Some results of derivatives of Reimann Xi function from analytical expression of Riemann Xi function $\xi(s)$

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Abstract : [In this paper some properties of integer order derivatives of Riemann Xi function os directly derived from expression of Riemann Xi function $\xi(s)$]

Key Words : Riemann Xi function, Riemann Zeta function, Functional equation, Derivatives.

The Riemann Xi function $\xi(s)$ and Riemann Zeta function $\zeta(s)$ are connected by the relation [1]

$$\xi(s) = \frac{1}{2} s(s - 1) \pi^{-\frac{s}{2}} \Gamma(s/2) \zeta(s) \quad \ldots \ (1)$$

The Riemann Xi function $\xi(s)$ satisfies a functional equation

$$\xi(s) = \xi(1-s) \quad \ldots \ (2)$$

It was shown by this author that Riemann Xi function can be analytically represented [2,3] as

$$\xi(s) = F_2(l_1) + F_1(l_1) \text{ Cosh } l_1 (s - \frac{1}{2}) \quad \ldots \ (3)$$

where $F_2(l_1)$ and $F_1(l_1)$ are two positive unknown constants and $l_1$ is a unknown positive parameter.

From (3) we can easily find the derivatives of $\xi(s)$.

$$\xi^{(1)}(s) = l_1 F_1(l_1) \text{ Sinh } l_1 (s - \frac{1}{2})$$
$$\xi^{(2)}(s) = l_1^2 F_1(l_1) \text{ Cosh } l_1 (s - \frac{1}{2})$$
$$\quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\xi^{(2n)}(s) = l_1^{2n} F_1(l_1) \text{ Cosh } l_1 (s - \frac{1}{2})$$
$$\xi^{(2n+1)}(s) = l_1^{2n+1} F_1(l_1) \text{ Sinh } l_1 (s - \frac{1}{2})$$

From (4) it easily follows that

$$\xi^{(2n)}(\frac{1}{2}) = l_1^{2n} F_1(l_1) \text{ which imply }$$
$$\xi^{(2)}(\frac{1}{2}), \xi^{(4)}(\frac{1}{2}), \ldots \ldots \text{ are all positive because } l_1, F_1(l_1) \text{ are positive } \ldots \ (5)$$

It also follows from (4) that

$$\xi^{(2n+1)}(\frac{1}{2}) = 0$$
$$\text{i.e., } \xi^{(1)}(\frac{1}{2}) = \xi^{(3)}(\frac{1}{2}) = \xi^{(5)}(\frac{1}{2}) = \ldots = 0 \quad \ldots \ (6)$$

It also follows directly from (4) that all the derivatives of Xi function are positive for $s > \frac{1}{2}$.
Other interesting results also follow from (4)
\[ \xi^{(2n)}(0) = l_1^{2n} F_1(l_1) \text{Cosh} \frac{l_1}{2} \]  
which imply
\[ \xi^{(2)}(0), \quad \xi^{(4)}(0), \ldots \text{are all positive} \]
And again from (4)
\[ \xi^{(2n + 1)}(0) = - l_1^{2n + 1} F_1(l_1) \text{Sinh} \frac{l_1}{2} \]  
which imply
\[ \xi^{(1)}(0), \quad \xi^{(3)}(0), \ldots \text{are all negative} \]
The above results were obtained by Coffey after a long calculation [4]. However as shown above that these results of derivatives of Riemann Xi function follow directly from analytic expression of Xi function \( \xi(s) \) [2,3].

References

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