determine the relative velocity of the home twin and the distance to the home twin, using only first principles, together with the fact that the speed of light relative to him is always equal to the constant "c". (Finished the above work on 8-8-19).

Revised 8-9-19. Revised the end of Section 10, which shows why the accelerating traveler also concludes that what the CADO equation tells him is completely meaningful and "real", based on (potential) observations and (potential) elementary first-principles calculations.

Revised 8-11-19. Revised the explanation of what "first-principles" means, in Section 10.

Michael L. Fontenot

PhysicsFiddler@gmail.com