Theory about rounding of real numbers.

Juan Elias Millas Vera
juanmillasgz@gmail.com
Zaragoza (Spain)
July 2021

0- Abstract:
With a very intuitive notation, we are going to see in this paper a solution of the approximation to a determinate position digit in whole and fractional numbers. We are going to use ceiling and floor functions to give precision to the rounding.

1- Introduction:
Firstly, we should see the main notation for the different digits number following the decimal positioning system, we are going to name $<a>$ to the whole part and $<b>$ to the fractional part.

$$a_m a_{(m-1)} \ldots a_2 a_1 a_0 \cdot b_1 b_2 b_3 \ldots b_{(n-1)} b_n$$

Where:
- $a_0$ correspond to ones, $a_1$ to the tens and $a_2$ to the hundreds.
- $b_1$ correspond to tenths, $b_2$ to the hundredths and $b_3$ to the thousandths.

Now with a given number $a_m \cdot b_n$ we will use the ceiling ( $\lceil a_m \cdot b_n \rceil$ ) or the floor ( $\lfloor a_m \cdot b_n \rfloor$ ) function. We will use a positional determinant $s$ in the super index, this will show us which is the relevant last number in the approximation. The super index will be an indeterminate variable $a_m$ or $b_n$ and it will precise which determinate position we should choose end with.

2- Theory:

**Theorem 1:** The main rounding of the number could be in the whole part, and it will be cut in a determinate last right position number $a_x$, we should substitute all numbers cutting off in the right part of the determinate number, with zeros until the position of ones $a_0$. If the last number chosen are in the fractional part we just delete (substituting for zeros) all numbers in the right part and rounding to the determinate $b_x$.

**Proposition 1:** The possible combinations of the method are four of them: 1- $\lceil (a_m \cdot b_n) \rceil^{(ax)}$ 2- $(a_m \cdot b_n)^{(ax)}$ 3- $\lfloor (a_m \cdot b_n) \rfloor^{(bx)}$ and 4- $(a_m \cdot b_n)^{(bx)}$. The first one express a ceiling function with and ending in the whole part, the second one a floor function with an ending in the whole part, the third a ceiling function with and ending in the fractional part and the fourth a floor function with and ending in the fractional part.
**Proposition 2:** In the ceiling function the rounding will be plus one (+1) to the determinate $a_x$ or $b_x$ not depending which is the $a_{(x-1)}$ or the $b_{(x+1)}$ previous or next. All $a_{(x-n)}$ until $a_0$ will be zeros (0s) and all $b_{(x+n)}$ will be zero until $-\infty$. Except when all the right numbers until $-\infty$ are zero (0) then we should add zero (+0) to the chosen number.

- **Lemma 1:** If the chosen determinate variable is in a determinate case a number nine (9) in $a_x$ instead of plus one (Proposition 2), we should substitute with a zero (0) and add one (+1) in $a_{(x+1)}$.
- **Lemma 1.1:** If $a_{(x+1)}$ is also a nine (9) we should substitute with a zero (0) $a_{(x+1)}$ and add one (+1) to $a_{(x+2)}$. If we have an iteration we should do this steps until $a_{(x+n)}$ is not a nine and then add one (+1) to these number.
- **Lemma 2:** If the chosen determinate variable is in a determinate case a number nine (9) in $b_x$ instead of plus one (Proposition 2) we should substitute with a zero (0) and add one (+1) in $b_{(x+1)}$.
- **Lemma 2.1:** If $b_{(x-1)}$ is also a nine (9) we should substitute with a zero (0) $b_{(x-1)}$ and add one (+1) to $b_{(x-2)}$. In this case too if we have an iteration we should continue doing this until $b_{(x-m)}$ is not a nine.
- **Lemma 2.2:** If we have a necessary iteration of substitutions of nines to zeros until $b_1$ and we pass to $a_0$ we should continue doing substitutions in the whole part and add one (+1) to the non-nine number. (Ex.11)

It will be interesting to notice that in some cases there are possibilities to obtain the same result in a different ceiling functions. ( $\lfloor [a_m, b_n] \rfloor^{(ax)}$ or $\lceil [a_m, b_n] \rceil^{(bx)}$). This happens in the exposed situations of Lemmas 1 and 2. In this situations $\lfloor [a_m, b_n] \rfloor^{(ax)} = \lceil [a_m, b_n] \rceil^{(ax+1)}$, when $a_9 = 9$ and $\lfloor [a_m, b_n] \rfloor^{(bx-1)} = \lceil [a_m, b_n] \rceil^{(bx)}$, when $b_9 = 9$. We will see in the examples part in (Ex.16 and Ex.17)

**Proposition 3:** The rounding will be plus zero (+0) in the $a_x$ or $b_x$ in the floor function. All $a_{(x-n)}$ until $a_0$ will be zeros (0s) and all $b_{(x+n)}$ will be zero until $-\infty$.

3-Examples:

1. $[32.0]^{(a)} = 32.0$
2. $[32.0]^{(b)} = 32.0$
3. $[32.0]^{(a+1)} = 40.0$
4. $[32.2]^{(a)} = 33.0$
5. $[32.2]^{(b)} = 32.2$
6. $[32.2]^{(a+1)} = 40.0$
7. $[538.79]^{(b+1)} = 538.7$
8. $[9.3873]^{(b+2)} = 9.38$
9. $[38.37521]^{(a+0)} = 39$
10. $[3259.395]^{(b+2)} = 3259.4$
11. $[59.9953]^{(b+2)} = 60$
12. $[7853.39]^{(a+2)} = 7800$
13. $[325379839.235]^{(a+6)} = 325000000$
14. $[3.141592]^{(b+5)} = 3.14159$
15. $[3.141592]^{(b+5)} = 3.1416$
16. $[89.59]^{(a+0)} = 90$
17. $[89.59]^{(a+1)} = 90$
4- Conclusions:

We have seen the main theorem and its related ideas and exemplified some key cases for understanding the theory. This theory can be interesting in applied sciences such as physics, chemistry or economics. It is a clear example of how an elementary idea that we all learned as children can be taken further in a precise and rigorous way.