On prime numbers in linear form

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Abstract

A lower bound is given for the number of primes in a special linear form less than N, under the assumption of the weakened Elliott-Halberstam conjecture.

1 Introduction

Using the weight function of the form (q,p_t - prime numbers)

$$(1.1)\quad v(2n) = 1 - \frac{1}{2}\sum_{\substack{z \le q < y \\ q^k \| 2n}} k - \frac{1}{2}\sum_{\substack{2p_1p_2p_3=2n \\ z\le p_1<y\le p_2\le p_3}} 1 - \frac{1}{2}\sum_{\substack{2p_1p_2=2n \\ z\le p_1<y\le p_2}} 1 - \sum_{\substack{2p_1p_2=2n \\ y\le p_1\le f\le p_2}} 1,$$

(such a weight function $v(2n)$ leaves only prime numbers when sifting (i.e. $v(2n) = 1$ with $n = p \ge z$ and $v(2n) \le 0$ for other values of n) for

$$\sum_{\substack{2n\in A \\ (2n,P(z))=1}} v(2n)$$

) , where

$$z \asymp N^{0.25001};\ y \asymp N^{\frac{1}{3}};\ f \asymp N^{\frac{1}{2}};\ (2n, N) = 1;\ 2n < N.$$

And the weakened conjecture of the Elliott-Halberstam

$$(1.2)\qquad \sum_{d\le D} \max_{(a,d)=1} \left| \psi(y; q, a) - \frac{y}{\phi(q)} \right| \ll \frac{x}{(\log x)^B},$$

where

$$|a| \le (\log N)^B;\ D = N^{1-C};\ C \approx 0.002;\ B \ge 3,$$

Key words and phrases: Sophie Germain prime, safe prime, strong prime, parity problem

it can be proved that there are infinitely many prime numbers in linear form

$$p_r = 2p_u + a.$$

The main role is played by the upper bound for the sum for numbers of the form

$$2p_1p_2 \in A = \{p - a;\ p \le N, p \in \mathbb{P},\ z \le p_1 < y \le p_2,\ |a| \le (\log N)^B,\ B \ge 3\}.$$

2 Main results

Theorem 2.1. *Assuming* (1.2) *there are infinitely many primes of the form*

$$p_r = 2p_u + a,$$

where a is an arbitrary fixed odd integer.

The proof of Theorem 2.1 is given at the end of the paper. We now give several intermediate theorems and lemmas.

Theorem 2.2. (See Theorem 9.7 (Jurkat-Richert) [1])

Let $J = \{a(n)\}_{n=1}^{\infty}$ be an arithmetic function such that

$$a(n) \ge 0 \qquad \text{for all } n$$

and

$$|J| = \sum_{n=1}^{\infty} a(n) < \infty.$$

Let $\mathbb{P}$ be a set of prime numbers ($2 \notin \mathbb{P}$) and, for $z \ge 2$, let

$$P(z) = \prod_{\substack{p \in \mathbb{P} \\ p < z}} p.$$

Let

$$S(J, \mathbb{P}, z) = \sum_{\substack{n=1 \\ (n, P(z))=1}}^{\infty} a(n).$$

For every $n \ge 1$, let $g_n(d)$ be a multiplicative function such that

$$0 \le g_n(p) < 1 \qquad \text{for all } p \in \mathbb{P}.$$

Define $r(d)$ by

$$|J_d| = \sum_{\substack{n=1 \\ d|n}}^{\infty} a(n) = \sum_{n=1}^{\infty} a(n) g_n(d) + r(d).$$

Let $\mathbb{Q}$ be a finite subset of $\mathbb{P}$, and let Q be the product of the primes in $\mathbb{Q}$. Suppose that, for some ϵ satisfying $0 < \epsilon < \frac{1}{200}$, the inequality

$$\prod_{\substack{p\in\mathbb{P}\setminus\mathbb{Q}\\ u\le p<z}} (1-g_n(p))^{-1} < (1+\epsilon)\frac{\log z}{\log u}$$

holds for all n and $1 < u < z$. Then for any $D \ge z$ there is the upper bound

$$S(J,\mathbb{P},z) < (F(s)+\epsilon e^{14-s})X + R, \tag{2.1}$$

and for any $D \ge z^2$ there is the lower bound

$$S(J,\mathbb{P},z) > (f(s)-\epsilon e^{14-s})X - R, \tag{2.2}$$

where

$$s = \frac{\log D}{\log z},$$

$F(s)$ and $f(s)$ are the continuous functions defined as

$$F(s) = 1 + \sum_{\substack{n=1\\ n\equiv 1 (mod 2)}}^{\infty} f_n(s) \ \ for\ s \ge 1; \quad f(s) = 1 - \sum_{\substack{n=2\\ n\equiv 0 (mod 2)}}^{\infty} f_n(s) \ \ for\ s \ge 2,$$

$$X = \sum_{n=1}^{\infty} a(n) \prod_{p|P(z)} (1-g_n(p)),$$

and the remainder term is

$$R = \sum_{\substack{d|P(z)\\ d<DQ}} |r(d)|.$$

If there is a multiplicative function $g(d)$ such that $g_n(d) = g(d)$ for all n, then

$$X = V(z)|J|,$$

where

$$V(z) = \prod_{p|P(z)} (1-g(p)).$$

Lemma 2.3. (See Theorem 4, Theorem 1 [3] and Lemma 2 [1])

An arithmetic function $\lambda(d)$ is said to be well-factorable of level $D \ge 1$ if for any $R, S \ge 1$ with $RS = D$ there are functions δ_r, η_s with $|\delta_r|$, $|\eta_s| \le 1$ supported on $r < R$, $s < S$, such that

$$\lambda(d) = \sum_{rs=d} \delta_r \eta_s.$$

Let $0 < \epsilon < \frac{1}{8}$, $2 \le z \le D$. *Then from* Theorem 4 [3] *it follows*

$$S(J,\mathbb{P},z) \le X(F(\frac{\log D}{\log z}) + E) + \sum_{l<L} \sum_{d|P(z)} \lambda_l^+(d) r(J,d). \tag{2.3}$$

J,$\mathbb{P}$,z,X *are defined as in* Theorem 2.2. *In this formula,* L *depends only on* ϵ *and* λ_l^+ *- is well factorable coefficient of order* 1 *and of level* D*, and the constant* E *satisfies*

$$E = O(\epsilon + \epsilon^{-8} e^K (\log D)^{-\frac{1}{3}}),$$

where K *is some constant* > 1. *Using the definition given in* Theorem 2.2

$$F(s) = \frac{2e^\gamma}{s} \quad for\ 0 < s \le 2$$

with γ *- EulerâĂŞMascheroni constant.*

Lemma 2.4. (See Lemma 6 [1])

We denote by $|\alpha_h|$, $|\beta_m| \le 1$ *two sequences with* $h \in [H, 2H)$ *and* $m \in [M, 2M)$*, also define* $\nu = \frac{\log H}{\log N}$, $N = 2HM$ *and the following equality*

$$\sum_{(d,a)=1} \lambda(d) \left(\sum_{hm \equiv a[d]} \alpha_h \beta_m - \frac{1}{\varphi(d)} \sum_{(hm,d)=1} \alpha_h \beta_m \right) = O_B \left(\frac{N}{(\log N)^B} \right) \tag{2.4}$$

is true for $B \ge 3$*, uniformly for* $|a| \le (\log N)^B$*, for any positive* ϵ*, for any* ν *(*$\epsilon \le \nu \le 1-\epsilon$*), with* $D = N^{\theta(\nu)-\epsilon}$*, where the function* $\theta(\nu)$ *has the following value:*

$$\begin{cases} \frac{2}{3} - \frac{\nu}{3} & for\ \ \frac{1}{4} < \nu \le \frac{2}{7}, \\ \frac{1}{2} + \frac{\nu}{4} & for\ \ \frac{2}{7} \le \nu \le \frac{2}{5}, \\ 1 - \nu & for\ \ \frac{2}{5} \le \nu \le \frac{1}{2}. \end{cases}$$

Lemma 2.5. *To estimate the sum*

$$\frac{1}{2} \sum_{2n \in A} \sum_{\substack{2n = 2p_1p_2 \\ z \le p_1 < y \le p_2}} 1$$

we pass from one set A *to another* F *(switching principle), we obtain*

$$\frac{1}{2} S(F,\mathbb{P},f) \le 0.1773748 \frac{e^\gamma}{2} \frac{N}{\log N} V(z) + O\left(\frac{N}{(\log N)^B} \right), \tag{2.5}$$

where

$$F = \{2p_1p_2 + a :\ \ z \le p_1 < y \le p_2,\ 2p_1p_2 < N,\ (2p_1p_2, N) = 1,\ |a| \le (\log N)^B,\ B \ge 3\}.$$

Proof. The remainder term of the sieving function $S(F, \mathbb{P}, f)$ by Lemma 2.3 will be equal to

$$\sum_{d} \lambda(d) \left(|F_d| - \frac{|F|}{\phi(d)} \right) = O\left(\frac{N}{(\log N)^B} \right)$$

with $(2, d) = 1$ and $D = N^{\theta(\nu)-\epsilon}$. The minimum value for $\theta(\nu)$ is defined in Lemma 2.4, i.e

$$\theta\left(\frac{\log p_1}{\log N} \right) \geq \frac{4}{7} \quad for\ z \leq p_1 \leq y.$$

Since $\frac{V(f)}{V(z)} = \frac{\log z}{\log f}\left(1 + O\left(\frac{1}{\log N}\right)\right) = 0.50002 + O\left(\frac{1}{\log N}\right)$ (using the definition for $V(z)$ in Theorem 2.2) we have

$$\frac{1}{2} S(F, \mathbb{P}, f) \leq 0.50002 \frac{7}{8} \frac{e^\gamma}{2} \int_{0.25001}^{1/3} \frac{dt}{t(1-t))} \frac{N}{\log N} V(z) + O\left(\frac{N}{(\log N)^B} \right).$$

□

3 Proof of Theorem 2.1

Let $z(2p + a, N)$ be the number of primes of the form $p_r = 2p_u + a \leq N$, where a is an arbitrary fixed odd integer and $N > e^{|a|^{\frac{1}{B}}}$; $B = 3$. We also denote $A = \{p - a;\ p \leq N, p \in \mathbb{P},\ |a| \leq (\log N)^B,\ B \geq 3\}$ and $P(z) = \prod_{\substack{p \in \mathbb{P} \\ p < z}} p$.

We give a lower bound for $z(2p + a, N)$ using the weight function (1.1).

$$z(2p + a, N) \geq \sum_{\substack{2n \in A \\ n \in \{1, p \geq z\}}} 1 \geq \sum_{\substack{2n \in A \\ (2n, P(z)) = 1 \\ n \in \{1, p \geq z\}}} 1 \geq \sum_{\substack{2n \in A \\ (2n, P(z)) = 1}} v(2n).$$

Now open the last sum and applying the switching principle for the set A we obtain

$$z'(2p+a, N) = S(A, \mathbb{P}, z) - \frac{1}{2} \sum_{z \leq q < y} S(A_q, \mathbb{P}, z) - \frac{1}{2} S(B, \mathbb{P}, f) - \frac{1}{2} S(F, \mathbb{P}, f) - S(E, \mathbb{P}, f) + O(N^{\frac{3}{4}}),$$

where $z(2p + a, N) \geq z'(2p + a, N)$,

$$B = \{2p_1p_2p_3 + a :\ z \leq p_1 < y \leq p_2 \leq p_3,\ 2p_1p_2p_3 < N,\ (2p_1p_2p_3, N) = 1,\ |a| \leq (\log N)^B,\ B \geq 3\}$$

and

$$E = \{2p_1p_2 + a :\ y \leq p_1 < f \leq p_2,\ 2p_1p_2 < N,\ (2p_1p_2, N) = 1,\ |a| \leq (\log N)^B,\ B \geq 3\}.$$

The first two sums in $z'(2p+a,N)$ are estimated using the Theorem 2.2 and the weakened Elliott-Halberstam conjecture (1.2) with

$$f(s) = \frac{2e^\gamma \log(s-1)}{s} \quad for\ s = \frac{\log D}{\log z} \in [3,4]; \quad F(s_q) = \frac{2e^\gamma}{s_q} \quad for\ s_q = \frac{\log \frac{D}{q}}{\log z} \in (0,3].$$

Acting as in Theorem 10.4 [2] and Theorem 10.5 [2], only with the value $|a| \le (\log N)^B,\ B \ge 3,\ z = N^{0.25001}$ and $D = N^{1-C},\ C \approx 0.002$ we obtain

$$S(A,\mathbb{P},z) \ge (f(s) - \epsilon e^{14-s}) V(z) \sum_{n=1}^{\infty} a(2n) + O\left(\frac{N}{(\log N)^B}\right) \ge 1.0981287 \frac{e^\gamma}{2} \frac{N}{\log N} V(z),$$

respectively

$$\frac{1}{2} \sum_{z \le q < y} S(A_q, \mathbb{P}, z) \le 0.50002 * e^\gamma N \sum_{z \le q < y} \frac{1}{\phi(q) \log\left(\frac{D}{q}\right)} + O\left(\frac{N}{(\log N)^B}\right) \le$$

$$\le 0.50002 * e^\gamma \int_{0.25001}^{1/3} \frac{dt}{t(0.998-t))} \frac{N}{\log N} V(z).$$

The third sum in $z'(2p+a,N)$ is estimated as for Theorem 10.6 [2], only with the value $|a| \le (\log N)^B,\ B \ge 3$ and $z = N^{0.25001}$ and since

$$\frac{V(f)}{V(z)} = \frac{\log z}{\log f}\left(1 + O\left(\frac{1}{\log N}\right)\right) = 0.50002 + O\left(\frac{1}{\log N}\right)$$

(using the definition for $V(z)$ in Theorem 2.2) we obtain

$$\frac{1}{2} S(B,\mathbb{P},f) \le 0.50002 * e^\gamma V(z) \sum_{\substack{z \le p_1 < y \le p_2 \le p_3 \\ 2p_1p_2p_3 \le N}} 1 + O\left(\frac{N}{(\log N)^B}\right) \le$$

$$\le 0.50002 \frac{e^\gamma}{2} \int_{0.25001}^{1/3} \int_{1/3}^{(1-\beta)/2} \frac{dt d\beta}{t\beta(1-t-\beta)} \frac{N}{\log N} V(z).$$

An estimate for the fourth sum is given in Lemma 2.5.

$$\frac{1}{2} S(F,\mathbb{P},f) \le 0.1773748 \frac{e^\gamma}{2} \frac{N}{\log N} V(z).$$

It remains to estimate the last sum in $z'(2p+a,N)$. Acting as in Lemma 2.5, we choose the minimum value of the function

$$\theta\left(\frac{\log p_1}{\log N}\right) \ge \frac{1}{2} \quad for\ y \le p_1 \le f,$$

so we have

$$S(E,\mathbb{P},f) \le 0.50002 * e^\gamma \int_{1/3}^{1/2} \frac{dt}{t(1-t))} \frac{N}{\log N} V(z) + O\left(\frac{N}{(\log N)^B}\right) \le$$

$$\leq 0.693175 \frac{e^{\gamma}}{2} \frac{N}{\log N} V(z).$$

Putting together estimates for the sums in $z'(2p + a, N)$, we obtain

$$z(2p+a, N) \geq (1.0981287 - 0.2032878 - 0.0240915 - 0.1773748 - 0.693175) \frac{e^{\gamma}}{2} \frac{N}{\log N} V(z) \geq$$

$$\geq 0.00019 \frac{e^{\gamma}}{2} \frac{N}{\log N} V(z), \tag{3.1}$$

with

$$V(z) = \prod_{\substack{p<z \\ (p,N)=1}} \left(1 - \frac{1}{p-1}\right) = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p>2}} \frac{p-1}{p-2} * \frac{e^{-\gamma}}{\log z} \left(1 + O\left(\frac{1}{\log N}\right)\right)$$

for sufficiently large N, this proves Theorem 2.1.

Acknowledgements

None.

References

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