Studies on Twin Primes in Goldbach Partitions of Even Numbers

Marcin Barylski (marcin.a.barylski@gmail.com)

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Abstract
Goldbach strong conjecture states that all even integers $n \geq 2$ can be expressed as the sum of two prime numbers (Goldbach partitions of $n$). This work is devoted to studies on twin primes present in Goldbach partitions. Based on executed experiments original Goldbach conjecture has been extended to a form that all even integers $n \geq 4$ can be expressed as the sum of twin prime and prime.

1 Introduction
Goldbach strong (also called binary) conjecture asserts that all positive even integer $n \geq 4$ can be expressed as the sum of two prime numbers. This hypothesis, formulated by Goldbach in 1742 in letter to Euler [1] and then updated by Euler to the form above is one of the oldest and still unsolved problems in number theory. Empirical verification showed that it is true for all $n \leq 4 \times 10^{18}$ [2] [3]. The expression of a given number $n$ as a sum of two primes $p_1$ and $p_2$ is called a Goldbach Partition (GP) of $n$. Let $r(n)$ be the number of GPs of $n$ and let $R(n)$ be a set of distinct GPs of $n$ (uniqueness guaranteed through $p_1 \leq p_2$). Goldbach strong conjecture may be rewritten that $r(n) > 0$ for all $n \geq 4$. Computational experiments show that bottom estimation for $\text{min}(r(n))$ is increasing with $n$ (Figure 1). [4] formulates conjecture that lower and upper bounds can be expressed as simple exponentials.

Prime number $p$ is a twin prime if either $p - 2$ or $p + 2$ is another prime. Let’s denote a set of twin primes as $\mathbb{P}_T$ and a set of primes as $\mathbb{P}$. OEIS A014574 [5] lists first twin pairs of form $n \pm 1$. Twin prime conjecture, still unsolved, states that there are an infinite number of twin primes. In twin pair $(p, p + 2)$ first number $p$ is called the lesser of twin primes (OEIS A001359 [6]), $p + 2$ - the greater of twin primes (OEIS A006512 [7]).

Lemma 1. 5 is the only prime which is both the lesser of twin primes (to 7) and the greater of twin primes (to 3).

Proof. In a set of numbers $n, n + 2, n + 4$ one is always divisible by 3. $n + 2$ can be both the lesser of twin primes and the greater of twin primes if and only if all $n, n+2, n+4$ are primes. 3 is the only prime which is divisible by 3, so 3 must be in sequence $n, n+2, n+4$. 3 must be the first number of the sequence, because two other combinations: $1, 3, 5$ and $-1, 1, 3$ are not containing primes only. This means that the only allowed sequence is 3, 5, 7 and 5 is the only prime which is a member of two distinct twin primes (3, 5) and (5, 7).

2 Primality tests
For the sake of this work, to check if a given number is prime or not, algorithm skewed in Listing 1 was used. Presented approach is taking advantage of preloaded prime and composite sets (containing prime and composite numbers found earlier) which gives instant result. Then, algorithm is testing if a candidate for prime is even (divides it by 2) or is a multiple of 3 - in case of success the candidate is confirmed as a composite number. Eventually, it is taking advantage of Lemma 2.

Listing 1: Primality test

```python
# in: integer n
# prime_set - a set of primes
# composite_set - a set of non-primes
# out: True if n is prime; False otherwise
def is_prime(n):
    if n<=1:
        return False
    elif n<=3:
        return True
    elif n in prime_set:
        return True
    elif n in composite_set:
        return False
    elif n%2 == 0 or n%3 == 0:
        return False
    i=5
    while i*i<=n:
        if n%i == 0 or n%(i+2) == 0:
            return False
        i+=6
    return True
```

Figure 1: $r(n)$ ($2 < n < 10^6$, $n = 2k, k \in \mathbb{N}$)
Listing 2 presents a simplified method how to classify a given prime number as a part of twin prime pair, including criterion for either the lesser or the greater of twin primes.

![Figure 2: Minimal prime in $R(n)$](image)

(2 < $n$ < $10^6$, $n = 2k, k \in \mathbb{N}$)

Listing 2: Tests for twin primes

```python
# in: integer n
# external dependencies:
# is_prime() method
# out: True if n is twin prime;
# False otherwise
def is_twin_prime(p):
    if n in twinprime_set:
        return True
    else:
        n in composite_set:
        return False
    if is_greater_twin_prime(p) or
    is_lesser_twin_prime(p):
        return True
    return False

# in: integer n
# out: True if n is the greater of twin primes;
# False otherwise
def is_greater_twin_prime(p):
    if is_prime(p) and is_prime(p-2):
        return True
    return False

# in: integer n
# out: True if n is the lesser of twin primes;
# False otherwise
def is_lesser_twin_prime(p):
    if is_prime(p) and is_prime(p+2):
        return True
    return False
```

**Lemma 2.** Every prime $p > 3$ can be written as $p = 6k \pm 1$ (where $k$ is a positive integer).

**Proof.** Every positive integer $n \geq 6$ can be expressed as $6k + m$, where $m = 0, 1, \ldots, 5, k \geq 1$. Numbers $6k, 6k + 2, 6k + 3$ and $6k + 4$ are always composite because they are divisible by either 2 or 3 or both ($6k = 2 \times 3k, 6k + 2 = 2 \times (3k + 1), 6k + 3 = 3 \times (2k + 1), 6k + 4 = 2 \times (3k + 2)$). 6k + 1 and 6k + 5 are either prime (i.e. $6 \times 1 + 1 = 7, 6 \times 2 + 1 = 11$) or composite (i.e. 6k + 5 is divisible by 5 if $k$ is multiple of 5, 6k + 1 is divisible by 3 if sum of decimal digits is divisible by 3). 6k + 5 can be rewritten as $6 \times (k + 1) - 1$. All primes < 5 (2 and 3) cannot be expressed as $p = 6k \pm 1$ where $k$ is a positive integer. This means that every prime $p \geq 5$ can be expressed as $6k \pm 1$ (where $k \in \mathbb{N}$).

As a consequence of Lemma 2, every twin prime pair different than (3, 5) is of form $(6k - 1, 6k + 1)$, where $k \in \mathbb{N}$.

3 Twin primes in Goldbach partitions

Detailed examination of $R(n)$ ($n < 10^6$) shows that the minimal prime in at least one of GPs is usually low (Figure 2). For all $n < 10^6$ it has been computationally verified that 523 is the biggest minimal prime (for $n = 503222$) in all possible GPs. [2] verified that 3325581707333960528 is the smallest number that has no GP with a prime below 9781. Among the smallest primes in GBs ($n < 10^6$, Table 1) the most popular is 3 (78497 occurrences), then 5 (70328), 7 (62185), 11 (48582), 13 (40916), 17 (31091), 19 (29791). 3, 5, 7, 11, 13, 17, 19 are members of $P_r$.

OEIS A294185 [8] lists first elements of sequence containing number of distinct greater of twin primes which are in Goldbach partitions of even $n$, while OEIS A294185 [9] lists first elements of sequence containing number of distinct greater of twin primes which are in Goldbach partitions of even $n$, while OEIS A294185 [10] lists first elements of sequence containing number of distinct lesser of twin primes which are in Goldbach partitions of even $n$, while OEIS A294186 [10] lists first elements of sequence containing number of distinct lesser of twin primes which are in Goldbach partitions of even $n$. OEIS A294185 and OEIS A294186, presenting number of distinct twin primes which are in Goldbach partitions of even $n$. OEIS A295424 is not exact sum of OEIS A294185 and OEIS A294186 because of Lemma 1 - 5 is both lesser and greater twin prime and is a frequent member of $R(n)$.

Further experiments show that there are in general two categories of even numbers $n$: category 1 - with 0, 1, or 2 distinct greater/lesser of twin primes in all $R(n)$, and category 2 - with fast increasing number of distinct greater/
Table 1: Appearances of smallest primes in $R(n)$, with twin primes bolded ($2 < n < 10^6$, $n = 2k, k \in \mathbb{N}$)

<table>
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<th>Prime</th>
<th>Appearances</th>
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lesser twin primes in $R(n)$ (Figure 3, Figure 4). From all twin primes standpoint such observation is not visible (Figure 6) - except 4 (which has $R(4) = \{(2, 2)\}$, without twin primes) even numbers $n$ have at least one twin prime in $R(n)$. Analysis of difference between number of the greater of twin primes and the lesser of twin primes show that there is balance between those two properties of $R(n)$, and average diff is close to 0 (Figure 5).

Let $T(n)$ be a number of twin primes in all $R(n)$ and $C(n) = |R(n)|$. Examination of $\frac{T(n)}{C(n)}$ for consecutive even $n$ shows that the sequence has a downward trend (Figure 7). This illustration may be yet another expression of the scarcity of twin primes, like Brun’s constant [11].
4 Extended strong Goldbach conjecture

Executed experiments, especially non-zero values observed in Figures 6 and 7 let build the hypothesis that strong Goldbach conjecture can be extended from original form (1):

\[ \forall n > 1, n \in \mathbb{N} \exists p, q \in \mathbb{P} \quad 2 \times n = p + q \quad (1) \]

to the new one (2), skipping just the first case \( n = 2 \) for which is not true:

\[ \forall n > 2, n \in \mathbb{N} \exists p, q \in \mathbb{P} \quad 2 \times n = p + q \quad (2) \]

Stronger version of (2) is not true - assumption (3):

\[ \forall n > 2, n \in \mathbb{N} \exists p, q \in \mathbb{P} \quad 2 \times n = p + q \quad (3) \]

has at least 35 counter-examples, listed in OEIS A007534 [12] and evaluated at [13] (and this is still an open question if this list is complete).

5 Summary and future work

Although extended strong Goldbach conjecture (2) does not help, directly, in proving the original problem stated by Goldbach and Euler (1), it depicts that there is even greater synergy between sum of primes and consecutive even numbers - twin primes are also important builders of even numbers. As a result of this work three integer sequences were submitted to OEIS database: OEIS A294185 [8], OEIS A294186 [9] and OEIS A295424 [10].

Additionally, presented work provides few more interesting questions which can be a foundation of further research work. Is \( \lim_{n \to \infty} \frac{T(n)}{C(n)} \) converging to a positive number or zero? Is it possible to find a pattern between even numbers for which number of the lesser or the greater of twin primes in \( R(n) \) is zero? Why there are two categories of numbers: first one with the \( \leq 2 \) lesser or the greater of twin primes in \( R(n) \), and second one - with the lesser and the greater twin primes >2, which is growing with \( n \)?

References