Some results of derivatives of Reimann Xi function from analytical expression of Riemann Xi function $\xi(s)$

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Abstract : [In this paper some properties of integer order derivatives of Riemann Xi function os directly derived from expression of Riemann Xi function $\xi(s)$]

Key Words : Riemann Xi function, Riemann Zeta function, Functional equation, Derivatives.

The Riemann Xi function $\xi(s)$ and Riemann Zeta function $\zeta(s)$ are connected by the relation [1]

$$\xi(s) = \frac{1}{2} s(s-1)\,\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) \qquad \ldots (1)$$

The Riemann Xi function $\xi(s)$ satisfies a functional equation

$$\xi(s) = \xi(1-s) \qquad \ldots (2)$$

It was shown by this author that Riemann Xi function can be analytically represented [2,3] as

$$\xi(s) = F_2(l_1) + F_1(l_1) \operatorname{Cosh} l_1 (s - ½) \qquad \ldots (3)$$

where $F_2(l_1)$ and $F_1(l_1)$ are two positive unknown constants and l_1 is a unknown positive parameter.

From (3) we can easily find the derivatives of $\xi(s)$.

$$\left.\begin{aligned}
&\xi^{(1)}(s) = l_1 F_1(l_1) \operatorname{Sinh} l_1(s - ½)\\
&\xi^{(2)}(s) = l_1^2\, F_1(l_1) \operatorname{Cosh} l_1(s - ½)\\
&\quad .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\\
&\quad .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\;\; .\\
&\xi^{(2n)}(s) = l_1^{2n}\, F_1(l_1) \operatorname{Cosh} l_1(s - ½)\\
&\xi^{(2n+1)}(s) = l_1^{2n+1}\, F_1(l_1) \operatorname{Sinh} l_1(s - ½)
\end{aligned}\right| \qquad \ldots (4)$$

From (4) it easily follows that

$$\xi^{(2n)}(½) = l_1^{2n}\, F_1(l_1) \text{ which imply}$$

$$\xi^{(2)}(½),\quad \xi^{(4)}(½)\;.\;\;.\;\;.\;\;.\text{ are all positive because } l_1, F_1(l_1) \text{ are positive} \qquad \ldots (5)$$

It also follows from (4) that

$$\xi^{(2n+1)}(½) = 0$$

$$\text{i.e., } \xi^{(1)}(½) = \xi^{(3)}(½) = \xi^{(5)}(½) = \;.\;\;.\;\;.\; = 0 \qquad \ldots (6)$$

It also follows directly from (4) that all the derivatives of Xi function are positive for $s > ½$

Other interesting results also follow from (4)

$$\xi^{(2n)}(0) = l_1^{2n}\ F_1(l_1)\ \mathrm{Cosh}\,\frac{l_1}{2} \qquad \ldots (7)$$

which imply

$$\xi^{(2)}(0),\quad \xi^{(4)}(0)\ .\ \ .\ \ .\ \ .\ \text{are all positive}$$

And again from (4)

$$\xi^{(2n+1)}(0) = -\ l_1^{2n+1}\ F_1(l_1)\ \mathrm{Sinh}\,\frac{l_1}{2} \qquad \ldots(8)$$

which imply

$$\xi^{(1)}(0),\quad \xi^{(3)}(0)\ .\ \ .\ \ .\ \ .\ \text{are all negative}$$

The above results were obtained by Coffey after a long calculation [4]. However as shown above that these results of derivatives of Riemann Xi function follow directly from analytic expression of Xi function ξ(s) [2,3].

References

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