Values of the Barnes function

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Abstract

In this paper, we give conjectural values for the Barnes function G at unsual points as $G(1/5)$, $G(2/5)$ or $G(3/10)$ for example.

Expressions for $G(k/2), G(k/3), G(k/4)$ or $G(k/6)$ with k natural integer are well-known but what about $G(k/5)$ or $G(k/10)$?

We give eight conjectural integral formulas and we see several applications.

There are 2 cases:

- Either there is an integral with the bounds (0 and $-1/5$) and two trigamma functions.

- Or there are just only two trigamma functions at 1/5 and at 2/5. (1)

1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G(1+z) = (2\pi)^{\frac{z}{2}} \, \mathrm{e}^{-\frac{z(1+z)}{2} - \frac{\gamma z^2}{2}} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k \mathrm{e}^{-z + \frac{z^2}{2k}} \qquad (2)$$

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

$$G(1) = 1 \quad (3)$$

$$G(1+z) = G(z)\,\Gamma(z) \quad (4)$$

$$\log(G(1+z)) = \frac{z\log(2\pi)}{2} - \frac{z(1+z)}{2} + z\log(\Gamma(1+z)) - \int_0^z \log(\Gamma(t+1))\,\mathrm{d}t \quad (5)$$

$$\int_0^z \log(\Gamma(t+1))\,\mathrm{d}t = \frac{z\log(2\pi)}{2} - \frac{z(1+z)}{2} + z\log(\Gamma(1+z)) - \log(G(z)) - \log(\Gamma(z)) \quad (6)$$

3 List of conjectural formulas (found experimentally)

Let A be the Glaisher–Kinkelin constant. Then

$$\log\left(G\left(\frac{1}{5}\right)\right) = \frac{2}{25} - \int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\,\mathrm{d}t - \frac{\log(\pi)}{10} - \frac{\log(2)}{10} + \frac{4\pi^2 + 4\pi^2\sqrt{5} - 2\sqrt{5}\Psi(1,2/5) + \Psi\left(1,\frac{1}{5}\right)\left(-\sqrt{5}-5\right)}{50\pi\sqrt{10+2\sqrt{5}}} - \frac{4\log\left(\Gamma\left(\frac{1}{5}\right)\right)}{5}$$

$$\log\left(G\left(\frac{2}{5}\right)\right) = \frac{3}{25} + \int_0^{\frac{-1}{5}} \log(\Gamma(t+1))\,\mathrm{d}t - \frac{12\log(A)}{5} + \frac{\log(\pi)}{10} + \frac{\log(2)}{10} + \frac{\log(5)}{120} + \frac{\left(5-\sqrt{5}\right)\Psi\left(1,\frac{1}{5}\right) + \left(3\sqrt{5}+5\right)\Psi\left(1,\frac{2}{5}\right) - 2\pi^2\sqrt{5} - 6\pi^2}{100\sqrt{10+2\sqrt{5}}\pi} - \frac{3\log\left(\Gamma\left(\frac{2}{5}\right)\right)}{5}$$

$$\log\left(G\left(\frac{3}{5}\right)\right) = \frac{3}{25} + \int_0^{\frac{-1}{5}} \log(\Gamma(t+1))\,\mathrm{d}t - \frac{12\log(A)}{5} - \frac{3\log(\pi)}{10} - \frac{\log(2)}{2} + \frac{13\log(5)}{120} + \frac{\log\left(\sqrt{5}+1\right)}{5} + \frac{\left(3\sqrt{5}+5\right)\Psi\left(1,\frac{1}{5}\right) + \left(\sqrt{5}-5\right)\Psi\left(1,\frac{2}{5}\right) - 6\pi^2\sqrt{5} - 2\pi^2}{100\sqrt{10+2\sqrt{5}}\pi} + \frac{2\log\left(\Gamma\left(\frac{2}{5}\right)\right)}{5}$$

$$\log\left(G\left(\frac{4}{5}\right)\right) = \frac{2}{25} - \int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\,\mathrm{d}t - \frac{3\log(\pi)}{10} - \frac{\log(2)}{5} + \frac{\log(5)}{20} - \frac{\log\left(\sqrt{5}+1\right)}{10} + \frac{\log\left(\Gamma\left(\frac{1}{5}\right)\right)}{5}$$

$$\log\left(G\left(\frac{1}{10}\right)\right) = \frac{9}{200} - \frac{3}{2}\int_0^{-\frac{1}{5}} \log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t + \frac{9\,\log\left(A\right)}{10} + \frac{3\,\log\left(\pi\right)}{10} + \frac{289\,\log\left(2\right)}{600} - \frac{7\,\log\left(5\right)}{30} - \frac{9\,\log\left(\sqrt{5}+1\right)}{20} + \frac{\left(-4\sqrt{5}-10\right)\Psi\left(1,\frac{1}{5}\right) + \left(-3\sqrt{5}+5\right)\Psi\left(1,\frac{2}{5}\right) + 10\,\pi^2\sqrt{5} + 6\,\pi^2}{100\sqrt{10+2\sqrt{5}}\pi} - \frac{9\,\log\left(\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\right)}{10}$$

$$\log\left(G\left(\frac{3}{10}\right)\right) = \frac{21}{200} + \frac{3}{2}\int_0^{-\frac{1}{5}} \log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t - \frac{27\,\log\left(A\right)}{10} - \frac{\log\left(\pi\right)}{5} - \frac{509\,\log\left(2\right)}{600} + \frac{\log\left(5\right)}{240} + \frac{7\,\log\left(\sqrt{5}+1\right)}{10} + \frac{\left(3\sqrt{5}+5\right)\Psi\left(1,\frac{1}{5}\right) + \left(\sqrt{5}-5\right)\Psi\left(1,\frac{2}{5}\right) - 6\,\pi^2\sqrt{5} - 2\,\pi^2}{200\sqrt{10+2\sqrt{5}}\pi} + \frac{7\,\log\left(\frac{\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{1}{5}\right)}\right)}{10}$$

$$\log\left(G\left(\frac{7}{10}\right)\right) = \frac{21}{200} + \frac{3}{2}\int_0^{-\frac{1}{5}} \log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t - \frac{27\,\log\left(A\right)}{10} - \frac{29\,\log\left(2\right)}{600} + \frac{\log\left(5\right)}{240} + \frac{\left(3\sqrt{5}+25\right)\Psi\left(1,\frac{1}{5}\right) + \left(11\sqrt{5}+5\right)\Psi\left(1,\frac{2}{5}\right) - 18\,\pi^2\sqrt{5} - 22\,\pi^2}{200\sqrt{10+2\sqrt{5}}\pi} - \frac{3\,\log\left(\frac{\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{1}{5}\right)}\right)}{10}$$

$$\log\left(G\left(\frac{9}{10}\right)\right) = \frac{9}{200} - \frac{3}{2}\int_0^{-\frac{1}{5}} \log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t + \frac{9\,\log\left(A\right)}{10} - \frac{3\,\log\left(\pi\right)}{10} - \frac{131\,\log\left(2\right)}{600} + \frac{\log\left(5\right)}{60} - \frac{\log\left(\sqrt{5}+1\right)}{20} + \frac{\left(\sqrt{5}-5\right)\Psi\left(1,\frac{1}{5}\right) + \left(-3\sqrt{5}-5\right)\Psi\left(1,\frac{2}{5}\right) + 2\,\pi^2\sqrt{5} + 6\,\pi^2}{100\sqrt{10+2\sqrt{5}}\pi} + \frac{\log\left(\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\right)}{10}$$

4 Examples of applications

First case

Consider the sum

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1) z^{2k+2} \left(2^{2k+1}-1\right)}{k+1} \quad (8)$$

and let $z = 3/10$. We obtain

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)\left(2^{2k+1}-1\right)}{k+1} \left(\frac{3}{10}\right)^{2k+2} =$$
$$-\frac{9}{100} - \frac{9\gamma}{100} - \frac{29 \log(2)}{300} - \frac{\log\left(\Gamma\left(\frac{4}{5}\right)\right)}{2} - \frac{\log\left(\Gamma\left(\frac{1}{5}\right)\right)}{2}$$
$$- \log\left(G\left(\frac{1}{5}\right)\right) - \log\left(G\left(\frac{4}{5}\right)\right) + 3\zeta(1,-1).$$

With the definition (5),we have:

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)\left(2^{2k+1}-1\right)}{k+1} \left(\frac{3}{10}\right)^{2k+2} = \int_0^{\frac{1}{5}} \log(\Gamma(t+1))\, dt + \int_0^{\frac{4}{5}} \log(\Gamma(t+1))\, dt$$
$$+ 3\zeta(1,-1) + \frac{3}{4} - \frac{9\gamma}{100} - \frac{4\log(\pi)}{5}$$
$$- \frac{176 \log(2)}{75} + \frac{43\log(5)}{40} - \frac{3\log\left(\sqrt{5}+1\right)}{20} + \frac{3\log\left(\Gamma\left(\frac{1}{5}\right)\right)}{5}$$

With the rule (6) and using formulas, finally

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)\left(2^{2k+1}-1\right)}{k+1} \left(\frac{3}{10}\right)^{2k+2} = 2\int_0^{-1/5} \log(\Gamma(t+1))\, dt - 3\log(A) - \frac{9\gamma}{100}$$
$$- \frac{\log(\pi)}{10} - \frac{7\log(2)}{150} + \frac{3\log(5)}{40} - \frac{3\log\left(\sqrt{5}+1\right)}{20}$$
$$+ \frac{\left(5+\sqrt{5}\right)\Psi\left(1,\frac{1}{5}\right) + 2\sqrt{5}\Psi(1,2/5) - 4\pi^2 - 4\pi^2\sqrt{5}}{50\sqrt{10+2\sqrt{5}}\pi}$$
$$+ \frac{3\log\left(\Gamma\left(\frac{1}{5}\right)\right)}{5}$$

Other example

Consider the sum

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1) z^{2k+2}}{k+1} \quad (9)$$

and let $z = 9/10$. We obtain

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}\left(\frac{9}{10}\right)^{2k+2}=-\frac{81}{100}-\frac{81\gamma}{100}-\log\left(G\left(\frac{1}{10}\right)G\left(\frac{19}{10}\right)\right)$$

With the definition (5), we have:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}\left(\frac{9}{10}\right)^{2k+2}=\int_0^{\frac{9}{10}}\log(\Gamma(t+1))\,\mathrm{d}t+\int_0^{\frac{-9}{10}}\log(\Gamma(t+1))\,\mathrm{d}t-\frac{81\gamma}{100}$$
$$-\frac{9\log(\pi)}{5}-\frac{9\log(2)}{25}-\frac{9\log(3)}{5}$$
$$+\frac{27\log(5)}{20}+\frac{9\log\left(\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\right)}{5}$$

With the rule (6) and using formulas, finally

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}\left(\frac{9}{10}\right)^{2k+2}=3\int_0^{-1/5}\log(\Gamma(t+1))\,\mathrm{d}t-\frac{9\log(A)}{5}-\frac{9}{10}-\frac{81\gamma}{100}$$
$$-\frac{3\log(\pi)}{2}-\frac{289\log(2)}{300}+\frac{7\log(5)}{15}$$
$$+\frac{\left(3\sqrt{5}+15\right)\Psi\left(1,\frac{1}{5}\right)+6\,\Psi(1,2/5)\sqrt{5}-12\pi^2\sqrt{5}-12\pi^2}{100\sqrt{10+2\sqrt{5}}\pi}$$
$$+\frac{9\log\left(\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\right)}{5}$$

Second case (there are only two trigamma functions at the end)

Consider and calculate the closed form of

$$2\int_0^{-\frac{9}{10}}\log(\Gamma(t+1))\,\mathrm{d}t+3\int_0^{-3/5}\log(\Gamma(t+1))\,\mathrm{d}t$$

We obtain

$$\frac{27\log(A)}{5}-\frac{9\log(\pi)}{5}-\frac{541\log(2)}{300}-\frac{\log(5)}{120}$$
$$+\frac{\left(11\sqrt{5}+5\right)\Psi\left(1,\frac{1}{5}\right)+\left(-3\sqrt{5}-25\right)\Psi\left(1,\frac{2}{5}\right)-14\pi^2\sqrt{5}+6\pi^2}{100\sqrt{10+2\sqrt{5}}\pi}$$

Other example with three integrals

Consider and calculate the closed form of

$$6\int_0^{\frac{7}{10}}\log(\Gamma(t+1))\,\mathrm{d}t-4\int_0^{\frac{13}{10}}\log(\Gamma(t+1))\,\mathrm{d}t+3\int_0^{9/5}\log(\Gamma(t+1))\,\mathrm{d}t$$

We obtain

$$\frac{27\,\log(A)}{5}-\frac{28}{5}+\frac{11\,\log(\pi)}{5}+\frac{2771\,\log(2)}{300}+\frac{48\,\log(3)}{5}-\frac{673\,\log(5)}{120}+\frac{21\,\log(7)}{5}-\frac{26\,\log(13)}{5}+\frac{\left(-3\sqrt{5}-65\right)\Psi\left(1,\frac{1}{5}\right)+\left(-31\sqrt{5}-25\right)\Psi\left(1,\frac{2}{5}\right)+42\,\pi^2\sqrt{5}+62\,\pi^2}{100\sqrt{10+2\sqrt{5}}\pi}$$

Consider and calculate the closed form of

$$\prod_{k=1}^{\infty}\left(\frac{(10\,k+1)\,(10\,k+5)^2\,(10\,k+7)}{(10\,k+2)\,(10\,k+4)^2\,(10\,k+8)}\right)^k \qquad (10)$$

We obtain

$$\mathrm{e}^{\frac{\left(3\sqrt{5}+5\right)\Psi\left(1,\frac{1}{5}\right)+\left(\sqrt{5}-5\right)\Psi\left(1,\frac{2}{5}\right)-2\,\pi^2-6\,\pi^2\sqrt{5}}{200\,\pi\sqrt{10+2\sqrt{5}}}}\,2^{\frac{1}{60}}\,5^{\frac{49}{240}}\left(\sqrt{5}+1\right)^{\frac{-7}{20}}$$

Consider and calculate the closed form of

$$\prod_{k=1}^{\infty}\left(\frac{(5\,k+11)\,(5\,k+7)\,(5\,k+4)\,(5\,k+2)}{(5\,k+10)\,(5\,k+8)\,(5\,k+5)\,(5\,k+1)}\right)^k \qquad (11)$$

We obtain

$$\mathrm{e}^{\frac{1}{5}+\frac{\left(-5\sqrt{5}+5\right)\Psi\left(1,\frac{1}{5}\right)+\left(5\sqrt{5}+15\right)\Psi\left(1,\frac{2}{5}\right)-10\,\pi^2+2\,\pi^2\sqrt{5}}{100\,\pi\sqrt{10+2\sqrt{5}}}}\,5^{\frac{-67}{24}}\left(\sqrt{5}+1\right)^{\frac{6}{5}}\pi^{-4/5}\Gamma\left(\frac{2}{5}\right)^{\frac{17}{5}}\Gamma\left(\frac{1}{5}\right)^{\frac{6}{5}}A^{-\frac{12}{5}}$$

Remark and conclusion: this is a first approach with these particular values of the Barnes function and any improvement for these eight expressions is welcome.

5 References

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