Abstract:
Argentest II is born, a personal research project that develops a new exclusive probabilistic primality test for Twin prime numbers. I present a test similar to Fermat’s little theorem.

Twin prime numbers

A twin prime is a prime number that is either 2 less or 2 more than another prime number—for example, either member of the twin prime pair (41, 43). In other words, a twin prime is a prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair.

Usually the pair (2, 3) is not considered to be a pair of twin primes. Since 2 is the only even prime, this pair is the only pair of prime numbers that differ by one; thus twin primes are as closely spaced as possible for any other two primes.

The first few twin prime pairs are:
(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), (107, 109), (137, 139), ... OEIS: A077800.

Five is the only prime that belongs to two pairs, as every twin prime pair greater than (3,5) is of the form (6n+1, 6n-1) for some natural number n.

Probabilistic primality test for Twin prime numbers

$$\exists k > 0 \in \mathbb{N}/2k + 1 = p$$

$$\frac{2^{p+2} - 8 \pmod{p}}{p} \equiv 3 \iff p, p + 2 \text{ are primes}$$

$$\therefore P \land P + 2 \text{ are Twin primes}$$
### Examples

When the two numbers are prime it has congruence.

**Examples**

A. Test for 3 and 5
\[
\frac{2^5 - 8}{3} \equiv 3 \, (\text{Mod} \ 5)
\]

B. Test for 5 and 7
\[
\frac{2^7 - 8}{5} \equiv 3 \, (\text{Mod} \ 7)
\]

C. Test for 11 and 13
\[
\frac{2^{13} - 8}{11} \equiv 3 \, (\text{Mod} \ 13)
\]

D. Test for 17 and 19
\[
\frac{2^{19} - 8}{17} \equiv 3 \, (\text{Mod} \ 19)
\]

E. Test for 29 and 31
\[
\frac{2^{31} - 8}{29} \equiv 3 \, (\text{Mod} \ 31)
\]

F. Test for 41 and 43
\[
\frac{2^{43} - 8}{41} \equiv 3 \, (\text{Mod} \ 43)
\]

G. Test for 59 and 61
\[
\frac{2^{61} - 8}{59} \equiv 3 \, (\text{Mod} \ 61)
\]

When at least one of the two numbers is not a prime number, it has no congruence.

**Examples**

H. Test for 9 and 11
\[
\frac{2^{11} - 8}{9} \not\equiv 3 \, (\text{Mod} \ 11)
\]

I. Test for 13 and 15
\[
\frac{2^{15} - 8}{13} \not\equiv 3 \, (\text{Mod} \ 15)
\]

J. Test for 15 and 17
\[
\frac{2^{17} - 8}{15} \not\equiv 3 \, (\text{Mod} \ 17)
\]

K. Test for 19 and 21
\[
\frac{2^{21} - 8}{19} \not\equiv 3 \, (\text{Mod} \ 21)
\]

L. Test for 21 and 23
\[
\frac{2^{23} - 8}{21} \not\equiv 3 \, (\text{Mod} \ 23)
\]

M. Test for 23 and 25
\[
\frac{2^{25} - 8}{23} \not\equiv 3 \, (\text{Mod} \ 25)
\]

N. Test for 27 and 29
\[
\frac{2^{29} - 8}{27} \not\equiv 3 \, (\text{Mod} \ 29)
\]

This test is probabilistic since there are pseudo-prime numbers that pass the test like 561.

Test for 561 and 563
\[
\frac{2^{563} - 8}{561} \equiv 3 \, (\text{Mod} \ 563)
\]

561 is a composite number.
563 is a prime number.
Therefore these numbers are not twin prime.

*Pseudo prime numbers (Psp) are a tiny portion of composite numbers that pass the test, these are known as Carmichael numbers.*
These Pseudoprime have a prime partner \( P = Psp + 2 \)

\[ Psp = \{561, 1905, 2465, 4371, 23001, 25761, 60701, 72249, 158369, \ldots \} \]

These prime have a pseudoprime partner \( Psp = P + 2 \)

\[ P = \{1103, 2699, 2819, 3643, 4679, 6599, 10259, 12799, 14489, 18719, \ldots \} \]

Probabilistic primality test for Twin prime numbers

Demonstration

\[ \exists k > 0 \in \mathbb{N} / 2k + 1 = p \]

\[ \frac{2^{p+2} - 8}{p} \equiv 3(\text{Mod } p + 2) \iff p, p + 2 \text{ are primes} \]

Demonstration when \( p = \text{prime number and } (p + 2) \) also.

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Fermat's theorem

Theorem: Fermat's Little Theorem, if \( p \) is a prime number, then, for each natural number \( a \), with \( a > 0 \)

\[ a^p \equiv a \ (\text{mod } p) \]
Program with Python 3.9

# Probabilistic primality test for Twin prime numbers.
# Author Gabriel M Zeolla

n = input("Enter Odd number: ")
if int(n) % 2 == 0:
    print("ERROR")
    n = input("Enter Odd number: ")
    if int(n) % 2 == 0:
        print("ERROR")

x = ((2**(int(n)+2) - 8) // (int(n)))
r = x % (int(n)+2)

p = r == 3

if p is True:
    print(n, "and", int(n)+2, " are probable Twin prime numbers")
else:
    print(n, "and", int(n)+2, 'are not Twin Prime!!')

Conclusion

Except for the difficulty generated by the pseudo-prime numbers, this test works correctly for all twin prime numbers without any exception.

Professor Zeolla Gabriel Martín

Other works of the author
https://independent.academia.edu/GabrielZeolla
References

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14. de Polignac, A. (1849). "Recherches nouvelles sur les nombres premiers" [New research on prime numbers]. Comptes rendus (in French). 29: 397–401. From p. 400: "1er Théorème. Tout nombre pair est égal à la différence de deux nombres premiers consécutifs d'une infinité de manières … " (1st Theorem. Every even number is equal to the difference of two consecutive prime numbers in an infinite number of ways … )
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