ON THE TWIN PRIME CONJECTURE

B. GENSEL AND J. SELLERS

Abstract. It’s well known that every prime number \( p \geq 5 \) has the form \( 6k - 1 \) or \( 6k + 1 \). We call \( k \) the generator of \( p \). Twin primes have a common generator and therefore it makes sense to consider twin primes on the level of their generators. The present paper considers increasing sections of the number line containing prospective twin prime generators and gives a lower bound on the number of such generators in periodic subsections. We prove further that the prospective twin prime generators are asymptotically, uniformly distributed over those periodic subsections. With these results the Twin Prime Conjecture finally can be proved.

Notations

We’ll use the following notations:

\[ \mathbb{N} \text{ the set of the positive integers}, \]
\[ \mathbb{P} \text{ the set of the primes}, \]
\[ \mathbb{P}^* \text{ primes } \geq 5. \]

1. Introduction

The question on the infinity of twin primes has kept many mathematicians busy for a long time. In 1919 V. Brun [3] proved that the sum of the series of the inverted twin primes converges, while attempting to prove the Twin Prime Conjecture. Several authors worked on bounds for the length of prime gaps (see f.i. [4, 5, 6]). In 2014 Y. Zhang [7] obtained great attention with his proof that there are infinitely many consecutive primes with a gap of at most 70,000,000. With the project ”PolyMath8” this bound was lessened down to 246 then assuming the validity of the Elliott–Halberstam Conjecture, to 12 [8].

Independently from each other both authors have found proofs for the Twin Prime Conjecture in the past (see [9], [10]). We present in this paper a combination of both approaches in one solution.

2. Twin Prime Generators

It’s well known that every prime number \( p \geq 5 \) has the form \( 6k - 1 \) or \( 6k + 1 \). We call \( k \) the generator of \( p \). Twin primes have a common generator and therefore it makes sense to consider twin primes on the level of their generators.

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Let
\[\kappa(p) = \begin{cases} 
p + 1 & \text{for } p \equiv -1 \pmod{6} 
\frac{p - 1}{6} & \text{for } p \equiv +1 \pmod{6}
\end{cases} \quad (2.1)\]
be the generator of the pair \((6\kappa(p) - 1, 6\kappa(p) + 1)\).

A number \(x\) is a twin prime generator (TPG) if \(6x - 1\) as well as \(6x + 1\) are primes. This is true if the following statement holds.

**Theorem 2.1.** A number \(x\) is a TPG if and only if there is no \(p \in \mathbb{P} \) with \(p < 6x - 1\) where one of the following congruences holds:

\[x \equiv -\kappa(p) \pmod{p} \tag{2.2}\]
\[x \equiv +\kappa(p) \pmod{p} \tag{2.3}\]

**Proof.**

(1) \(p = 6\kappa(p) - 1\): If (2.2) is true then there is a \(n \in \mathbb{N}\):

\[x = -\kappa(p) + n \cdot (6\kappa(p) - 1)\]
\[6x = -6\kappa(p) + 6n \cdot (6\kappa(p) - 1)\]
\[6x + 1 = -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) + 1\]
\[= (6n - 1)(6\kappa(p) - 1)\]
\[\Rightarrow 6x + 1 \equiv 0 \pmod{(6\kappa(p) - 1)} \Rightarrow x \text{ is no TPG.}\]

For (2.3) the proof will be done with \(6x - 1\):

\[6x - 1 = 6\kappa(p) + 6n \cdot (6\kappa(p) - 1) - 1\]
\[= (6n + 1)(6\kappa(p) - 1)\]
\[\Rightarrow 6x - 1 \equiv 0 \pmod{(6\kappa(p) - 1)} \Rightarrow x \text{ is no TPG.}\]

(2) \(p = 6\kappa(p) + 1\): We go the same way with (2.2) and \(6x - 1\) as well as (2.3) and \(6x + 1\):

\[6x - 1 = (6n - 1)(6\kappa(p) + 1) \Rightarrow 6x - 1 \equiv 0 \pmod{(6\kappa(p) + 1)}\]
\[6x + 1 = (6n + 1)(6\kappa(p) + 1) \Rightarrow 6x + 1 \equiv 0 \pmod{(6\kappa(p) + 1)}\]

These show that \(x\) is not a TPG if the congruences (2.2) or (2.3) hold. They cannot be true both because they exclude each other.

If on the other hand \(x\) is not a TPG, then at least one of \(6x - 1\) or \(6x + 1\) is not a prime. Let \(6x - 1 \equiv 0 \pmod{p}\) and \(p \equiv -1 \pmod{6}\), then we have:

\[6x - 1 \equiv p \pmod{p}\]
\[\equiv (6\kappa(p) - 1) \pmod{p}\]
\[6x \equiv 6\kappa(p) \pmod{p}\]
\[x \equiv \kappa(p) \pmod{p}.
\]
For $p \equiv +1(\text{mod } 6)$ we have

$$6x - 1 \equiv -p(\text{mod } p)$$
$$\equiv -(6\kappa(p) + 1)(\text{mod } p)$$
$$6x \equiv -6\kappa(p)(\text{mod } p)$$
$$x \equiv -\kappa(p)(\text{mod } p).$$

We can handle both other cases in the same way. Therefore either (2.2) or (2.3) holds if $x$ is not a TPG. □

If we consider that the least proper divisor of a number $6x - 1$ or $6x + 1$ is less or equal to $\sqrt{6x + 1}$ then $p$ in the congruences (2.2) and (2.3) can be further limited by

$$\hat{p}(x) = \max(p \in \mathbb{P}^* \mid p \leq \sqrt{6x + 1}).$$

(2.4)

Because we consider only primes as moduli we have independent congruences.

It is evident that

$$\tau(x, p) := (x + \kappa(p)) \text{ Mod } p, \text{ for } 5 \leq p \leq \hat{p}(x)$$

(2.5)

has the $p$ values $0, 1, 2, \ldots, p - 1$ and if $\tau(x, p) = 0$ or $\tau(x, p) = 2\kappa(p)$ then $x$ is no TPG since (2.2) or (2.3) is fulfilled. We will call this function $\tau(x, p)$ as position function. Obviously $\tau(x, p)$ is in $x$ a periodic function with a period length of $p$.

The congruences in (2.2) and (2.3) can be combined in the following way:

$$x^2 \equiv \kappa(p)^2(\text{mod } p) \text{ for } p \in \mathbb{P}^*, p \leq \hat{p}(x),$$

(2.6)

because if $x \equiv \pm\kappa(p)(\text{mod } p)$ then there is a number $t$ with $x = \pm\kappa(p) + tp$. Squared this produces $x^2 = \kappa(p)^2 + p(t^2p \pm 2t\kappa(p))$ and we get $x^2 \equiv \kappa(p)^2(\text{mod } p)$. This defines the so called indicator function $\psi(x, p)$ for which holds

$$x^2 - \kappa(p)^2 \equiv \psi(x, p)(\text{mod } p)$$

or

$$\psi(x, p) = (x^2 - \kappa(p)^2) \text{ Mod } p$$

(2.7)

for $p \in \mathbb{P}^*, p \leq \hat{p}(x)$.

Obviously also $\psi(x, p)$ is a periodic function in $x$ with a period length of $p$. Between the indicator function $\psi(x, p)$ and the position function $\tau(x, p)$ there is the following relationship:

$$\psi(x, p) = \tau(x, p) \cdot (x - \kappa(p)) \text{ Mod } p$$
$$= \tau(x, p) \cdot (\tau(x, p) - 2\kappa(p)) \text{ Mod } p$$

(2.8)

with $\psi(x, p) = 0$ if and only if (2.2) or (2.3) is true for $p$, which means that either $\tau(x, p) = 0$ or $\tau(x, p) = 2\kappa(p)$. In what follows we will use the letter $p$ for general prime numbers and $p_i$ with $p_1 = 2$ if we consider a prime of a sequence of primes.
Because we have to consider all primes \(5 \leq p \leq \hat{p}(x)\) as possible prime factors of \(6x \pm 1\) we’ll build the aggregate indicator function (abbreviated as AIF)

\[
\Psi(x, p_n) = \prod_{i=3}^{n} \frac{\psi(x, p_i)}{p_i}
\]

and

\[
\hat{\Psi}(x) = \Psi(x, \hat{p}(x)).
\]

Because of practical reasons we will expand the domain of this function to

\[
\Psi(x, p_n) = \Psi(x, p_{n-1}) \cdot \begin{cases} 
\frac{\psi(x, p_n)}{p_n} & \text{for } p_n \leq \hat{p}(x) \\
1 & \text{for } p_n > \hat{p}(x)
\end{cases}
\]

Since the co-domain of \(\psi(x, p)\) consists of positive integers between 0 and \(p - 1\), \(\Psi(x, p)\) and \(\hat{\Psi}(x)\) have rational values between 0 and \(<1\).

It holds \(\hat{\Psi}(x) > 0\) if and only if \(x\) is a TPG. Analogously, we call \(x\) a prospective TPG with respects to \(p_n\) if and only if \(\Psi(x, p_n) > 0\). In what follows we denote it shortly as \(p\text{TPG}_{p_n}\).

3. Generic Extensions

Since the congruences (2.5)

\[
x + \kappa(p_i) \equiv \tau(x, p_i)(\text{mod } p_i), \quad 3 \leq i \leq n
\]

are independent they satisfy the requirements of the Chinese Remainder Theorem (see [1, p. 89]). Therefore they are modulo \(5 \cdot 7 \cdot \ldots \cdot p_n\) uniquely resolvable. With

\[
p_n\#_5 := \prod_{i=3}^{n} p_i = 5 \cdot 7 \cdot \ldots \cdot p_n
\]

they are \((\text{mod } p_n\#_5)^1\) uniquely resolvable. Therefore the AIF \(\Psi(p_n)\) has the period length of \(p_n\#_5\) and it holds:

\[
\Psi(x + m \cdot p_n\#5, p_n) = \Psi(x, p_n) \mid m \in \mathbb{N}.
\]

**Corollary 3.1.** Because of this periodicity there exist infinitely many \(p\text{TPG}_{p_n}\) for each natural \(n \geq 3\).

**Definition 3.2.** With respect to the periodicity of the AIF \(\Psi(x, p_n)\) we define the following sections of the number line:

\[
\mathbb{G}_n := \{1, 2, \ldots, p_n\#5\} = \mathbb{G}_{n}^{(0)} \cup \mathbb{G}_{n}^{(1)} \cup \ldots \cup \mathbb{G}_{k}^{(p_n-1)}
\]

with

\[
\mathbb{G}_{n}^{(m)} := \{m \cdot p_{n-1}\#5 + 1, \ldots, (m + 1) \cdot p_{n-1}\#5\}, 0 \leq m \leq p_n - 1.
\]

Then \(\mathbb{G}_n\) is the section of the first period of \(\Psi(x, p_n)\) and contains \(p_n\) period sections with respect to \(\Psi(x, p_{n-1})\).

\(^1\)It is \(p_n\#5 = \frac{p_n\#}{6}\), with the primorial \(p_n\#.\)
It follows that:

\[ |G_n| = p_n \#_5 \quad \text{and} \quad |G_n^{(m)}| = p_{n-1} \#_5 \]

and

\[ G_{n+1}^{(0)} = G_n. \] (3.3)

**Definition 3.3.** We define the number of pTPG\(_{p_n}\) in \(G_n\) as:

\[ \phi(p_n) := |\{ x \in G_n \mid \Psi(x, p_n) > 0 \}| \]

and in \(G_n^{(m)}\) as

\[ \phi^{(m)}(p_n) := |\{ x \in G_n^{(m)} \mid \Psi(x, p_n) > 0 \}|, 0 \leq m \leq p_n - 1. \]

It follows that:

\[ \sum_{m=0}^{p_n-1} \phi^{(m)}(p_n) = \phi(p_n). \] (3.4)

**Proposition 3.4.** The number of pTPG\(_{p_n}\) in \(G_n\) is given by:

\[ \phi(p_n) = \prod_{k=3}^{n} (p_k - 2) \] (3.5)

**Proof.** The section \(G_n\) contains

\[ p_n \#_5 = \prod_{k=3}^{n} p_k \]

successive natural numbers \(1, 2, \ldots, p_n \#_5\). Each sequence of \(p_k|3 \leq k \leq n\) successive members of them obtains due to (2.5) two members \(x_\pm\) with \(\psi(x_\pm, p_k) = 0\) \((\tau(x_-, p_k) = 0 \text{ and } \tau(x_+, p_k) = 2\kappa(p_k))\) and \(p_k - 2\) members \(y\) with \(\psi(y, p_k) > 0\). A pTPG\(_{p_n}\) \(y \in G_n\) is determined by the requirement

\[ \psi(y, p_3) > 0, \ldots, \psi(y, p_n) > 0, \]

for all primes \(p_3, \ldots, p_n\) which results by virtue of (2.9) in \(\Psi(y, p_n) > 0\), while already \(\psi(x, p_j) = 0\) for one \(p_j \leq p_n\) results in \(\Psi(x, p_n) = 0\). And since for each modul \(p_k\) there are \(p_k - 2\) members with \(\psi(y, p_k) > 0\) then \(G_n\) with \(\prod_{k=3}^{n} p_k\) members contains

\[ \phi(p_n) = \prod_{k=3}^{n} (p_k - 2) \]

members with \(\Psi(y, p_n) > 0\). These are the pTPG\(_{p_n}\) in \(G_n\). \(\Box\)

**Definition 3.5.** We’ll denote with

\[ \xi_n(x, m) := x + m \cdot p_n \#_5 \]

the \(m^{th}\) **generic extension** of \(x\) of the order \(p_n\).

Obviously we have:

\[ G_n^{(m)} = \{ \xi_{n-1}(x, m) \mid x \in G_{n-1} \}. \] (3.6)
Lemma 3.6. Let $x$ be a fixed member of $\mathbb{G}_{n-1}$ and $m$ an integer varying between 0 and $p_n - 1$ and $\xi_{n-1}(x,m)$ the $m$th generic extension of order $p_{n-1}$. Then the value of
\[ \tau(\xi_{n-1}(x,m), p_n) = \tau(x + m \cdot p_{n-1} \#_5, p_n) \]
is uniquely determined by $m$.

Proof. Contrarily we assume that two different values $m_1, m_2$ of $m$ results in the same value of the position function
\[ \tau(x + m_1 \cdot p_{n-1} \#_5, p_n) = \tau(x + m_2 \cdot p_{n-1} \#_5, p_n). \]
By virtue of (2.5) we have
\[ \tau(x + m \cdot p_{n-1} \#_5, p_n) = (x + m \cdot p_{n-1} \#_5 + \kappa(p_n)) \mod p_n \]
\[ = (\tau(x, p_n) + m \cdot p_{n-1} \#_5) \mod p_n. \]
Since $x$ is a fixed member of $\mathbb{G}_{n-1}$ it must hold
\[ (m_1 \cdot p_{n-1} \#_5) \mod p_n = (m_2 \cdot p_{n-1} \#_5) \mod p_n. \]
Because of $m_1, m_2 < p_n$ and $\gcd(p_{n-1} \#_5, p_n) = 1$ this equation is only solvable with
\[ m_1 = m_2. \]

Corollary 3.7. Since the conditions for (3.7) are also fulfilled for all $p > p_n$ the following holds
\[ \tau(x, p) \neq \tau(\xi_{n}(x,m), p) \] while $p > p_n$ and $0 \leq m \leq p_n - 1$
since $\gcd(p_n \#_5, p) = 1$.

Lemma 3.8. Let $x$ be a member of $\mathbb{G}_n$ and $\Psi(x, p_n) > 0$, which means that $x$ is a $pTPG_{p_n}$. Then $p_{n+1} - 2$ generic extensions of $x$
\[ \xi_n(x,m) \] for $0 \leq m \leq p_{n+1} - 1$
are $pTPG_{p_{n+1}}$ and two aren’t.

Proof. Because of the periodicity of the AIF $\Psi(x, p_n)$ it’s evident that $\xi_n(x,m)$ for
$0 \leq m \leq p_{n+1} - 1$ are $pTPG_{p_n}$, it holds
\[ \Psi(\xi_n(x,m), p_n) > 0 \] since $\Psi(x + m \cdot p_{n} \#_5, p_n) = \Psi(x, p_n) > 0$.

We consider for a fixed $x_o$ the position function
\[ \tau(\xi_n(x_o,m), p_{n+1}) = \tau(x_o + m \cdot p_{n} \#_5, p_{n+1}). \]
It has $p_{n+1}$ different values and in virtue of Lemma 3.6 exactly one value 0 for
$m = m_-$ and one value $2\kappa(p_{n+1})$ for $m = m_+$
\[ \tau(\xi_n(x_o,m_-), p_{n+1}) = 0 \] and
\[ \tau(\xi_n(x_o,m_+), p_{n+1}) = 2\kappa(p_{n+1}). \]
By virtue of (2.8) we get for the indicator function
\[ \psi(\xi_n(x_o,m_\pm), p_{n+1}) = 0 \]
and since
\[ \Psi(\xi_n(x_o, m_{\pm}), p_{n+1}) = \Psi(\xi_n(x_o, m_{\pm}), p_n) \cdot \frac{\psi(\xi_n(x_o, m_{\pm}), p_{n+1})}{p_{n+1}} \]
we get finally for 2 values \( m_{\pm} \)
\[ \Psi(\xi_n(x_o, m_{\pm}), p_{n+1}) = 0, \]
while for \( p_{n+1} \) 2 values of \( m \) holds \( \psi(\xi(x_o, m), p_{n+1}) > 0 \), which results in
\[ \Psi(\xi(x_o, m), p_{n+1}) > 0. \]

\[ \Box \]

**Theorem 3.9.** The number \( \phi^{(r)}(p_n) \) of pTPG in any subsection \( \mathbb{G}_n^{(r)} \) with \( 0 \leq r \leq p_n - 1 \) meets the following criterion
\[ \phi^{(r)}(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}). \]

**Proof.** We consider a fixed pTPG \( x_o \in \mathbb{G}_{n-2} \) with \( \Psi(x_o, p_{n-2}) > 0 \) and their generic extensions of order \( p_{n-2} \)
\[ \xi_{n-2}(x_o, m) \text{ for } 0 \leq m \leq p_{n-1} - 1. \]

By virtue of Lemma 3.8 we know that there are \( p_{n-1} - 2 \) generic extensions with \( \Psi(\xi_{n-2}(x_o, m), p_{n-1}) > 0 \), which means they are pTPG. As generic extensions they are members of \( \mathbb{G}_{n-1} \). Since (3.3) they are also members of \( \mathbb{G}_n^{(0)} \). In what follows we will denote these pTPG as \( x_o \)-candidates. If \( y \) is an \( x_o \)-candidate then \( \Psi(y, p_{n-1}) > 0 \) and we can use the position function to check whether they are also pTPG. By virtue of Corollary 3.7 all the \( p_{n-1} - 2 \) \( x_o \)-candidates have different \( \tau \)-values. But since the position function \( \tau(y, p_n) \) has \( p_n > p_{n-1} - 2 \) values, the disallowed values do not necessarily occur among the \( x_o \)-candidates, but they can occur single or both. The disallowed values would occur among \( x_o \)-candidates if
\[ \tau(y, p) = 0 \text{ or } \tau(y, p) = 2k(p). \]

Hence there are at least \( (p_{n-1} - 2) - 2 = p_{n-1} - 4 \) \( x_o \)-candidates that are also pTPG in \( \mathbb{G}_n^{(0)} \). This holds for one (fixed) pTPG \( x_o \). But since there are \( \phi(p_{n-2}) \) pTPG in \( \mathbb{G}_{n-2} \) we have
\[ \phi^{(0)}(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}) \tag{3.8} \]
pTPG in the subsection \( \mathbb{G}_n^{(0)} \).

The above described situation related to the uniqueness of the \( \tau \)-values holds also for the generic extensions of order \( p_{n-1} \)
\[ \xi_{n-1}(\xi_{n-2}(x_o, m), r) \mid 0 \leq m \leq p_{n-1} - 1, \ 1 \leq r \leq p_n - 1. \]

This is true in the \( p_n - 1 \) subsections \( \mathbb{G}_n^{(1)}, \mathbb{G}_n^{(2)}, \ldots, \mathbb{G}_n^{(p_n - 1)} \) because the \( p_{n-1} - 2 \) extensions of the \( x_o \)-candidates inside of these subsections have the same position differences and hence the same situation related to the \( \tau \)-values in virtue of Corollary 3.7 like in \( \mathbb{G}_n^{(0)} \). Therefore and since the right side of (3.8) is independent from the number of a subsection it holds in all subsections
\[ \phi^{(r)}(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}), \ 0 \leq r \leq p_n - 1. \]

\[ \Box \]
Proposition 3.10. The \( p\text{TPG}_{p_n} \) in \( \mathbb{G}_n \) are over the subsections \( \mathbb{G}_n^{(0)}, \ldots, \mathbb{G}_n^{(p_n-1)} \) asymptotically, uniformly distributed

\[
\phi^{(m)}(p_n) \sim \frac{\phi(p_n)}{p_n} \text{ for } 0 \leq m \leq p_n - 1.
\]

Proof. We consider the following ratio and get in virtue of Theorem 3.9

\[
\frac{p_n \cdot \phi^{(m)}(p_n)}{\phi(p_n)} \geq \frac{p_n(p_n-1-4) \cdot \phi(p_n-2)}{\phi(p_n)} = \frac{p_n(p_n-1-4)}{(p_n-2)(p_n-2)}
\]

and with \( p_n-1 = p_n - d \)

\[
= \frac{p_n (p_n - (d + 4))}{(p_n - (d + 2))(p_n - 2)}
\]

\[
= \frac{1 - \frac{d+4}{p_n}}{(1 - \frac{d+2}{p_n})(1 - \frac{2}{p_n})} \rightarrow 1. \quad p_n \rightarrow \infty
\]

Additionally we have

\[
\frac{p_n(p_n-1-4)}{(p_n-2)(p_n-2)} < 1
\]

because the numerator is less than the denominator:

\[
p_n(p_n-1-4) - ((p_n-1-2)(p_n-2)) = p_n p_n - 4p_n - p_n p_n - 2p_n - 2p_n - 4 = 2p_n - 4 < 0.
\]

Finally since (3.4) holds

\[
\frac{p_n \cdot \phi^{(m)}(p_n)}{\phi(p_n)} \sim 1.
\]

\[\square\]

As supplement to the number functions in Definition 3.3 we define

\[
\phi_k(p_n) := |\{x \in \mathbb{G}_k \mid \Psi(x, p_n) > 0\}|, k < n \quad (3.9)
\]

as the number of \( p\text{TPG}_{p_n} \) in the section \( \mathbb{G}_k \). Because of (3.3) and Theorem 3.9 we have:

\[
\phi_{n-1}(p_n) = \phi^{(0)}(p_n) \geq (p_n-1-4) \cdot \phi(p_n-2). \quad (3.10)
\]

It is evident that holds

\[
\phi_n(p_n) = \phi(p_n). \quad (3.11)
\]

With an average density of \( p\text{TPG}_{p_n} \) like in [9, p. 54, eq.(4.7)]

\[
\eta(p_m) := \frac{\phi(p_m)}{p_m^{\#5}} = \frac{m}{\prod_{j=3}^{m} \frac{p_j - 2}{p_j}} \quad (3.12)
\]

we obtain the following corollary:
Corollary 3.11. Because of Proposition 3.10 and equations (3.10) and (3.11) hold the following relations

\[ \phi_n(p_n) = \phi(p_n) = p_n \#_5 \cdot \eta(p_n) \quad \text{and} \]
\[ \phi_n(p_{n+1}) = \phi^{(0)}(p_{n+1}) \sim \frac{\phi(p_{n+1})}{p_{n+1}} = p_n \#_5 \cdot \eta(p_{n+1}) \]

and hence

\[ \frac{\phi_n(p_{n+1})}{\phi_n(p_n)} \sim \frac{\eta(p_{n+1})}{\eta(p_n)}. \quad (3.13) \]

4. On the Twin Prime Conjecture

Lemma 4.1. Let

\[ x_n := \frac{p_n^2 - 1}{6}. \quad (4.1) \]

Then for each integer \( k \geq 4 \) there is a unique integer \( n \) such that

\[ x_n < p_k \#_5 < x_{n+1} \quad \text{with} \quad n > k. \]

Proof. Let be \( p_n = \max(p \in \mathbb{P}^* \mid p^2 < p_k \#) \). Then holds

\[ p_n^2 < p_k \# < p_{n+1}^2. \quad (4.2) \]

First we prove the left inequality. Since \( p_k \# = 6p_n \#_5 \) we get

\[ 6p_k \#_5 > p_n^2 > p_n^2 - 1 \quad \implies \quad p_k \#_5 > \frac{p_n^2 - 1}{6} = x_n. \]

For the right inequality we have since \( p_k \# < p_{n+1}^2 \) and \( p_{n+1} = 6 \kappa(p_{n+1}) \pm 1 \)

\[ 6p_k \#_5 \leq p_{n+1}^2 - 1 \]
\[ = 6 \kappa(p_{n+1}) (6 \kappa(p_{n+1}) \pm 2) \]
\[ = 6u_{n+1} \]
and hence

\[ u_{n+1} = 2 \kappa(p_{n+1}) (3 \kappa(p_{n+1}) \pm 1) \]
\[ \equiv 0 \pmod{2}. \]

On the other hand holds \( p_k \#_5 \equiv 1 \pmod{2} \). Therefore we get

\[ 6p_k \#_5 < 6u_{n+1} = p_{n+1}^2 - 1 \quad \implies \quad p_k \#_5 < \frac{p_{n+1}^2 - 1}{6} = x_{n+1}. \]

From both it concludes

\[ x_n < p_k \#_5 < x_{n+1}. \]

\[ \square \]

Proposition 4.2. All \( pTPG_{p_n} \) inside of \( \mathbb{G}_k \) with

\[ x_n < p_k \#_5 < x_{n+1} \quad \text{and} \quad 4 \leq k < n. \]

are TPGs, with \( x_n \) given by (4.1).
Proof. By virtue of (2.4) \( \hat{p}(x) \) determines the greatest possible prime factor of \( 6x \pm 1 \). If \( x \geq x_{n+1} \), then by virtue of (4.1)

\[
x \geq x_{n+1} = \frac{p_{n+1}^2 - 1}{6} \implies p_{n+1} = \sqrt{6x_{n+1} + 1} = \hat{p}(x_{n+1}) \leq \hat{p}(x).
\]

Hence \( x_{n+1} \) is the least generator of pairs \( (6x - 1, 6x + 1) \) where \( p_{n+1} \) could be a prime factor of them. That means that for all \( x \geq x_{n+1} \) a pTPG \( p_n \) is not necessarily a TPG and on the other hand that all \( x < x_{n+1} \) with \( \Psi(x, p_n) > 0 \) are surely TPGs since no prime \( p > p_n \) can be a prime factor of \( 6x \pm 1 \). And because \( p_k \#_5 \) as upper limit of the section \( G_k \) is less than \( x_{n+1} \), all members of \( G_k \) are TPGs and the number function \( \phi_k(p_n) \) counts in this case the TPGs in \( G_k \). \( \square \)

**Theorem 4.3.** For each integer \( k \geq 4 \) there is an integer \( n > k \) with \( x_n < p_k \#_5 < x_{n+1} \) such that in \( G_k \) there are at least

\[
\phi(p_k) \cdot \prod_{j=k}^{n-1} \frac{p_j - 4}{p_j - 2} \text{ TPGs.}
\]

Proof. Lemma 4.1 guarantees that for each integer \( k \) such integer \( n \) exists. Proposition 4.2 shows that in this case the function \( \phi_k(p_n) \) counts the TPGs in \( G_k \). Hence we have to prove that:

\[
\phi_k(p_n) \geq \phi(p_k) \cdot \prod_{j=k}^{n-1} \frac{p_j - 4}{p_j - 2}.
\]

As consequence of Corollary 3.11 and the meaning of \( \eta(p) \) as average density of pTPG\( _p \) we can assume

\[
\frac{\phi_k(p_{k+r})}{\phi_k(p_{k+r-1})} \sim \frac{\eta(p_{k+r})}{\eta(p_{k+r-1})} = \frac{p_{k+r} - 2}{p_{k+r}}, \quad (4.3)
\]

for any positive integers \( k, r \). We set \( r = 2 \) and get

\[
\phi_k(p_{k+2}) \sim \frac{p_{k+2} - 2}{p_{k+2}} \cdot \phi_k(p_{k+1}).
\]

On the other hand by virtue of Proposition 3.10 holds

\[
\frac{\phi^{(0)}(p_{k+2})}{\phi(p_{k+1})} \sim \frac{\phi(p_{k+2})}{p_{k+2} \cdot \phi(p_{k+1})} = \frac{p_{k+2} - 2}{p_{k+2}}, \quad (4.4)
\]

Ergo we obtain

\[
\phi_k(p_{k+2}) \sim \frac{\phi^{(0)}(p_{k+2})}{\phi(p_{k+1})} \cdot \phi_k(p_{k+1})
\]

\[= \frac{\phi^{(0)}(p_{k+1})}{\phi(p_{k+1})} \cdot \phi^{(0)}(p_{k+2})\]

The validity of (4.4) is preserved if we substitute \( k + 2 \) by \( k + r \)

\[
\frac{\phi^{(0)}(p_{k+r})}{\phi(p_{k+r-1})} \sim \frac{\phi(p_{k+r})}{p_{k+r} \cdot \phi(p_{k+r-1})} = \frac{p_{k+r} - 2}{p_{k+r}}
\]
and hence from (4.3) we get
\[ \phi_k(p_{k+r}) \sim \frac{\phi^{(0)}(p_{k+r})}{\phi(p_{k+r-1})} \cdot \phi_k(p_{k+r-1}). \]

Finally we set \( r = n - k \) and obtain
\[ \phi_k(p_n) \sim \frac{\phi^{(0)}(p_n)}{\phi(p_n-1)} \cdot \phi_k(p_{n-1}) \]
\[ \sim \frac{\phi^{(0)}(p_n)\phi^{(0)}(p_{n-1})}{\phi(p_{n-1})\phi(p_{n-2})} \cdot \phi_k(p_{n-2}) \]
\[ = \phi^{(0)}(p_n) \cdot \frac{\phi^{(0)}(p_{n-1})\phi_k(p_{n-2})}{\phi(p_{n-1})\phi(p_{n-2})} \]
\[ \sim \phi^{(0)}(p_n) \cdot \frac{\phi_k(p_{k+1})}{\phi(p_{k+1})} \cdot \prod_{j=k+1}^{n-1} \frac{\phi^{(0)}(p_j)}{\phi(p_j)} \]

and since \( \phi_k(p_{k+1}) = \phi^{(0)}(p_{k+1}) \)
\[ \sim \phi^{(0)}(p_n) \cdot \prod_{j=k+1}^{n-1} \frac{\phi^{(0)}(p_j)}{\phi(p_j)}. \]

In virtue of Theorem 3.9 we obtain
\[ \phi_k(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}) \cdot \prod_{j=k+1}^{n-1} \frac{(p_{j-1} - 4)\phi(p_{j-2})}{\phi(p_j)} \]
\[ = (p_{n-1} - 4) \cdot \phi(p_{n-2}) \cdot \frac{\phi(p_{k-1})\phi(p_k)}{\phi(p_{n-2})\phi(p_{n-1})} \cdot \prod_{j=k+1}^{n-1} (p_{j-1} - 4) \]
\[ = (p_{n-1} - 4) \cdot \phi(p_k) \cdot \frac{\phi(p_{k-1})}{\phi(p_{n-1})} \cdot \prod_{j=k}^{n-2} (p_j - 4) \]
\[ = \phi(p_k) \cdot \frac{\phi(p_{k-1})}{\phi(p_{n-1})} \cdot \prod_{j=k}^{n-1} (p_j - 4) \]
\[ = \phi(p_k) \cdot \prod_{j=k}^{n-1} \frac{p_j - 4}{p_j - 2} \]

because
\[ \frac{\phi(p_{k-1})}{\phi(p_{n-1})} = \frac{\prod_{j=3}^{k-1} (p_j - 2)}{\prod_{j=3}^{n-1} (p_j - 2) \cdot \prod_{j=k}^{n-1} (p_j - 2)} = \frac{1}{\prod_{j=k}^{n-1} (p_j - 2)}. \]

\( \square \)

Expanding the Definition 3.2 of the period sections we build a chain of gapless and disjunct sections of the number line
\[ \mathbb{G}_j : = \{x \in \mathbb{N} \mid p_{j-1} \#_5 + 1 \leq x \leq p_j \#_5\} \text{ for } \forall j \geq 5 \]
\[ = \bigcup_{m=1}^{p_j-1} \mathbb{G}_j^{(m)}. \]
Obviously is
\[ \mathbb{N} = \mathbb{G}_4 \cup \bigcup_{j=5}^{\infty} \hat{\mathbb{G}}_j \] and \( \hat{\mathbb{G}}_j \cap \hat{\mathbb{G}}_{j+1} = \emptyset, j \geq 5. \)

**Theorem 4.4 (Twin Prime Conjecture).** There are infinitely many primes \( p \in \mathbb{P}^* \) such that \( p + 2 \) is prime too.

**Proof.** Let the integer \( k \) and \( n \) be determined like in Theorem 4.3. Then for an arbitrary \( k \geq 5 \)

\[ \phi_k(p_n) \geq \phi(p_k) \cdot \prod_{j=k}^{n-1} \frac{p_j - 4}{p_j - 2}, \]

in which in this case all \( \text{pTPG}_{p_n} \) in \( \mathbb{G}_k \) are TPGs. By virtue of Proposition 3.10 these TPGs are asymptotically, uniformly distributed over the subsections \( \mathbb{G}_k^{(0)}, \ldots, \mathbb{G}_k^{(p_k - 1)} \).

Hence each of the in (4.5) defined sections

\[ \hat{\mathbb{G}}_k \text{ contains at least } \frac{p_k - 1}{p_k} \cdot \phi_k(p_n) \text{ TPGs.} \]

Ergo the following holds

\[ |\hat{\mathbb{G}}_k \cap T| \geq \nu(p_k) := \frac{(p_k - 1)\phi(p_k)}{p_k} \cdot \prod_{j=k}^{n-1} \frac{p_j - 4}{p_j - 2}, \quad (4.6) \]

where \( T = \{x \in \mathbb{N} : \hat{\Psi}(x) > 0\} \) is the set of all TPGs.

We look at the product in (4.6)

\[
\prod_{j=k}^{n-1} \frac{p_j - 4}{p_j - 2} > \frac{p_k - 4}{p_k - 2} \cdot \frac{p_k - 2}{p_k - 2} \cdot \frac{p_k - 4 + 2m}{p_k - 2 + 2m} \cdot \frac{p_{n-1} - 4}{p_{n-1} - 2}
\]

\[= \frac{p_k - 4}{p_{n-1} - 2} \]
since all prime \( p > 2 \) are odd numbers and each factor \( \frac{p_k - 4 + 2m}{p_k - 2 + 2m} \) is less than 1. With it we get

\[
\nu(p_k) > \frac{p_k - 1}{p_k} \cdot \frac{p_k - 4}{p_{n-1} - 2} \cdot \phi(p_k)
\]

and since \( p_n = \max(p < \sqrt{p_k\#}) \) (see eq.(4.2))

\[
> \frac{p_k - 1}{p_k} \cdot \frac{p_k - 4}{\sqrt{p_k\#}} \cdot \phi(p_k)
\]

\[
= \frac{p_k - 1}{p_k} \cdot \frac{p_k - 4}{\sqrt{p_k\#}} \cdot \phi(p_k)
\]

and with \( \eta(p_k) \) in virtue of (3.12)

\[
= \frac{p_k - 1}{p_k} \cdot \frac{p_k - 4}{\sqrt{6}} \cdot \sqrt{\eta(p_k) \cdot \sqrt{\phi(p_k)}}
\]

and since \( \eta(p) > \frac{3}{p} \) (see [9, Theorem 3])

\[
> \frac{(p_k - 1)(p_k - 4)}{p_k \cdot \sqrt{2p_k}} \cdot \sqrt{\phi(p_k)} \sim \frac{\sqrt{p_k \cdot \phi(p_k)}}{2}.
\]

And this is evidently a function increasing with \( p_k \). Ergo \( \nu(p_k) \) is an increasing function too with a least value \( > 39 \) for \( k = 5 \) and hence \( n = 15 \). Consequently it holds for infinitely many integers \( k \geq 5 \)

\[
|\tilde{G}_k \cap T| > 1.
\]

Therefore there are infinitely many disjunct sections \( \tilde{G}_k \) containing all at least more than one TPG. Hence there exist infinitely many twin primes on the number line. \( \square \)

References

[3] Brun, V.: La série \( \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \ldots \) où les dénominateurs sont “nombres premiers jumeaux” est convergente finie. Bull. Sci. Math. (2) 43 (1919), 100-104 and 124-128.
Prof. Dr. B. Gensel, Carinthia University of Applied Sciences, Spittal on Drau, Austria
E-mail address: b.gensel@fh-kaernten.at

Dr. John K. Sellers, Colorado, USA
E-mail address: jksellers43@gmail.com