A TOPOLOGICAL APPROACH TO THE TWIN PRIME AND DE POLIGNAC CONJECTURES

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Abstract. We introduce a topology in the set of natural numbers via a subbase of open sets. With this topology, we obtain an irreducible (hyperconnected) space with no generic points. This fact allows proving that the cofinite intersections of subbasic open sets are always empty. That implies the validity of the Twin Prime Conjecture. On the other hand, the existence of strictly increasing chains of subbasic open sets shows that the Polignac Conjecture is false for an infinity of cases.

1. Introduction

We will begin by introducing a topology in the set X of the odd prime numbers via a subbase of open sets. With this topology, X becomes an irreducible (or hyperconnected) space and also a T_1 space. Moreover, using also a subbase, a topology is defined in the set X^* of the strictly positive integers. With this new topology, X^* also turns out to be an irreducible space.

In order to shorten, in this abstract, we call CISOS the cofinite intersections of subbasic open sets of the space X^*. We see immediately that if the twin primes conjecture was false then there would be a non-empty CISOS.

First we show that X^* has no generic points. A direct consequence of this is that no CISO can be a dense set. However, from the study of the notion of extremality introduced in section 3, it is obtained that if a CISOS is not the empty set then it must be dense (in fact it is seen that every non-empty intersection of subbasic open sets, not necessarily cofinite, must be dense). From all this, we deduce that all CISO must be empty and, so, the TPC must be true.

Finally, a strictly increasing chain of open subbasic sets of X^* is constructed. The existence of this chain shows that X^* is not a noetherian space and provides proof that de Polignac conjecture is false for an infinity of cases.

2. First properties of topological spaces X and X^*

Let X be the set of the odd prime numbers and $\mathbb{N}$ the set of the strictly positive integers.

For every $m \in \mathbb{N}$ we consider the subset we consider the subset defined as

$$O_m = \{p \in X : p + 2m \notin X\}$$

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We will denote H_m the complementary set

$$X - O_m = \{p \in X : p + 2m \in X\}$$

In order to clarify the notation, we will sometimes write $O(m)$ and $H(m)$ instead of O_m and H_m respectively.

We introduce in X the topology τ generated by the set of all the O_m as a subbase ([3]). This means that the open sets of this topology are all the reunions of finite intersections of the sets O_m, namely all sets of the form

$$\bigcup (O(i_1) \cap ... \cap O(i_n))$$

We start by proving some properties of the topological space (X, τ)

Proposition 2.1. *The space X is irreducible.*

Proof. It is enough to see that the intersection of a finite number of O_n,s is non-empty.

If we have $O(m_1) \cap ... \cap O(m_r) = \emptyset$ then $H(m_1) \cup ... \cup H(m_r) = X$. Let $\mu = \max\{m_1, ...m_r\}$. We know that there are arbitrarily large intervals of natural numbers that do not contain prime numbers (as an example of length $n-1$ of these intervals we can take $[n!+2, n!+n] \cap \mathbb{N}$). More specifically, for each natural number m there is a prime number q such that $s(q) - q > m$ where $s(q)$ is the prime number that follows q.

In particular, there must be a q such that $s(q) - q > 2\mu$ and, for this q, $\min\{j : q \in H_j\} > \mu \geq m_i$ for every $i = 1, ..., r$ or, in other words, $q \notin H(m_1) \cup H(m_r)$ which is a contradiction. □

Proposition 2.2. *The space X is* T_1

Proof. Let $p \in X$. Let's see that $\{p\}$ is a closed set. given a j such that $1 \leq j < p$, because $GCD(j, p) = 1$, there is a $\lambda > 1$ such that $j + \lambda p \in X$. In fact, the Dirichlet theorem relative to prime numbers in arithmetic progressions ([1]; chapter 7) ensures the existence of an infinite amount of these λ.

For all $j = 1, ..., p-1$ let $\nu_j = \min\{\lambda > 1 : j + \lambda p \in X\}$. We have:

$$p \in H\left(\frac{1}{2}(1 + (\nu_1 - 1)p)\right) \cap ... \cap H\left(\frac{1}{2}(p - 1 + (\nu_{p-1} - 1)p)\right)$$

Let $q \in H\left(\frac{1}{2}(1 + (\nu_1 - 1)p)\right) \cap ... \cap H\left(\frac{1}{2}(p - 1 + (\nu_{p-1} - 1)p)\right)$ and suppose that $q \equiv i \neq 0 \pmod p$ where $i < p$ and let say that $q = i + \mu p$, $i < p$.

Because $q \in H\left(\frac{1}{2}(p - i + (\nu_{p-i} - 1)p)\right)$ we have that

$$q + p - i + (\nu_{p-i} - 1)p = i + \mu p + p - i + (\nu_{p-i} - 1)p = (\mu + \nu_{p-i})p \in X$$

which is absurd because $\mu + \nu_{p-i} > 1$. We deduce that it must be $i = 0$ i.e. $q \equiv 0 \pmod p$ which implies $q = p$. So that $\{p\} = \bigcap_{j=1}^{p-1} H\left(\frac{1}{2}(j + (\nu_j - 1)p)\right)$ which is closed.

(We observe that T_1 is the largest separation that allows an irreducible space). $\square$

Remark 1: We emphasize that we have proved that every point in X can be written as a finite intersection of the closed sets H_m that contain it.

Proposition 2.3. *For every $m \in \mathbb{N}$ we have that $\bigcap_{\lambda\in N} O_{\lambda m}$ is the set of the odd primes that divide m. In particular, if p is a prime number then $\bigcap_{\lambda\in N} O_{\lambda p} = \{p\}$*

Proof. Suppose that p is a prime divisor of m. If p belonged to some $H_{\lambda m}$ then $p + 2\lambda m = p\left(1 + 2\lambda\dfrac{m}{p}\right) \in X$ which is absurd so that $p \in \bigcap_{\lambda\in N} O_{\lambda m}$

Let's see now that if p is an odd prime number that does not divide m then $p \notin \bigcap_{\lambda\in N} O_{\lambda m}$. If p does not divide m then $GCD(p, m) = 1$ and so, $GCD(p, 2m) = 1$. Applying again the Dirichlet theorem mentioned in proposition 2.2, there must be a λ such that $p+2\lambda m \in X$, that is, $p \in H_{\lambda m}$.

Now we introduce a topology τ^* in the set $X^* = \mathbb{N}$ taking the sets

$$O_p^* = \{m \in \mathbb{N} : p + 2m \notin X\}$$

(where p is any prime number) as subbase of open sets. In a similar way to what we have previously done with the space X we will write $H_p^* = \{m \in \mathbb{N} : p + 2m \in X\}$ to denote the complementary sets. Sometimes we will write $O^*(p)$ and $H^*(p)$ instead of O_p^* and H_p^*, respectively. $\square$

Let's start now the study of the space X^*.

Remark 2: Before starting the study of this space, we note that we now have a new language in order to enunciate the conjectures that are the object of this paper. Indeed, with this language, the de Polignac conjecture says that, for all m, the set $H_m \cap O_1 \cap ... \cap O_{m-1}$ has infinite elements and the twin primes conjecture simply says that the closed set H_1 is infinite.

Proposition 2.4. *X^* is an irreducible topological space.*

Proof. Again it is sufficient to prove that the intersection of a finite number of open sets of the form O_q^* is non empty.

Specifically, if $q_1, ...q_r$ are different primes then

$$q_1 \cdot ... \cdot q_r \in O^*(q_1) \cap ... \cap O^*(q_r)$$

because if for some $j = 1, ...r$ were $q_1 \cdot ... \cdot q_r \in H^*(q_j)$ then

$$q_j + 2q_1 \cdot ... \cdot q_r = q_j\left(1 + 2q_1 \cdot q_2 \cdot ... \cdot \hat{q}_j \cdot ... \cdot q_r\right) \in X$$

which is absurd. In fact, if ( ) means "ideal" then

$$(q_1 \cdot ... \cdot q_r) \cap \mathbb{N} = \{\lambda q_1 \cdot ... \cdot q_r : \lambda \in \mathbb{N}\} \subset O^*(q_1) \cap ... \cap O^*(q_r)$$

□

Proposition 2.5. *We have:*
a) If $H_p^* \subset H_q^*$ the $p=q$
b) The subspaces H_p^* are irreducible.

Proof. a) If $H_p^* \subset H_q^*$ then every m in H_p^* belongs to H_q^* or, in other words, if $p \in H_m$ then $q \in H_m$ or, put in other way, $\{m : p \in H_m\} \subset \{m : q \in H_m\}$ and that implies

$$\bigcap_{q \in H_m} H_m \subset \bigcap_{p \in H_m} H_m$$

From remark 1 after proposition 2.2, we deduce that, for every r,

$$\bigcap_{q \in H_r} H_r = \{r\}$$

and, so, the previous inclusion implies $\{q\} \subset \{p\}$ or, what is the same, $p = q$.

b) We must to prove that if T and T' are closed sets then $H_p^* \subset T \cup T'$ implies either $H_p^* \subset T$ or $H_p^* \subset T'$. We begin proving that if $H_p^* \subset \bigcup_{i=1}^{r} H^*(q_i)$ then there is i such that $p = q$.

If $H_p^* \subset \bigcup_{i=1}^{r} H^*(q_i)$ then $\bigcap_{i=1}^{r} O^*(q_i) \subset O^*(p)$ which means that for all m such that $\{q_1, ...q_r\} \subset O_m$ we must have that $p \in O_m$, but for all $\lambda \in \mathbb{N}$,

$$\{q_1, ...q_r\} \subset O(\lambda q_1 \cdot ... \cdot q_r)$$

so that, for all $\lambda \in \mathbb{N}$, $p \in O(\lambda q_1 \cdot ... \cdot q_r)$ that is $p \in \bigcap_{\lambda \in N} O(\lambda q_1 \cdot ... \cdot q_r)$. By proposition 2.3, the last intersection is equal to $\{q_1, ..., q_r\}$ and therefore $p = q_i$ for some $i = 1, ..., r$

Suppose that $H_p^* \nsubseteq T \cup T' =$

$$\left(\bigcap_i \left(H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)\right)\right) \cup \left(\bigcap_j \left(H^*(q_{j_1}) \cup ... \cup H^*\left(q_{j_{s(j)}}\right)\right)\right)$$

and that $H_p^* \nsubseteq T'$. Then there must be a j such that
$H_p^* \nsubseteq H^*(q_{j_1}) \cup ... \cup H^*\left(q_{j_{s(j)}}\right)$ but, for every i, we have

$$H_p^* \subset \left(H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)\right) \cup \left(H^*(q_{j_1}) \cup ... \cup H^*\left(q_{j_{s(j)}}\right)\right)$$

This implies that for each i exists a t, $(1 \le t \le r_i)$ such that $H_p^* = H^*(p_{i_t})$ since we can't have $H_p^* = H^*(q_{j_k})$. Therefore, for every i,
$H_p^* \subset H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)$ and, finally,

$$H_p^* \subset \bigcap_i \left(H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)\right) = T$$

□

Proposition 2.6. *$H_n \subset H_m$ if and only if $\overline{\{m\}} \subset \overline{\{n\}}$ where the upper bar means closure in the space X^**

Proof. $\Rightarrow$)
Suppose that $m \notin \overline{\{n\}}$. Then there will be $p_1, ..., p_r$ such that $m \in O^*(p_1) \cap ... \cap O^*(p_r)$ and $n \in H^*(p_1) \cup ... \cup H^*(p_r)$. So, for each $i = 1, ..., r$, p_i belongs to O_m and there is a j such that p_j belongs to H_n. As $H_n \subset H_m$, this p_j belongs to H_m which is a contradiction with $p_i \in O_m$ for every $i = 1, ..., r$
$\Leftarrow$)
$\overline{\{m\}} \subset \overline{\{n\}}$ implies $m \in \overline{\{n\}}$ and, therefore, for every p such that $m \in O_p^*$ we must have that $n \in O_p^*$ or, what is the same, for every p such that $n \in H_p^*$ we must have that $m \in H_p^*$. That means that, for every p, $p \in H_n$ implies $p \in H_m$. In other words, $H_n \subset H_m$. □

Proposition 2.7. $\overline{\{m\}} = \bigcap_{p \in H_m} H_p^*$

Proof. $\subset$)
Obviously, for every $p \in H_m$, we have $m \in H_p^*$ and therefore $m \in \bigcap_{p \in H_m} H_p^*$ so that

$$\overline{\{m\}} \subset \overline{\bigcap_{p \in H_m} H_p^*} = \bigcap_{p \in H_m} H_p^*$$

$\supset$)
Be $n \in \bigcap_{p \in H_m} H_p^*$. If $n \notin \overline{\{m\}}$ then will exist $q_1, ..., q_s$ such that $n \in O^*(q_1) \cap ... \cap O^*(q_s)$ and $m \in H^*(q_1) \cup ... \cap \cup H^*(q_s)$. This tell us that there is an i such that $m \in H^*(q_i)$, that is such that $q_i \in H_m$. So, in particular,we have that $n \in H^*(q_i)$ because $n \in \bigcap_{p \in H_m} H_p^*$. That contradicts the fact that $n \in O^*(q_1) \cap ... \cap O^*(q_s)$. □

Proposition 2.8. *$H_m \cap H_n = \emptyset$ if and only if $m \in \bigcap_{p \in H_n} O_p^*$*

Proof. We have successively: $H_m \cap H_n = \emptyset \Leftrightarrow H_n \subset O_m \Leftrightarrow$ for every $p \in H_n$ we must have that $p \in O_m \Leftrightarrow$ for all $p \in H_n$ it must have that $m \in O_p^* \Leftrightarrow m \in \bigcap_{p \in H_n} O_p^*$ □

Proposition 2.9. *The following conditions are equivalent:*
i) $I \subset \mathbb{N}$ is a finite set.
ii) $\bigcap_{i \in I} O^*(p_i)$ is an open non-empty set.
iii) The interior of $\bigcap_{i \in I} O^*(p_i)$ is non-empty.

Proof. $i) \Rightarrow ii)$ and $ii) \Rightarrow iii)$ are obvious (note that in $i) \Rightarrow ii)$, $\bigcap_{i\in I} O^*(p_i)$ is non empty because I is finite and, by proposition 1.4, X^* is an irreducible space). We're going to see $iii) \Rightarrow i)$.
If the interior of $\bigcap_{i\in I} O^*(p_i)$ is non-empty then there are $q_1, ...q_s$ such that $O^*(q_1) \cap ... \cap O^*(q_s) \subset \bigcap_{i\in I} O^*(p_i)$ or, which is equivalent, $\bigcup_{i\in I} H^*(p_i) \subset H^*(q_1) \cup ... \cup H^*(q_s)$ and, therefore, for every $i \in I$ we have that $H^*(p_i) \subset H^*(q_1) \cup ... \cup H^*(q_s)$. Using the proposition 2.5, we obtain that, for every $i \in I$, there is a j $(1 \leq j \leq s)$ such that $H^*(p_i) = H^*(q_j)$. So that, for every $i \in I$, there is a j $(1 \leq j \leq s)$ such that $p_i = q_j$ and so, $\{p_i : i \in I\} \subset \{q_1, ...q_s\}$ and I is finite. □

Proposition 2.10. *Are equivalent:*

i) For every $C \subset \mathbb{N}$ cofinite, $\bigcap_{q\in C} O_q^* = \emptyset$

ii) For every $m \in \mathbb{N}$, the closed set H_m has infinite points.

Proof. $i) \Rightarrow ii)$ If there is a $m \in \mathbb{N}$ such that H_m is finite then O_m is cofinite and, nevertheless, $\bigcap_{q\in O_m} O_q^* \neq \emptyset$ because $m \in \bigcap_{q\in O_m} O_q^*$.

Indeed, $n \in \bigcap_{q\in O_m} O_q^* \Leftrightarrow n \in O_q^*$ for every $q \in O_m \Leftrightarrow q \in O_n$ for every $q \in O_m \Leftrightarrow O_m \subset O_n$ and so, $m \in \bigcap_{q\in O_m} O_q^*$ because, obviously $O_m \subset O_m$.

$ii) \Rightarrow i)$ We have that $m \in O_q^* \Leftrightarrow q \in O_m$ and so, given any $I \subset X$, $m \in \bigcap_{q\in I} O_q^* \Leftrightarrow m \in O_q^*$ for every $q \in I$ which is equivalent to $q \in O_m$ for all $q \in I$ or, what is the same, $I \subset O_m$. Now, if there is a cofinite set C such that $\bigcap_{q\in C} O_q^* \neq \emptyset$ and $m \in \bigcap_{q\in C} O_q^*$, then $C \subset O_m$ and, therefore, $H_m \subset X - C$ which is finite. □

Remark 3: We observe that if we prove $\bigcap_{q\in C} O_q^* = \emptyset$ for every cofinite C, in particular, we will have proven that H_1 is an infinite set which, as we have already explained in remark 2 after proposition 2.3, is equivalent to the Twin Prime Conjecture.

We will write $A_n = \{m : H_m \cap H_n = \emptyset\}$ which, from proposition 2.8. is equal to $\bigcap_{q\in H_n} O_p^*$.

Proposition 2.11. *H_n is finite if and only if A_n is open.*

Proof. If H_n is finite then $A_n = \{m : H_m \cap H_n = \emptyset\} = \bigcap_{q\in H_n} O_p^*$ is open because is an intersection of a finite number of open sets. The reciprocal follows from proposition 2.9. □

Proposition 2.12. *$H_m = \emptyset$ if and only if $\overline{\{m\}} = X^*$ namely if and only if m is a generic point in X^*.*

Proof. We have that $H_m = \emptyset \Leftrightarrow O_m = X$ and this is equivalent to $p \in O_m$ for every $p \in X$ or, in other words, $m \in O_p^*$ for every $p \in X$. That is the same as saying that m belongs to all open sets $U \subset X^*$ which is equivalent to say $\overline{\{m\}} = X^*$ since $\{m\}$ is dense if and only if m belongs to the all open sets. □

We will call Z the set $\{m \in X^* : H_m = \emptyset\}$. Let's see that $Z = \emptyset$ namely X^* has no generic points.

Proposition 2.13. $Z = \bigcap_{q\in X} O_q^*$

Proof. We have $m \in Z \Leftrightarrow O_m = X \Leftrightarrow q \in O_m$ for every $q \in X \Leftrightarrow m \in O_q^*$ for every $q \in X \Leftrightarrow m \in \bigcap_{q\in X} O_q^*$. □

Proposition 2.14. $Z = \bigcap_{m\in X^*} A_m$

Proof. We have:

$$\bigcap_{m\in X^*} A_m = \bigcap_{m\in X^*} \bigcap_{q\in H_m} O_q^* = \bigcap_{q\in \bigcup_{m\in X^*} H_m} O_q^* = \bigcap_{q\in X} O_q^* = Z$$

□

Proposition 2.15. $Z = \bigcap_{n\in X^*-Z} A_n$

Proof. From the proposition 2.8. we know that $A_n = \bigcap_{p\in H_n} O_p^*$ and by proposition 2.7, $\overline{\{n\}} = \bigcap_{p\in H_n} H_p^*$. Obviously, for every $n \in X^* - Z$ we have $A_n \cap \overline{\{n\}} = \emptyset$ (note that $H_n \neq \emptyset$ because $n \in X^* - Z$). That is, for every $n \in X^* - Z$ we have $A_n \subset X^* - \overline{\{n\}}$ so that

$$Z = \bigcap_{n\in X^*} A_n \subset \bigcap_{n\in X^*-Z} A_n \subset \bigcap_{n\in X^*-Z} \left(X^* - \overline{\{n\}}\right) = X^* - \bigcup_{n\in X^*-Z} \overline{\{n\}} \subset$$
$$\subset X^* - \bigcup_{n\in X^*-Z} \{n\} = X^* - (X^* - Z) = Z$$

□

Proposition 2.16. *$Z=\emptyset$ namely X^* has no generic points. In particular, $H_m \neq \emptyset$ for every $m \in X^*$*

Proof. $A_n = \bigcap_{p\in H_n} O_p^*$ implies $X^* - A_n = \bigcup_{p\in H_n} H_p^*$ but $\overline{\{n\}} = \bigcap_{p\in H_n} H_p^*$ so that $\overline{\{n\}} \subset X^* - A_n$ and, therefore,

$$\bigcup_{n\in X^*} \overline{\{n\}} = X^* \subset \bigcup_{n\in X^*} (X^* - A_n) = X^* - \bigcap_{n\in X^*} A_n$$

From this we deduce that $X^* - \bigcap_{n\in X^*} A_n = X^*$ and, finally, that

$$Z = \bigcap_{n\in X^*} A_n = \emptyset$$

. □

Proposition 2.17. *We have:*

a) X^* is T_0 if and only if $O_m \neq O_n$ for every m,n such that $m\neq n$.

b) X^* is T_1 if and only if $O_m \not\subset O_n$ for every m,n such that $m\neq n$.

Proof. We will show *b)*. The proof of a) is analogous.
X^* is not T_1 if and only if there are m,n with $m\neq n$ and such that $n \in O_{p_1}^* \cap ... \cap O_{p_r}^*$ for every $p_1, ...p_r$ such that $m \in O_{p_1}^* \cap ... \cap O_{p_r}^*$. That is equivalent to that exist m,n with $m\neq n$ and such that $\{p_1,...p_r\} \subset O_n$ for every $p_1,...p_r$ with $\{p_1,...p_r\} \subset O_m$. Again, this amounts to that for each finite set F, $F \subset O_m$ implies $F \subset O_n$ and that is equivalent to $O_m \subset O_n$ □

3. Extremality

Definition Let $I \subset X$. We will say that I is an *extremal set* or, simply, an *extremal* if $\bigcap_{q\in I} O_q^* \neq \emptyset$ and $O_p^* \cap \bigcap_{q\in I} O_q^* = \bigcap_{q\in I\cup\{p\}} O_p^* = \emptyset$ for every $p \in X - I$.

Proposition 3.1. *We have:*

a) If $\bigcap_{q\in I} O_q^* \neq \emptyset$ then $\bigcap_{q\in X-I} H_q^* \subset \overline{\bigcap_{q\in I} O_q^*}$

b) $I \subset X$ is an extremal set if and only if $\bigcap_{q\in I} O_q^* \neq \emptyset$ and

$$\bigcap_{q\in X-I} H_q^* = \overline{\bigcap_{q\in I} O_q^*}$$

Proof. a) If $\bigcap_{q\in X-I} H_q^* = \emptyset$ there is nothing to prove. Otherwise,

$m \in \bigcap_{q\in X-I} H_q^* \Leftrightarrow m \in H_q^*$ for every $q \in X - I \Leftrightarrow q \in H_m$ for every $q \in X - I \Leftrightarrow X - I \subset H_m \Leftrightarrow O_m \subset I^{(1)}$. Consider $q_1, ...q_r$ such that

$m \in O^*(q_1) \cap ... \cap O^*(q_r)$. This is equivalent to $\{q_1, ...q_r\} \subset O_m$ and, by **(1)**, we have that $\{q_1, ...q_r\} \subset I$ so that $O^*(q_1) \cap ... \cap O^*(q_r) \cap \bigcap_{q \in I} O_q^* = \bigcap_{q \in I} O_q^* \neq \emptyset$. Therefore, $m \in \overline{\bigcap_{q \in I} O_q^*}$

b) We know that in every topological space, if A is an open set and B is any set, then $A \cap \overline{B} \subset \overline{A \cap B}$ (see [2]; 1.7, prop. 5). In our case this tells us that for every $p \in X - I$, we have $O_p^* \cap \overline{\bigcap_{q \in I} O_q^*} \subset \overline{\bigcap_{q \in I \cup \{p\}} O_q^*}$ but this last set is empty because I is an extremal and, so, $\overline{\bigcap_{q \in I} O_q^*} \subset H_p^*$ for every $p \in X - I$, that is $\overline{\bigcap_{q \in I} O_q^*} \subset \bigcap_{p \in X - I} H_p^*$. Part *a)* completes the proof. Let's see the reciprocal of part *b)*.

$\overline{\bigcap_{q \in I} O_q^*} = \bigcap_{p \in X - I} H_p^* \Rightarrow \bigcap_{q \in I} O_q^* \subset \bigcap_{p \in X - I} H_p^*$ which means that for every m such that $I \subset O_m$ we have that $X - I \subset H_m$ that is to say that for every m such that $I \subset O_m$ we have that $I = O_m$. If there was $p \in X - I$ such that $\bigcap_{q \in I \cup \{p\}} O_q^* \neq \emptyset$ then there would be n such that $I \subset I \cup \{p\} \subset O_n$ and by the previous observation we have that $I = I \cup \{p\} \subset = O_n$ which implies $p \in I$. This is a contradiction. Therefore, $\bigcap_{q \in I \cup \{p\}} O_q^* = \emptyset$ for every $p \in X - I$ and I is an extremal set. □

Proposition 3.2. *I is extremal $\Leftrightarrow$ exists n such that $I = O_n$ and $\bigcap_{q \in O_n} O_q^* = \{m : O_m = O_n\}$.*

Proof. $\Rightarrow$)

We have $n \in \bigcap_{q \in I} O_q^* \Leftrightarrow I \subset O_n$. In addition, if I is extremal then $\bigcap_{q \in I} O_q^* \subset H_p^*$ for every $p \in X - I$ which implies that, for every $p \in X - I$, if $I \subset O_n$, then $p \in H_n$. Therefore, if $I \subset O_n$ then $X - I \subset H_n$. That means that, if $I \subset O_n$ then $O_n \subset I$ and, so, $O_n = I$. In short, we must have $O_n = I$ for every $n \in \bigcap_{q \in I} O_q^*$ and so, $\bigcap_{q \in I} O_q^* = \{m : O_m = O_n\}$.

$\Leftarrow$)

$\bigcap_{q \in I} O_q^* \neq \emptyset$ because $I = O_n$ and therefore $n \in \bigcap_{q \in O_n} O_q^* = \bigcap_{q \in I} O_q^*$. If there was a $p \in X - I$ such that $\bigcap_{q \in I \cup \{p\}} O_q^* \neq \emptyset$ then there should be m that

$I \cup \{p\} \subset O_m$ but $I \subsetneqq I \cup \{p\} \subset O_m$ and, in particular, $I \subset O_m$ which implies $m \in \bigcap_{q \in I} O_q^*$ and $I = O_n \nsubseteq O_m$ which is absurd.

Definition: If $I \subset X$ we will call $N(I)$ the set

$$N(I) = \left\{ p \in X : O_p^* \cap \bigcap_{q \in I} O_q^* = \emptyset \right\} = \left\{ p \in X : \bigcap_{q \in I \cup \{p\}} O_q^* = \emptyset \right\}$$

We have:

$a)$ If $\bigcap_{q \in I} O_q^* = \emptyset$ then $N(I) = X$ and reciprocally. Indeed, $N(I) = X$ implies that $\bigcap_{q \in I} O_q^* \subset H_p^*$ for every $p \in X$, namely $\bigcap_{q \in I} O_q^* \subset \bigcap_{p \in X} H_p^*$. But $m \in \bigcap_{p \in X} H_p^* \Leftrightarrow p \in H_m$ for every $p \in X$ which is equivalent to $H_m = X$ and this is absurd due to the existence of gaps of prime numbers of arbitrary length.
$b)$ If $\bigcap_{q \in I} O_q^* \neq \emptyset$ is dense, then $N(I) = \emptyset$. □

Proposition 3.3. *If* $\bigcap_{q \in X - N(I)} O_q^* \neq \emptyset$ *then* $\bigcap_{q \in N(I)} H_q^* = \overline{\bigcap_{q \in X - N(I)} O_q^*}$ *and so, by proposition* 3.1*,* $X - N(I)$ *is extremal.*

Proof. $\bigcap_{q \in X - N(I)} O_q^* \neq \emptyset$ implies $N(I) \neq \emptyset$ because, otherwise, $\bigcap_{q \in X - N(I)} O_q^* = \bigcap_{q \in X} O_q^* = Z = \emptyset$.

We have that $\bigcap_{q \in I} O_q^* \subset H_p^*$ for all the $p \in N(I)$ and so, $\bigcap_{q \in I} O_q^* \subset \bigcap_{p \in N(I)} H_p^*$ which implies $\overline{\bigcap_{q \in I} O_q^*} \subset \bigcap_{p \in N(I)} H_p^{*(1)}$ because $\bigcap_{p \in N(I)} H_p^*$ is a closed set.

Given that $\bigcap_{q \in X - N(I)} O_q^* \neq \emptyset$, by the proposition 3.1$a$), we have that

$$\bigcap_{q \in N(I)} H_q^* \subset \overline{\bigcap_{q \in X - N(I)} O_q^{*}}^{(2)}$$

On the other hand, $N(I) \subset X - I \Rightarrow I \subset X - N(I) \Rightarrow$

$$\Rightarrow \bigcap_{q \in X - N(I)} O_q^* \subset \bigcap_{q \in I} O_q^* \Rightarrow \overline{\bigcap_{q \in X - N(I)} O_q^*} \subset \overline{\bigcap_{q \in I} O_q^{*}}^{(3)}$$

Therefore, $\overline{\bigcap_{q\in I} O_q^*} \subset \bigcap_{p\in N(I)} H_p^* \subset \overline{\bigcap_{q\in X-N(I)} O_q^*} \subset \overline{\bigcap_{q\in I} O_q^*}$ where inclusions come, respectively, from **(1)**, **(2)** and **(3)**. We finally obtain that

$$\bigcap_{p\in N(I)} H_p^* = \overline{\bigcap_{q\in X-N(I)} O_q^*}$$

. $\square$

Proposition 3.4. *If* $\bigcap_{q\in I} O_q^* \neq \emptyset$ *then* I *is extremal if and only if* $N(I) = X - I$.

Proof. We have $X - I \subset N(I) \Leftrightarrow \bigcap_{q\in I} O_q^* \cap O_p^* = \emptyset$ for every $p \in X - I \Leftrightarrow I$ is extremal. But if $\bigcap_{q\in I} O_q^* \neq \emptyset$ then $X - I \subset N(I)$ which is equivalent to $X - I = N(I)$ because we always have $N(I) \subset X - I$. $\square$

Proposition 3.5. $N(I) = \bigcap_{I\subset O_m} H_m$

Proof. We have $I \subset O_m \Leftrightarrow m \in \bigcap_{q\in I} O_q^*$ and therefore

$p \in \bigcap_{I\subset O_m} H_m \Leftrightarrow p \in H_m$ for every $m \in \bigcap_{q\in I} O_q^* \Leftrightarrow m \in H_p^*$ for every

$m \in \bigcap_{q\in I} O_q^* \Leftrightarrow O_p^* \cap \bigcap_{q\in I} O_q^* = \emptyset \Leftrightarrow p \in N(I)$. $\square$

Proposition 3.6. *If* $\bigcap_{q\in I} O_q^* \neq \emptyset$ *then* I *is an extremal* $\Leftrightarrow I = \bigcup_{I\subset O_m} O_m$

Proof. By proposition 3.4, I is an extremal if and only if $N(I) = X - I$ and, by proposition 3.5, $N(I) = \bigcap_{I\subset O_m} H_m$. $\square$

Proposition 3.7. $\bigcup_{I\subset O_m} O_m \neq X \Leftrightarrow J = \bigcup_{I=O_m} O_m$ *is an extremal.*

Proof. $\Rightarrow$)

If J was not an extremal then it should exists $p \in X - J$ such that $O_p^* \cap \bigcap_{q\in J} O_q^* \neq \emptyset$ and so there would be one $p \in X - J$ and one $s \in \mathbb{N}$ such that $J \cup \{p\} = O_s$ and therefore $J \subsetneqq O_s$, but $I \subset \bigcup_{I\subset O_m} O_m = J \subsetneqq O_s$ so that $I \subset O_s$. We deduce that s is one of the m such that $I \subset O_m$ and, so,

$O_s \subset \bigcup_{I\subset O_m} O_m = J$ which is contradictory to $J \subsetneqq O_s$.

$\Leftarrow$) If we had $J = \bigcup_{I\subset O_m} O_m = X$ than $\bigcap_{q\in J} O_q^* = \emptyset$ which would go against the definition of extremal set. $\square$

Corollary 3.8. *Suppose that* $\bigcap_{q\in I} O_q^* \neq \emptyset$. *Then* $N(I) \neq \emptyset \Leftrightarrow X - N(I)$ *is extremal.*

Proof. $N(I) \neq \emptyset \Leftrightarrow X - N(I) = \bigcup_{I\subset O_m} O_m \neq X$ (proposition 3.5). In addition, by proposition 3.7, $X - N(I) = \bigcup_{I\subset O_m} O_m$ and $J = X - N(I)$. $\square$

Proposition 3.9. *The following conditions are equivalent:*
i) There are no extremal sets.
ii) All sets $\bigcap_{q\in I} O_q^*$ *that are non-empty are dense.*

Proof. $i) \Rightarrow ii)$ If there no extremal sets then, by proposition 3.7, for all I such that $\bigcap_{q\in I} O_q^* \neq \emptyset$ we must have $\bigcup_{I\subset O_m} O_m = X$. In other words, for every $p \in X$ has to exist a m such that $I \subset O_m$ and $p \in O_m$. This means that for every $p \in X$, $\bigcap_{q\in I\cup\{p\}} O_q^* \neq \emptyset$ that is $O_p^* \cap \bigcap_{q\in I} O_q^* \neq \emptyset$ or, put it another way, for every I such that $\bigcap_{q\in I} O_q^* \neq \emptyset$ we must have $N(I) = \emptyset$

If $\bigcap_{q\in I} O_q^* \neq \emptyset$ was not dense then there should be $p_1, ..., p_k$ such that

$$\bigcap_{q\in I} O_q^* \cap (O^*(p_1) \cap ... \cap O^*(p_k)) = \emptyset$$

Suppose that $\{p_1, ...p_k\}$ is minimal with this property, that is

$$\bigcap_{q\in I} O_q^* \cap (O^*(p_1) \cap ... \cap O^*(p_{k-1})) \neq \emptyset$$

That implies that $p_k \in N(I \cup \{p_1, ..., p_{k-1}\})$ which is a contradiction and, therefore, $\bigcap_{q\in I} O_q^*$ is dense.

$ii) \Rightarrow i)$ If there was an extremal O_μ then $\bigcap_{q\in O_\mu} O_q^* \neq \emptyset$ which is not a dense set. $\square$

Proposition 3.10. *If* $I \subset J$ *then* $N(I) \subset N(J)$

Proof. $I \subset J$ implies $\bigcap_{q\in J} O_q^* \subset \bigcap_{q\in I} O_q^*$. If $p \in N(I)$ then $O_p^* \cap \bigcap_{q\in I} O_q^* = \emptyset$ and so $O_p^* \cap \bigcap_{q\in J} O_q^* = \emptyset$ which means that $p \in N(J)$. □

Proposition 3.11. *Suppose that* $\bigcap_{q\in I} O_q^* \neq \emptyset$. *Then* $N(I) \neq \emptyset \Rightarrow N(I) = N(X - N(I))$.

Proof. By the corollary 3.8, $N(I) \neq \emptyset$ if and only if $X - N(I)$ is an extremal so that by proposition 3.4, $N(X - N(I)) = X - (X - N(I)) = N(I)$. □

Proposition 3.12. *If* $N(I) \neq \emptyset$, $I \subset J$ *and* $\bigcap_{q\in J} O_q^* \neq \emptyset$ *then* $N(I) = N(J)$

Proof. $I \subset J \Rightarrow \bigcap_{q\in J} O_q^* \subset \bigcap_{q\in I} O_q^* \Rightarrow \bigcap_{q\in I} O_q^* \neq \emptyset$ because $\bigcap_{q\in J} O_q^* \neq \emptyset$. By the proposition 3.11, $N(I) = N(X - N(I))$ and by proposition 3.10, $N(J) \neq \emptyset$ because, by hypothesis, we have $N(I) \neq \emptyset$ and $N(I) \subset N(J)$.

Again by proposition 3.11, we have that $N(J) = N(X - N(J))$ and we obtain

$$N(X - N(I)) = N(I) \subset N(J) = N(X - N(J))$$

In addition, $X - N(J) \subset X - N(I)$ and, again by 3.10, we have that $N(X - N(J)) \subset N(X - N(I))$.

Finally we have $N(X - N(I)) = N(I) \subset N(J) = N(X - N(J)) \subset N(X - N(I))$ and in particular, $N(I) = N(J)$. □

Proposition 3.13. *If* $\bigcap_{q\in I} O_q^* \neq \emptyset$ *is not dense then* $N(X - N(I)) \neq \emptyset$.

Proof. If $\bigcap_{q\in I} O_q^* \neq \emptyset$ is not dense then there are $p_1, ..., p_k$ such that

$$\bigcap_{q\in I} O_q^* \cap (O^*(p_1) \cap ... \cap O^*(p_k)) = \emptyset$$

and

$$\bigcap_{q\in I} O_q^* \cap (O^*(p_1) \cap ... \cap O^*(p_{k-1})) \neq \emptyset$$

Let's write $F = \{p_1, ..., p_k\}$ (F is empty if $k = 1$). We have that $\bigcap_{q\in I\cup F} O_q^* \neq \emptyset$ and $N(I \cup F) \neq \emptyset$ (because $p_k \in N(I \cup F)$).

By the proposition 3.11, we have that

$$\emptyset \neq N(I \cup F) = N(X - N(I \cup F))$$

but $X - N(I \cup F) \subset X - N(I)$ and therefore

$$\emptyset \neq N(X - N(I \cup F)) \subset N(X - N(I))$$

and we deduce that
$$N\left(X-N\left(I\right)\right)\neq\emptyset$$
. $\square$

Proposition 3.14. $\bigcap\limits_{q\in I} O_q^* \neq \emptyset$ *is not dense if and only if there is a finite set F such that $I \cup F$ is an extremal.*

Proof. First we will show that $\bigcap\limits_{q\in I} O_q^* \neq \emptyset$ is not dense if and only if there is a finite set F such that $\bigcap\limits_{q\in I\cup F} O_q^* \neq \emptyset$ and $N\left(I\cup F\right)\neq\emptyset$.

$\Rightarrow$) Suppose $\bigcap\limits_{q\in I} O_q^* \cap \left(O^*\left(p_1\right)\cap ... \cap O^*\left(p_k\right)\right)=\emptyset$ where $F=\{p_1,...,p_k\}$ is minimal whit this propriety. Let $F'=\{p_1,...,p_{k-1}\}$. Then $\bigcap\limits_{q\in I\cup F'} O_q^* \neq \emptyset$ and $N\left(I\cup F'\right)\neq\emptyset$ because $p_k\in N\left(I\cup F'\right)$

$\Leftarrow$) Let $F=\{p_1,...,p_k\}$ such that $N\left(I\cup F\right)\neq\emptyset$ and $\bigcap\limits_{q\in I\cup F} O_q^* \neq \emptyset$. If $p\in N\left(I\cup F\right)$ then $\bigcap\limits_{q\in I\cup F} O_q^* \cap O^*\left(p\right)=\emptyset$ that is
$$\bigcap_{q\in I\cup F} O_q^* \cap \left(O^*\left(p_1\right)\cap ... \cap O^*\left(p_k\right)\right)\cap O^*\left(p\right)=\emptyset$$
and so, $\bigcap\limits_{q\in I} O_q^* \neq \emptyset$ is not dense.

Now let's see the thesis of 3.14.

$\Rightarrow$) Let's suppose that $\bigcap\limits_{q\in I} O_q^* \neq \emptyset$ is not dense. We just proved that there must be a finite set F such that $\bigcap\limits_{q\in I\cup F} O_q^* \neq \emptyset$ and $N\left(I\cup F\right)\neq\emptyset^{(1)}$. Let $J=I\cup F$ and let's suppose that for every finite set G, $I\cup G$ is not extremal. This implies that for every finite set G, $J\cup G$ is not extremal because $J\cup G=I\cup\left(F\cup G\right)$ and $F\cup G$ is finite.

Let's take any finite set G. Then, the fact that $J\cup G$ is not an extremal implies that there is p such that $\bigcap\limits_{q\in J\cup G\cup\{p\}} O_q^* \neq \emptyset$. Because $I\cup F=J\subset J\cup G$ and $\emptyset\neq N\left(I\cup F\right)\subset N\left(J\cup G\right)$ (see **(1)**), we have $N\left(J\right)=N\left(J\cup G\right)$ (proposition 3.12).

So, $J\cup G\subset X-N\left(J\cup G\right)=X-N\left(J\right)$ for every finite set G and we obtain $X=\bigcup\limits_{all\ finite\ G}\left(J\cup G\right)\subset X-N\left(J\right)$.

Therefore $N(J) = N(I \cup F) = \emptyset$ which is absurd.

$\Leftarrow$) If $I \cup F$ is extremal then, by definition, $\bigcap\limits_{q \in I \cup F} O_q^* \neq \emptyset$ and so $\bigcap\limits_{q \in I} O_q^* \neq \emptyset$

If $F = \{p_1, ..., p_k\}$ and $p \in X - (I \cup F)$ then

$$\emptyset = \bigcap_{q \in I \cup F} O_q^* \cap O_p^* = \bigcap_{q \in I} O_q^* \cap \left(O^*(p_1) \cap ... \cap O^*(p_k) \cap O_p^*\right)$$

and $\bigcap\limits_{q \in I} O_q^*$ is not dense. □

Proposition 3.15. *If* $\bigcap\limits_{q \in I} O_q^* \neq \emptyset$ *then* $N(I) = \emptyset$

Proof. If $\bigcap\limits_{q \in I} O_q^* \neq \emptyset$ is dense then there is nothing to prove. Let's suppose that $\bigcap\limits_{q \in I} O_q^*$ is not dense.

By the proposition 3.14, there is a finite set $F \subset X - I$ such that $I \cup F = O_\mu$ is an extremal$^{(1)}$. We will distinguish two cases:

a) There is $p' \in F$ such that $N(I \cup \{p'\}) = \emptyset$

b) $N(I \cup \{p'\}) \neq \emptyset$ for every $p' \in F$.

a) Then $N(I) \subset N(I \cup \{p'\}) = \emptyset$ and we're done.

b) Then for every $p' \in F$:

$\emptyset \neq N(I \cup \{p'\}) \subset N(I \cup F) = X - (I \cup F) = H_\mu{}^{(2)}$ where $N(I \cup F) = X - (I \cup F)$ comes from the proposition 3.4. However, $\bigcap\limits_{q \in I \cup \{p'\}} O_q^* \neq \emptyset$ because $\bigcap\limits_{q \in I \cup F} O_q^* \subset \bigcap\limits_{q \in I \cup \{p'\}} O_q^*$ and $\bigcap\limits_{q \in I \cup F} O_q^* \neq \emptyset$ given that $I \cup F = O_\mu$ and, therefore $\mu \in \bigcap\limits_{q \in I \cup F} O_q^*$. By the corollary 3.8, $X - N(I \cup \{p'\})$ is an extremal so that there is ν such that $X - N(I \cup \{p'\}) = O_\nu$

We deduce that $N(I \cup \{p'\}) = H_\nu$ for every $p' \in F$ because, from **(2)**,

$$O_\mu \subset X - N(I \cup \{p'\}) = O_\nu$$

and O_μ is extremal.

We obtain that, for every $p' \in F$ and all $p \in H_\mu$, we have $\bigcap\limits_{q \in I \cup \{p'\}} O_q^* \cap O_p^* = \emptyset$ or, in other words, for every $p' \in F$ and all $p \in H_\mu$, $\bigcap\limits_{q \in I \cup \{p\}} O_q^* \cap O_{p'}^* = \emptyset$

That means that for every $p \in H_\mu$, $F \subset N(I \cup \{p\}) \subset N(I \cup H_\mu)^{(3)}$

In addition, for every $p \in H_\mu$:

$$\emptyset \neq N(I \cup \{p\}) \subset N(I \cup H_\mu) \subset X - (I \cup H_\mu) =$$

$$= (X - I) \cap O_\mu = (X - I) \cap (I \cup F) = F^{(4)}$$

where the penultimate equality comes from **(1)**. From **(3)** and **(4)** we obtain $N(I \cup H_\mu) = F^{(5)}$

Besides, $N(I \cup F) = N(O_\mu) = X - O_\mu = H_\mu{}^{(6)}$, but

$$I \subset I \cup H_\mu \Rightarrow N(I) \subset N(I \cup H_\mu) = F \text{ (from } \mathbf{(5)}\text{) and}$$

$$I \subset I \cup F \Rightarrow N(I) \subset N(I \cup F) = H_\mu \text{ (from}\mathbf{(6)}\text{)}.$$

Intersecting these last inclusions we get $N(I) \subset F \cap H_\mu = \emptyset$ because

$$I \cup F = O_\mu \Rightarrow F \subset O_\mu \Rightarrow F \cap H_\mu = \emptyset$$

. □

Proposition 3.16. $\bigcap_{q \in I} O_q^* \neq \emptyset \Rightarrow \bigcap_{q \in I} O_q^*$ *is dense.*

Proof. If $\bigcap_{q \in I} O_q^*$ is not dense, by the proposition 3.14, then must be a finite set F such that $I \cup F = O_\mu$ is an extremal, In particular, by the definition of extremal, we must have $\bigcap_{q \in I \cup F} O_q^* \neq \emptyset$ and then, by the proposition 3.15, $N(I \cup F) = \emptyset$. In addition, the fact that $I \cup F = O_\mu$ is an extremal implies that $N(I \cup F) = N(O_\mu) = X - O_\mu = H_\mu \neq \emptyset$ because if we have $H_\mu = \emptyset$ the $O_\mu = X$ and then we must have $\bigcap_{q \in O_\mu} O_q^* = \bigcap_{q \in I \cup F} O_q^* = \emptyset$ which is a contradiction. □

Corollary 3.17. *There are no extremal sets*

Proof. The extremal are open sets O_μ such that $\bigcap_{q \in O_\mu} O_q^* \neq \emptyset$ but, by definition, they are not dense. This contradicts the proposition 3.16. □

Corollary 3.18. *For every cofinite set C we have that* $\bigcap_{q \in C} O_q^* = \emptyset$.

Proof. If for some cofinite set C we had $\bigcap_{q \in C} O_q^* \neq \emptyset$ then, by proposition 3.16, $\bigcap_{q \in C} O_q^*$ should be dense but it is not because

$\bigcap_{q \in C} O_q^* \cap \bigcap_{q \in X-C} O_q^* = \bigcap_{q \in X} O_q^* = Z = \emptyset$ and $\bigcap_{q \in X-C} O_q^*$ is an open set because $X - C$ is finite. □

Theorem 3.19. *H_m is an infinite set for every $m \in \mathbb{N}$. In particular, H_1 is an infinite set and so, the Twin Prime Conjecture is true.*

Proof. If not, for some m, O_m would be cofinite and $\bigcap_{q\in O_m} O_q^* \neq \emptyset$ because, at least, $m \in \bigcap_{q\in O_m} O_q^*$ □

Theorem 3.20. *The de Polignac Conjecture is false.*

Proof. Let's take any O_ν. As $\bigcap_{q\in O_\nu} O_q^* \neq \emptyset$, by proposition 3.16, $\bigcap_{q\in O_\nu} O_q^*$ is dense and, therefore, if $p \in H_\nu$, we must have $\bigcap_{q\in O_\nu\cup\{p\}} O_q^* \neq \emptyset$. If $n_1 \in \bigcap_{q\in O_\nu\cup\{p\}} O_q^*$ then $O_\nu \subsetneqq O_{n_1}$. Iterating this process we get a strict chain:

$$O_\nu \subsetneqq O_{n_1} \subsetneqq O_{n_2} \subsetneqq O_{n_3} \subsetneqq \ldots$$

which shows that the space X is not noetherian ([4]) and, in particular, by proposition 1.17, X^* is not a T_1 space. Let's consider the complementary chain

$$H_\nu \supsetneqq H_{n_1} \supsetneqq H_{n_2} \supsetneqq H_{n_3} \supsetneqq \ldots$$

All the n_j are different because the H_{n_j} are and so, there must be a k such that $n_k > \nu$. We have $H_\nu \supsetneqq H_{n_k}$ which implies $H_{n_k} \cap O_\nu \subset H_\nu \cap O_\nu = \emptyset$ so that

$$H_{n_k} \cap O_1 \cap O_2 \cap \ldots \cap O_\nu \subset H_{n_k} \cap O_\nu = \emptyset$$

and, therefore

$$H_{n_k} \cap O_1 \cap O_2 \cap \ldots \cap O_\nu \cap O_{\nu+1} \cap \ldots \cap O_{n_k-1} = \emptyset$$

(note that $n_k > \nu \Rightarrow n_{k-1} \geq \nu$ which means that the subscript n_{k-1} is either ν or "comes" after ν).
The thesis is derived from remark 2 after the proposition 2.3. □

Corollary 3.21. *There are no gaps of prime numbers of length $2n$ for infinite values of n.*

Proof. It suffices to take $\nu = n_k$ and repeat the previous reasoning indefinitely.

Note however that, since H_m is always an infinite set, for each $m \in \mathbb{N}$ there are infinite couples of prime numbers that differ in $2m$ units (although, perhaps, they are not couples of consecutive prime numbers). □

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