Title:

An Inconsistency Between the Gravitational Time Dilation Equation and the Twin Paradox

Author: Michael Leon Fontenot

email: physicsfiddler@gmail.com

Abstract:

It is shown in this monograph that the Gravitational Time Dilation Equation, together with the well-known Equivalence Principle relating gravitation and acceleration, produce results that contradict the required outcome at the reunion of the twins in the famous twin 'paradox'. The Equivalence Principle Version of the Gravitational Time Dilation Equation (the "EPVGTD" equation) produces results that say that, when the traveling twin (he) instantaneously changes his velocity, in the direction TOWARD the distant home twin (her), that he will conclude that her age instantaneously becomes INFINITE. It is well known that, according to her, at their reunion, she will be older than him, but both of their ages will be FINITE. The twins clearly MUST be in agreement about their respective ages at the reunion, because they are co-located there.

Section 1. The Gravitational Time Dilation Equation

The Gravitational Time Dilation Equation is described in Wikipedia:

https://en.wikipedia.org/wiki/Gravitational_time_dilation

It says, in particular, that for two clocks in a constant and uniform gravitational field of force per unit mass "g", separated by the constant distance "d" in the direction of the field, the clock that is closer to the source of the field will run slower than the other clock, by the factor exp(g d).

The equivalence principle then says that for two clocks that are accelerating with the same acceleration "A", separated by the constant distance "d" in the direction of the acceleration, the trailing clock will run slower than the other clock, by the factor exp(A d). The two values "g" and "A" are numerically the same.

Section 2. A Possible Proof that Negative Aging Doesn't Occur in Special Relativity

Consider the following scenario:

At some instant, the perpetually-inertial "home twin" (she) is 20 years old, and is holding a display that always shows her current age. Facing (and co-located with) her is the "helper friend" (the "HF") of an observer (he) who is "d" ly away to her right. Both the HF and he are also 20 years old, and are stationary with respect to her at that instant. Like her, he and the HF are each holding a display that always shows their current ages.

Now, suppose that he and his helper friend then both start accelerating at a constant "A" ly/y/y toward the right. He knows that his helper friend (the HF) is then ageing at a constant rate that is slower than his own rate of ageing, by the factor exp(A d).

An instant later, his display shows the time 20 + epsilon_1, where epsilon_1 is a very small positive number. He knows that HF's display shows the time 20 + epsilon_2, where epsilon_2 = epsilon_1 / exp(A d).

She can still see HF's display (because HF has only moved an infinitesimal distance away from her, to her right). She will see that HF's display reads 20 + epsilon_1 / exp(A d). And likewise, HF can still see her display. What does HF see on her display? Does HF see that she is now slightly YOUNGER than 20? No! It would clearly be absurd for someone essentially co-located with her to see her get younger. What HF would see her display reporting is that she was some very small positive amount epsilon_3 OLDER than she was at the instant before the acceleration. HF then sends a message to him, telling him that she was 20 + epsilon_3 years old right then. When he receives that message, he then knows that her current age, when he was 20 + epsilon_1 years old, was 20 + epsilon_3. So he KNOWS that she didn't get younger when he accelerated away from her. That contradicts what the well-known Co-Moving Inertial Reference Frames (CMIF) simultaneity method says.

In the above, I asked

"What does HF see on her display?”.

And I answered

"HF would see her display reporting that she was some very small amount epsilon_3 OLDER that she was at the instant before the acceleration.”

Since the above argument makes use of very small (unspecified) quantities, it could be argued that time delays due to the speed of light might also need to be taken into account when describing what the HF sees on her display.

But I think any such concerns can be addressed by pointing out that the separation "d" between him and her can be made arbitrarily large, and CMIF simultaneity says that the amount of negative ageing that occurs is proportional to their separation. Since the errors involved due to the finite speed of light between her and the HF are essentially independent of the distance "d", those errors become negligible for sufficiently large “d".

There is another argument that shows that the HF ("Helper Friend") can't conclude that the home twin (she) is less than 20 years old when the HF is 20 + epsilon_2. We can require that she transmits NO light messages to him when she is 20 years old or younger. Suppose the HF receives a light message from her when he is 20 + epsilon_2 years old. By the requirement, she must have been older than 20 years old when she sent that message. When the HF receives that message, he knows that she must be older than when she sent the message, so she must definitely be older than 20 years old when the HF is 20+epsilon_2. Therefore, she did NOT get younger, according to him, when he accelerated away from her. [Note: we can’t KNOW that the HF received that light message; we’re just supposing it, so I now don’t consider this additional argument to be valid.]

A still simpler argument is that, if the HF ever concluded that she got younger when he accelerated away from her, he would be concluding that she was less than 20 years old, when he himself was more than 20 years old, at that instant of his acceleration. But BEFORE his acceleration, the HF was co-located and stationary with her, and they always shared the same exact age then (less than 20). So, we first CONJECTURE that after they separated, he was over 20 when she was under 20. But we KNOW that for EACH instant in her life when she was under 20, he was co-located with her and they shared the same age. So the conjecture is false.

It seems to me that, once the distant accelerating observer (the "AO") has a way to set up an array of clocks (with attending helper observers) that he can use to define his concept of "NOW" (analogous to how Einstein did it for perpetually-inertial observers), it becomes IMPOSSIBLE for the home twin to age negatively, according to the distant accelerating observer. It's true that those clocks aren't synchronized as they are in the perpetually-inertial case, but they don't have to be, since the distant AO knows exactly how the rates of those clocks compare to his own clock, and he can compensate for their different readings.

The way the accelerating observer defines his "NOW" instant at distant locations comes directly from the gravitational time dilation equation, via the equivalence principle. It says that a "helper friend" (HF), who always is accelerating exactly as the AO is accelerating, with acceleration "A", will age at a rate that is a fixed known ratio of the AO's rate. The given HF and the AO are always a constant distance "d" apart. If the chosen HF is BEHIND the AO (compared to the direction of the acceleration), that HF will age SLOWER than the AO by the factor exp(A d). To keep things as simple as possible, we can always let all of the HF's and the AO's ages be the same, immediately before they all start accelerating. Then the ratio of the age of the "behind" HF's age to the AO's age is just 1/exp(A d). And if, instead, another HF is AHEAD of the AO (compared to the direction of the acceleration), then the ratio of that "ahead" HF's age to the AO's age is just exp(A d). (Of course, different "behind" HF's will have different distances "d" to the AO, and likewise for the "ahead" HF's.) So, at some instant T in the AO's life, he computes that the original "behind" HF's current age is (T / exp(A d)). Or, alternatively, he computes that the "ahead" HF's current age is (T exp(A d)). The way he SELECTS the HF from among all possible HF's (both ahead and behind him) is such that the chosen HF is momentarily co-located with the home twin (her) at the instant the AO wants to know her current age.

So, if all of the above is correct, that allows the AO to construct an array of (effectively) synchronized clocks, with attending helper observers, that are "attached" to him, similar to what a perpetually-inertial observer can do, and that can put an observer momentarily co-located with the distant twin (her) at the instant in the AO's life when he wants to know her current age. And in both the perpetually-inertial and the accelerated cases, it would be ABSURD for that momentarily co-located observer to observe a large and abrupt change (either positive or negative) in her age at that instant.

Section 3. Instantaneous Velocity Changes in the Equivalence Principle Version of the Gravitational Time Dilation Equation

When using the CMIF simultaneity method, the analysis is GREATLY simplified by using instantaneous velocity changes, rather than finite accelerations that last for a finite amount of time. So I decided to try using instantaneous velocity changes in the Equivalence-Principle Version of the Gravitational Time Dilation equation (the "EPVGTD" equation). The result (assuming I haven't made a mistake somewhere) is unexpected and disturbing. My analysis found that the age change of the HF, produced by an instantaneous velocity change by the AO and the HF, from zero to 0.866 lightseconds/second (ls/s), directed TOWARD the home twin (her), is INFINITE!

I'll describe my analysis, and perhaps someone can find an error somewhere.

Before the instantaneous velocity change, the AO (he), the HF, and the home twin (she) are all mutually stationary. She and the HF are initially co-located, and the AO (he) is "d" lightseconds away from her and the HF.

I start by considering a constant acceleration "A" ls/s/s that lasts for a very short but finite time of "tau" seconds. That acceleration lasting tau seconds causes the rapidity, theta, (which starts at zero) to increase to

theta = (A tau) ls/s,

and so we get the following relationship:

A = theta / tau.

We will need the above relationship shortly.

(Rapidity has a one-to-one relationship to velocity. Velocity of any object that has mass can never be equal to or greater than the velocity of light in magnitude, but rapidity can vary from -infinity to +infinity.)

We want the velocity, beta, to be 0.866 ls/s after the acceleration (because that results in the gamma factor having the nice value 2.0). Rapidity, theta, is related to velocity, beta, by the equation

theta = arctanh (beta) = (1/2) ln [(1 + beta) / (1 - beta)].

("arctanh" just means the inverse of the hyperbolic tangent function.)

So velocity = 0.866 ls/s corresponds to a rapidity of about 1.317 ls/s.

When the acceleration is directed TOWARD the home twin (her), the "EPVGTD" equation says that the acceleration “A” will cause the HF to age FASTER than the AO by the factor exp(A d), where d is the constant separation between the AO and the HF.

Note that the argument in the exponential exp(A d) can be separated like this:

exp(A d) = [exp(d)] sup A,

where "sup A" means "raise the quantity exp(d) to the power "A" ". The rationale for doing that is because the quantity exp(d) won't change as we make the acceleration greater and greater, and the duration of the acceleration shorter and shorter. That will make the production of the table below easier.

The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds, is just

tau [exp(d)] sup A,

because [exp(d)] sup A is the constant RATE at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

But we earlier found that A = theta / tau, so we get tau [exp(d)] sup {theta / tau} = tau [exp(d theta)] sup {1/tau}

for the change in the age of the HF due to the short acceleration. So we have an expression for the change in the age of the HF that is a function of only the single variable tau ... all other quantities in the equation (d and theta) are fixed. We can now use that equation to create a table that shows the change in the age of the HF, as a function of the duration tau of the acceleration (while keeping the area under the acceleration-versus-tau graph constant).

In order to make the table as easy to produce as possible, I chose the arbitrary value of the distance "d" to be such that

exp(d theta) = 20000.

Therefore we need

ln[exp (d theta)] = d theta = ln (20000) = 9.903,

and since theta = 1.317, d = 7.52 lightseconds.

If we were creating this table for the CMIF simultaneity method, we would find that as the duration tau of the acceleration decreases (with a corresponding increase in the magnitude of the acceleration, so that the product remains the same), the amount of ageing by the HF approaches a finite limit. I.e., in CMIF, eventually it makes essentially no difference in the age of the HF when we halve the duration of the acceleration, and make the acceleration twice as great.

But here is what I got for the EPVGTD simultaneity method:

(In the table, "10sup4" means "10 raised to the 4th power".)

tau	(tau) (2000)sup(1/tau)
1.0	2x10sup4 = 20000
0.5	2x10sup8
0.4	2.26x10sup10
0.3	6.3x10sup13
0.2	0.64x10sup21
0.1	1.02x10sup42
0.01	1.27x10sup428
0.001	? (My calculator overflowed at 10sup500)

Clearly, for the EPVGTD simultaneity method, when the acceleration is directed TOWARD the home twin, the HF's age goes to infinity as the acceleration duration goes to zero. That seems like an absurd result to me. And it is radically different from what happens with CMIF simultaneity, where the HF's age quickly approaches a finite limit as tau goes to zero.

What does the EPVGTD simultaneity method say, when the acceleration is directed AWAY FROM the home twin? In that case, the "EPVGTD" equation says that the acceleration “A” will cause the HF to age SLOWER than the AO by the factor exp(-A d). The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds, is just

tau [exp(-d A)] = tau [exp(-d theta / tau)] .

As tau goes to zero, the first factor (tau) goes to zero, and so does the second factor (the exponential), since its argument again goes to infinity, but is now negative. So, when the acceleration is directed AWAY FROM the home twin, the HF’s age goes to zero as the acceleration duration tau goes to zero.

Section 4. Instantaneous Velocity Changes in the Equivalence Principle Version of the LINEARIZED Gravitational Time Dilation Equation - (the LGTD Model)

I repeated my previous analysis that found, when the acceleration is directed TOWARD the home twin, that the age of the HF, according to the AO, goes to infinity as the duration tau of the acceleration goes to zero, according to the Equivalence Principle Version of the Gravitational Time Dilation Equation, (the "EPVGTD" equation). But this time, instead of using the EPVGTD equation, I used the new equation, which I'll call the "Linearized Gravitational Time Dilation Equation", (the "LGTD" equation). I simply replace the exponential exp(A d) with the quantity (1 + A d). This is the same approximation that Einstein used in his 1907 paper. It is the first two terms of the power-series expansion of the exponential. But it is only accurate for small (A d), which is NOT the case in this example (since we are using an infinite (A) that lasts for an infinitesimal time: a Dirac “delta function”). But for SOME reason, it gives results that are more reasonable than the exponential. (Einstein used the approximation correctly, because its argument in his case was very small).

In what follows below, I'll repeat each affected calculation that I made in the above section, and show the revised calculation:

[Previous]:

The "EPVGTD" equation says that the acceleration “A” will cause the HF to age faster than the AO by the factor exp(A d), where d is the constant separation between the AO and the HF.

[Revised]:

The "LGTD" equation says that the acceleration “A” will cause the HF to age faster than the AO by the factor (1 + A d), where d is the constant separation between the AO and the HF.

(Both of the above are for the case where the AO accelerates TOWARD the unaccelerated person (her).)

[…]

[Previous]:

The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds, is just

tau [exp(d)] sup A,

because [exp(d)] sup A is the constant RATE at which the HF is ageing, during the acceleration, and “tau" is how long that rate lasts.

[Revised]:

The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau (1 + A d),

because (1 + A d) is the constant RATE at which the HF is ageing, during the acceleration, and “tau" is how long that rate lasts.
[…]

[Previous]:

But we earlier found that A = theta / tau, so we get

tau [exp(d)] sup {theta / tau}

[Revised]:

But we earlier found that A = theta / tau, so we get

tau (1 + [(theta d) / tau] = tau + (theta d)

[…]

It is still true that d = 7.52 lightseconds and theta = 1.317.

Therefore the REVISED result is that the change in HF's age during the acceleration is equal to

tau + (theta d) = tau + (1.317)(7.52) = tau + 9.904.

So, in the revised model, and when the acceleration is directed TOWARD the home twin, then as tau approaches zero (to give an instantaneous velocity change), the change in the HF's age during the speed change approaches 9.904 seconds from above. So the HF's age increased by a FINITE amount, unlike the INFINITE increase that the EPVGTD equation gave.

Before the instantaneous velocity change, the AO, the HF, and the home twin (she) were all the same age. She and the HF were co-located. So after the instantaneous speed change, the AO hasn't aged at all, but the HF is 9.904 seconds older than he was before the speed change, according to the AO. And since she and the HF have been colocated during the instantaneous speed change, they couldn't have ever differed in age during the speed change ... it would be absurd for either of them to see the other have an age different from their own age at any instant. So after the instantaneous speed change, the AO must conclude that she and the HF both instantaneously got 9.904 seconds older than they were immediately before the speed change.

By comparison, the CMIF simultaneity method says that the AO will conclude that her age instantaneously increases by 6.51 seconds, so the LGTD and CMIF don't agree quantitatively, but are similar qualitatively.

Section 5. LGTD, When the Direction of the Velocity Change is AWAY FROM Her

I just repeated my previous analysis of instantaneous velocity changes in the "linearized" (LGTD) version of the equivalence principle version of the gravitational time dilation equation, but for the case where the instantaneous velocity change is AWAY FROM the home twin (her). The result is exactly like the previous result, except that she instantaneously gets YOUNGER, not older. This contrasts with my previous possible proof (in Section 2) that negative ageing doesn't occur. And it also contrasts with the result of the EPVGTD equation, where the instantaneous velocity change is AWAY FROM her, that said her age doesn't change at all.

Below, I'll repeat the previous calculations, and show the changes.

[Previous]: (where the instantaneous velocity change is TOWARD her)

I simply replace the exponential exp(A d) with the quantity (1 + A d).

[New]: (where the instantaneous velocity change is AWAY FROM her)

I simply replace the exponential exp(-A d) with the quantity (1 - A d).

[Previous]: (where the instantaneous velocity change is TOWARD her)

The "LGTD" equation says that the acceleration "A" will cause the HF to age FASTER than the AO by the factor (1 + A d), where d is the constant separation between the AO and the HF.

(The above is for the case where the AO accelerates TOWARD the unaccelerated person (her).)

[New]: (where the instantaneous velocity change is AWAY FROM her)

The "LGTD" equation says that the acceleration "A" will cause the HF to age SLOWER than the AO by the factor (1 - A d), where d is the constant separation between the AO and the HF.

(The above is for the case where the AO accelerates AWAY FROM the unaccelerated person (her).)

[…]

[Previous]: (where the instantaneous velocity change is TOWARD her)

The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau (1 + A d),

because (1 + A d) is the constant rate at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

[New]: (where the instantaneous velocity change is AWAY FROM her)

The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau (1 - A d),

because (1 - A d) is the constant rate at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

[…]

[Previous]: (where the instantaneous velocity change is TOWARD her)

But we earlier found that A = theta / tau, so we get

tau (1 + [(theta d) / tau]) = tau + (theta d)

[New]: (where the instantaneous velocity change is AWAY FROM her)

But we earlier found that A = theta / tau, so we get

tau (1 - [(theta d) / tau]) = tau - (theta d)

[both Previous and New]:

It is still true that d = 7.52 lightseconds and theta = 1.317.

[Previous]: (where the instantaneous velocity change is TOWARD her)

Therefore the revised result is that the change in HF's age during the acceleration is equal to

tau + (theta d) = tau + (1.317)(7.52) = tau + 9.904.

[New]: (where the instantaneous velocity change is AWAY FROM her)

Therefore the revised result is that the change in HF's age during the acceleration is equal to

tau - (theta d) = tau - (1.317)(7.52) = tau - 9.904.

[…]

[Previous]: (where the instantaneous velocity change is TOWARD her)

So, in the revised model, as tau approaches zero (to give an instantaneous velocity change), the change in the HF's age during the speed change approaches 9.904 seconds from above. So with an instantaneous velocity change, the HF's age INCREASED instantaneously by a finite amount.

[New]: (where the instantaneous velocity change is AWAY FROM her)

So, in the revised model, as tau approaches zero (to give an instantaneous velocity change), the change in the HF's age during the speed change approaches -9.904 seconds from above.

So with an instantaneous velocity change, the HF's age DECREASED instantaneously by a finite amount.

[…]

[Previous]: (where the instantaneous velocity change is TOWARD her)

Before the instantaneous velocity change, the AO, the HF, and the home twin (she) were all the same age. She and the HF were co-located. So after the instantaneous speed change, the AO hasn't aged at all, but the HF is 9.904 seconds OLDER than he was before the speed change, according to the AO. And since she and the HF have been colocated during the instantaneous speed change, they couldn't have ever differed in age during the speed change ... it would be absurd for either of them to see the other have an age different from their own age at any instant. So after the instantaneous speed change, the AO must conclude that she and the HF both instantaneously got 9.904 seconds OLDER than they were immediately before the speed change.

By comparison, the CMIF simultaneity method says that the AO will conclude that her age instantaneously increases by 6.51 seconds, so the LGTD and CMIF don't agree.

[New]: (where the instantaneous velocity change is AWAY FROM her)

Before the instantaneous velocity change, the AO, the HF, and the home twin (she) were all the same age. She and the HF were co-located. So after the instantaneous speed change, the AO hasn't aged at all, but the HF is 9.904 seconds YOUNGER than he was before the speed change, according to the AO. And since she and the HF have been colocated during the instantaneous speed change, they couldn't have ever differed in age during the speed change ... it would be absurd for either of them to see the other have an age different from their own age at any instant. So after the instantaneous speed change, the AO must conclude that she and the HF both instantaneously got 9.904 seconds YOUNGER than they were immediately before the speed change.

By comparison, the CMIF simultaneity method says that the AO will conclude that her age instantaneously decreases by 6.51 seconds, so again the LGTD and CMIF don't agree quantitatively, but are similar qualitatively.

Section 6. What to Make of All These Different and Contradictory Results?

The "EPVGTD Equation" (the one with the exponential), says that, if the AO (he) instantaneously changes his velocity in the direction TOWARD the home time (her), she instantaneously gets INFINITELY older, according to him. That's nonsense, because it gives incorrect ages for the twins when they are reunited in the twin ‘paradox’. The twin ‘paradox’ outcome at the reunion is based ONLY on the time dilation equation (TDE) for perpetually-inertial observers, which is one of the most trusted equations in special relativity. The TDE is sacrosanct. So the EPVGTD equation must be rejected.

What about the LGTD equation? The linearized equation (the LGTD equation) gives results that are qualitatively similar to the CMIF simultaneity method: her age instantaneously changes, according to him, during his instantaneous velocity change (instantaneously INCREASING when his momentarily infinite acceleration is TOWARD her, and instantaneously DECREASING when his momentarily infinite acceleration is AWAY FROM her). But the AMOUNT of her instantaneous age change is greater than CMIF says it should be, and that amount (given by the LGTD equation) is inconsistent with the required ages of the twins at the reunion, which are based on the sacrosanct time dilation equation (TDE) for perpetually-inertial observers. So the LGTD equation is incorrect and must be rejected.

It is interesting that the amount of the instantaneous age changes would be exactly the same for CMIF and LGTD if the linearized equation multiplied the distance "d" by the velocity "v", rather than by the rapidity "theta". But, in determining the velocity effect obtained by integrating the acceleration "A", it IS necessary to use the rapidity "theta", not the velocity "v", as the variable of integration. (Taylor and Wheeler go over this in detail in their book “Spacetime Physics”).

WHY does the EPVGTD equation fail so miserably in this example? Isn't the GTD equation a well-established result in general relativity? And the equivalence principle is certainly well-established. Is the GTD equation WRONG? Or have I made a mistake somewhere?

And WHY goes the LGTD equation work better than the EPVGTD equation, at least qualitatively? The LGTD equation should be a justified approximation of the EPVGTD equation ONLY when the argument (A d) is small, and an infinite "A" (even though it lasts only an infinitesimal time) certainly isn't small! The LGTD equation shouldn't give results that are even qualitatively correct, but it does. Why?

Section 7. Modifying the LGTD Equation So That It Agrees with CMIF

The LGTD equation can be modified so that it agrees with the CMIF simultaneity method. I’ll show the case where the AO accelerates TOWARD the unaccelerated person (her).

[Original LGTD Equation]:

The "LGTD" equation says that the acceleration “A” will cause the HF to age faster than the AO by the factor (1 + A d), where d is the constant separation between the AO and the HF.

[Modified LGTD Equation]

The “Modified LGTD" equation says that the acceleration “A” will cause the HF to age faster than the AO by the factor (1 + alpha A d), where d is the constant separation between the AO and the HF, and alpha = v / theta = [tanh(theta)] / theta. So the HF ages faster than the AO by the factor

1 + [d A tanh(theta) / theta].

[…]

[Original LGTD Equation]:

The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau (1 + A d),

because (1 + A d) is the constant rate at which the HF is ageing, during the acceleration, and “tau" is how long that rate lasts.

[Modified LGTD Equation]:

The CHANGE in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau { 1 + [d A tanh(theta) / theta] },

because 1 + [d A tanh(theta) / theta] is the constant rate at which the HF is ageing, during the acceleration, and “tau" is how long that rate lasts.

[…]

[Original LGTD Equation]:

But we earlier found that A = theta / tau, so we get

tau (1 + [(theta d) / tau] = tau + (theta d)

for the CHANGE in the age of the HF.

[Modified LGTD Equation]:

But we earlier found that A = theta / tau, so we get

tau { 1 + [d tanh(theta) / tau] } = tau + [d tanh(theta)]

for the CHANGE in the age of the HF.

[…]

It is still true that d = 7.52 lightseconds and theta = 1.317.

[Original LGTD Equation]:

Therefore the original LGTD result is that the change in HF's age during the acceleration is equal to

tau + (theta d) = tau + (1.317)(7.52) = tau + 9.904.

So, according to the original LGTD equation, when the acceleration is directed TOWARD the home twin (her), then as tau approaches zero (to give an instantaneous velocity change), the change in the HF's age (and her age) during the speed change approaches 9.904 seconds from above. The CMIF method says that her age changes by 6.51 seconds.

[Modified LGTD Equation]:

Since tanh(theta) is just the velocity “v”, and since we chose v = 0.866 at the outset of this example because it gives the nice value of 2.0 for gamma, therefore the revised result is that the change in HF's age during the acceleration is equal to tau + [d tanh(theta)] = tau + (7.52)(0.866) = tau + 6.51,

and as tau goes to zero, the change in HF's age (and her age) during the instantaneous velocity change is just 6.51 years, which is the same result that CMIF gets, as we intended.

If we hadn't already known what "v" is in this example, but had started only by choosing theta, we could have computed "v" using the identity

tanh(theta) = { [exp(theta) - exp(-theta)] / [exp(theta) + exp(-theta)] .

The derivation of the case where the AO accelerates AWAY FROM the unaccelerated person (her) is essentially the same. Theta and "v" are just negative in that case, instead of positive, and so the HF and she instantaneously get 6.15 years YOUNGER rather than OLDER.

Section 8. Some Miscellaneous Background Information

Subsection 8.1 Disagreements About Simultaneity at a Distance

Special Relativity is widely considered to be a completed discipline … a "done deal". It's been more than a hundred years since Einstein presented it to us in 1905. Simultaneity at a distance, for a perpetually-inertial observer, isn't in dispute. Specifically, the question "How old is that distant person, RIGHT NOW", when asked by a perpetually-inertial observer, is never in dispute (even in the case where the distant person is NOT perpetually inertial).
We can call the perpetually-inertial observer "the home twin", and refer to her as "she". We can refer to "the traveler" who may sometimes accelerate, as "he" or "him". At any instant, he has a velocity relative to her of "v" lightyears/year. The quantity "gamma" depends only on "v", and has the value

gamma = 1 / sqrt { 1 - (v v) },

and according to her, at any instant in her life, he is aging slower than she is, by the factor gamma. For example, if their relative velocity is v = 0.866 ly/y, gamma is equal to 2.0. If he is continually changing his velocity (i.e., continually accelerating), gamma will be changing continually, and so she will conclude that his rate of aging is continually changing. So she will have to integrate that changing rate to compute his current age. But a much easier situation is when he just changes their relative velocity instantaneously, which keeps his rate of aging (compared to hers) constant between his instantaneous velocity changes.

For example, in the standard twin paradox, immediately after they are born, he changes his velocity with respect to her from zero to 0.866 ly/y, and maintains that velocity until he is ready to do his turnaround. At the turnaround, he instantaneously changes his velocity to -0.866 ly/y, and is then heading back toward her. The factor gamma doesn't depend on the direction or the sign of the velocity, so gamma = 2.0 for the entire trip. So she concludes that he is aging half as fast as she is, during the entire trip. Therefore she knows that he will be half as old as she is when he returns at the reunion. If she is 80 years old when they are reunited, he must be 40 years old then. She is just making use of the Time Dilation Equation (TDE) for a perpetually-inertial observer, which is the "gold standard" in special relativity.

Since they are co-located at the reunion, they MUST agree about their respective ages at the reunion. But what is HIS conclusion about how their ages compare during the parts of the trip when they are NOT co-located? I.e., what is HIS answer to the general question, "How old is she (that distant person) right now", at each instant of his life during the trip? He can't just use the time dilation equation (the TDE) during his entire trip (like SHE was able to do), because he is NOT perpetually inertial like she is.

There is disagreement among physicists about the answer to that question. As far as I know, Einstein never addressed that question. Some physicists believe that simultaneity at a distance, according to an accelerating observer, is a meaningless concept, and the question shouldn't even be asked. Some others think that any particular observer is free to choose from among an infinite number of possible answers to that question. I.e., some think simultaneity at a distance should just be regarded as a convention that can be chosen on a whim.

Probably the most popular simultaneity method is to specify that the traveling twin (he) should, at each instant of his life, always agree with the answer given, about the home twin's (her) current age, by the perpetually-inertial observer who is co-located and co-stationary with him at that instant. That method is usually called "the co-moving inertial frames", or "CMIF" method. In the case of the twin paradox scenario, the CMIF method says that on the outbound and inbound legs, he says she ages slower than him by the factor gamma, but that at the turnaround, when he instantaneously changes his velocity in the direction TOWARD her, he says she instantaneously gets OLDER by an amount that is just large enough so that he will agree with her about their respective ages at the reunion. The CMIF method also says that, if he instantaneously changes his velocity in the direction AWAY FROM her, he says she instantaneously gets YOUNGER. The amount of that instantaneous ageing, either positive or negative, is fairly easy to determine. One can draw a Minkowski diagram, with the two (straight) lines of simultaneity (LOS's) of slope 1/v shown, corresponding to the two different perpetually-inertial observers, one immediately BEFORE and one immediately AFTER the velocity change. Where each of those LOS's cross her worldline gives her age at the turnaround, according to each of the two perpetually-inertial observers. A line of simultaneity (LOS) is just the "RIGHT NOW" line for a perpetually-inertial observer (PIO).

Here is some more detail about how to draw that Minkowski diagram. I prefer to draw her worldline as the horizontal axis, and the distance "X" of objects from her, according to her, on the vertical axis. And so each point "T" on the horizontal axis corresponds to some instant in her life. When both twins are born, they are each located at the origin of the diagram (where the two axes intersect on the left side of the diagram). Immediately after they are born, he instantaneously changes his velocity, with respect to her, from zero to 0.866 ly/y. His distance from her (according to her) then increases linearly according to the equation

$$X = v\,T = 0.866\,T.$$

So, on the outbound leg, HIS worldine is a straight line starting from the origin and sloping upward to the right with a slope of 0.866. That straight line continues until the turnaround point … let's say she says she is 40 years old then. So draw a vertical line that starts at the point T = 40 on the horizontal axis, and extends upward until it intersects his worldline. His distance from her at the turn point, according to her, is (0.866 40) = 34.64, so mark and write that distance on the vertical axis. Using the time dilation equation (TDE) for a perpetually-inertial observer, she knows that he is 20 years old at the turnaround, so label that point on his worldline as 20.

Now, we want to determine the his line of simultaneity (LOS) that passes through his worldline immediately before he reverses his velocity. (That is the LOS of the perpetually-inertial observer (PIO) who is co-located and mutually stationary with him at that instant. That line has a slope of 1/v, or 1/0.866, or 1.155, sloping downward to the left. That LOS forms the hypotenuse of a right triangle, with a vertical side of length 34.64, and with a horizontal base side, extending to the LEFT of the vertical side, whose length we need to determine. The height of the triangle

(34.64), divided by the length “L” of the base of the triangle equals the slope 1.155 of the hypotenuse, so we have

$$34.64 / L = 1.155,$$

or

$$L = 34.64 / 1.155 = 30.$$

So he says her age immediately before he turns around is $40 - L = 40 - 30 = 10$ years old.

Note that, in this outbound case, we could have gotten that result immediately from the time dilation equation for a perpetually-inertial observer, because on the outbound leg, he can be considered to be an inertial observer (until he changes his velocity). (If there is any doubt about that, we can say that he and she aren’t really twins. Their respective mothers are perpetually inertial, and they just happened to be momentarily co-located when their babies were born. They have always had a relative velocity of 0.866. So in that case he never has accelerated before, and he is certainly entitled to use the time delay equation.) We already have determined that he is 20 years at the turnaround, and according to him, she has been ageing half as fast as he has on the outbound leg, so he says she must be 10 years old when he is 20 years old, immediately before the turnaround. But, nevertheless, it was important to show how he determines her age from his line of simultaneity (LOS).

Next, we need to use the same process to determine how old she is, according to him, immediately AFTER he changes his velocity to -0.866 ly/y. His new line of simultaneity (which is the LOS of the PIO he is NOW co-located with and co-stationary with) forms the hypotenuse of a right triangle, with a vertical side of length 34.64, and with a horizontal base side, extending to the RIGHT of the vertical side, whose length we need to determine. That length is again equal to 30, so now he says that her age immediately after he turns around is $40 + L = 40 + 30 = 70$ years old. So he says she instantaneously got 60 years older when he instantaneously changed his velocity from +0.866 to -0.866 (from going AWAY FROM her to going TOWARD her).

Note that he COULD, if he wanted, immediately decide to switch his velocity back to +0.866 from -0.866 (from going TOWARD her to going AWAY FROM her). If he did that, he would conclude that she instantaneously gets 60 years YOUNGER, from 70 years old to 10 years old. Such ”back-to-back” equal velocity changes (with no finite time between them) are equivalent to no velocity change at all … the velocity changes cancel each other out. For that reason, if she can instantaneously get older (according to him), it must also be possible that she can instantaneously get younger (according to him). Otherwise, a long series of back-to-back instantaneous velocity changes could make her age (according to him) be arbitrarily large, which would be inconsistent with her certain knowledge of his and her ages at the reunion.

The above back-to-back instantaneous velocity changes are of course not the only scenario where she gets younger, according to him. Whenever there is a finite amount of time between the two opposite velocity changes, the effects don’t cancel out. And it is generally true that anytime they are separated and he does a Dirac delta function acceleration (producing an instantaneous velocity change) in the direction TOWARD her, she will instantaneously get YOUNGER.

Besides the Minkowski diagram described above, there is another diagram (that I call the “Age Correspondence Diagram”) that is even more important. It (the “ACD”) basically graphically shows what the answer is to the question: “For each instant in the life of the accelerating observer (him), what is the current age of the distant person (her)?” For each instant in his life, it plots her corresponding current age, according to him. For example, in the well-known twin paradox scenario where the twins are colocated when they are born, and he immediately instantaneously changes his velocity from zero to 0.866 ly/y (so that his outbound velocity is 0.866 ly/y) and where he does an instantaneous turnaround when he is 20 years old, heading back to her at -0.866 ly/y, the ACD plot starts out at the origin (both twins aged zero), and then rises linearly to the right with a slope of 0.5. That represents the fact that he says she ages gamma times slower than he does on the outbound leg, and gamma equals 2.0. So it says she is 10 years when he is 20 years old, immediately before he changes his velocity to -0.866. Then, at his instantaneous velocity change, the plot goes straight up vertically by 60 years … indicating that she instantaneously gets 60 years older during his velocity change. So at the end of that vertical increase in her age, she is 70 years old, according to him. On the inbound leg, he again says that she is ageing half as fast as he is (because gamma is equal to 2.0). He ages by 20 years on the inbound leg, so he says she ages by10 years. So at the reunion, he is 40 and she is 80. So the last segment of the ACD plot slopes upward to the right with slope 0.5, and her age increases from 70 to 80 years old.

Some physicists object to that instantaneous ageing in CMIF simultaneity. And the negative ageing in CMIF simultaneity is even MORE abhorrent to a lot of physicists. (Some physicists even DISALLOW negative ageing, while allowing positive ageing, but that is inconsistent with the requirement that back-to-back velocity changes must cancel each other out.) But other physicists have no problem with negative ageing. A prime example of that latter group is Brian Greene, who in his NOVA show "The Fabric of the Cosmos" discusses it very clearly and enthusiastically:

(https://www.pbs.org/wgbh/nova/video/the-fabric-of-the-cosmos-the-illusion-of-time/

(scan forward to the 23:15 point).

(Brian also gives the same example in his book of the same title.)

Besides the CMIF simultaneity method, there are at least three other simultaneity methods that have been proposed. One is the Dolby and Gull "Radar" method (arXiv:gr-qc/0104077), another is the Minguizzi method (arXiv:physics/0411233v1), and a third is my method (http://viXra.org/abs/2109.0076). None of those three methods produce any discontinuities in her age, according to him. So that means there are no vertical rises or vertical drops in the ACD for any of these three simultaneity methods. But both the Dolby and Gull method, and Minguizzi's method, are non-causal: they have an effect on her age (according to him) well BEFORE he decides to change his velocity! In my opinion, that is a disqualifier for a simultaneity method. So the only simultaneity methods that I know of that are causal are the CMIF method and my method. Note: Even though my simultaneity method has no discontinuities in her age, I actually PREFER the CMIF method (because of its simplicity, and because I don't have a problem with instantaneous negative or positive ageing). But I don't know which (if either) method is correct.

The ACD for MY simultaneity method, for the scenario I gave above when I described the ACD for the CMIF method, is similar to the CMIF ACD. The plot for the outbound leg is the same for both methods. But in my method, there is no vertical rise in the plot at the turnaround. Instead, in my method, the plot rises linearly from the turnaround point with a steep but finite slope, until it intersects the final section of the diagram where the slope is equal to 0.5 as in the CMIF method. The slope of that steep section can be determined either with a fairly simple equation, or even easier by graphical means. Those details can be found either in my Amazon monograph ("A New Simultaneity Method for Accelerated Observers in Special Relativity", which you can search for on Amazon under my name, Michael Leon Fontenot), or else on viXra at https://vixra.org/abs/2106.0133 .

Subsection 8.2 The CADO Equation

Instead of plotting lines of simultaneity (LOS's), an easier and quicker way to answer the question "How old is that distant person, right now, according to an observer who sometimes accelerates", is to use the "CADO Equation". And even quicker is the "Delta CADO Equation". ("CADO" is just an abbreviation for the "Current Age of a Distant Object".)

For example, instead of using the LOS's of the two perpetually-inertial observers (PIO's) immediately before and immediately after the turnaround as we did above, we can just do this:

First, we need to know their separation "D", according to her, when he instantaneously changes his velocity: D = 34.64 ly.

Then, we need to know what the instantaneous CHANGE in his velocity is:

delta_v = v2 - v1 = (-0.866) - (0.866) = -1.732 ly/y.

Then the instantaneous change in her age, denoted "delta_CADO" is

delta_CADO = (-D) (delta_v) = (-34.64) (-1.732) = 60.0,

so the delta_CADO equation says that she instantaneously gets 60 years older (according to him) when he instantaneously changes his velocity from 0.866 ly/y to -0.866 ly/y. VERY EASY!

The CADO reference frame is the same as the CMIF reference frame. It just uses some new terminology that is designed to reduce errors that are commonly made in working with special relativity. And it also makes use of the very useful (and not well-known) CADO and delta_CADO equations. I first derived the CADO equation in a paper I published more than 20 years ago:

Fontenot, Michael L., "Accelerated Observers in Special Relativity", Physics Essays, December 1999, pp. 629-648.

Below, I've excerpted the first few sections of my webpage:

https://sites.google.com/site/cadoequation/cado-reference-frame

which is a followup of my paper. And it is also available on viXra at

https://vixra.org/pdf/2106.0122v1.pdf

The CADO Reference Frame for an Accelerating Observer

The CADO reference frame[1] is defined for an observer who accelerates in any manner whatsoever. Specifically, the observer's acceleration *a(t)*, where *t* is any instant in the observer's life, can be whatever the accelerating observer wants it to be, without restriction.

The CADO Frame, for the Standard Twin Paradox Scenario

Although the CADO frame is applicable to *any* acceleration profile, the concepts and terminology needed to describe the CADO reference frame are most quickly and easily understood if they are initially couched in the context of the standard well-known twin "paradox" scenario.
First, consider the even simpler scenario where two perpetually-inertial observers are moving at some fixed velocity *v* relative to one another, and when they momentarily are co-located, they just happen to be exactly the same age then. For example, it could just happen that they are both born at that instant of their co-location, even though their mothers could have had a relative velocity of *v* at that instant. Since each of those newborns is a perpetually-inertial observer, they are each entitled to use the Lorentz equations to determine, at any instant of their own life, the current age of the other.[2] And each of them is entitled to use the well-known time-dilation result of special relativity[3] to determine how fast or how slowly the other is currently ageing, relative to their own ageing.

In the standard twin paradox, the "home twin" is perpetually inertial by assumption, and thus is entitled to use either the Lorentz equations or the time-dilation result (or both) to determine the current age of the "traveling twin". To allow more brevity and less clutter in the writing which follows, the home-twin will always be referred to as a "she", and the traveling twin will always be referred to as a "he".

The traveling twin must accelerate, in order to accomplish his turnaround, so he is *not* a perpetually-inertial observer, and his reference frame during his trip *cannot* be an inertial frame. Specifically, he is *not* allowed, during his *entire* trip, to use the time-dilation result to determine the current age of his twin. And, depending on exactly how his reference frame is defined, he *might* or *might not* be allowed to use the Lorentz equations at each instant of his life during his trip.

So what *is* the reference frame of the traveling twin? There are five requirements that any such frame must have.

1. It must be such that the traveler is perpetually located at its spatial origin.
2. It must specify how the traveler, at each instant of his life, is to determine the current age and the current position of each and every object (or person) in the (assumed flat) universe.
3. It must be internally consistent.
4. It must not contradict special relativity.
5. It must be such that the traveler and the home-twin agree with one another about the correspondence between their ages, when they are reunited.

More than one reference frame for an accelerating observer have been defined, and there is not yet a consensus about which one is most appropriate. This article describes one such reference frame: the CADO frame.

The CADO frame was originally inspired by an example (Example 49) in Taylor's and Wheeler's classic book.[4] The results of their example are consistent with those obtained from the common gravitational time dilation explanation, but do not depend on the use of any fictitious gravitational fields. Their basic approach is clearly applicable to scenarios with finite accelerations, although they didn't pursue that generalization. The CADO frame accomplishes that generalization.

Even though the frame of the traveling twin, since he accelerates during some portion his trip, cannot be an inertial frame, there is, at each instant *t*of the traveler's life, a unique inertial frame which is momentarily stationary with respect to the traveler at that instant, with a spatial axis pointing in the same direction as the home-twin's spatial axis, and such that the traveler is located at the spatial origin of that frame at that instant. Furthermore, for uniqueness, we require that the time coordinate of that inertial frame be equal to the traveler's age, at that instant. That unique inertial frame is called the "momentarily stationary inertial reference frame, at the instant *t* in the traveler's life", abbreviated as the MSIRF*(t)*. In general, MSIRF*(t)* will correspond to a *different* inertial frame from one instant in the traveler's life to the next. It is only during unaccelerated segments of the traveler's life that the MSIRF*(t)* will consist of the *same* inertial frame for the entire segment.

Given this (generally infinite) collection of inertial frames, the CADO frame is defined to be the single unique frame having the property that its conclusions about the current age and location of all objects or persons in the (assumed flat) universe, at any instant t of the traveler's life, is the same as the corresponding conclusions of the MSIRF(t). I.e., at each instant of his life, the traveler adopts the viewpoint (about the simultaneity and location of distant objects) of the inertial frame with which he is momentarily stationary at that instant. The acronym "CADO" originates from the phrase "the current age of a distant object".

The CADO Equation

Given the above definition of the CADO frame, it is possible to derive a very simple, and very useful equation, called "the CADO equation",[5] which allows the traveler to determine, at each instant t in his life, the current age of any given distant perpetually-inertial object or person (the "home-twin" in the twin paradox scenario).

First, it is important to understand that, for any given instant t in the traveler's life, the home-twin and the traveler will generally *disagree* with one another about how old the home-twin is at that instant of the traveler's life. There are two quantities in the CADO equation which represent each of the twins' conclusions about the home-twin's age when the traveler's age is t. The quantity *CADO_T* denotes the *traveler's conclusion* about the home-twin's age, when the traveler's age is t, whereas the quantity *CADO_H* denotes the *home-twin's conclusion* about the home-twin's age, when the traveler's age is t.

The CADO equation can be written (most simply) as

$$CADO\_T = CADO\_H - v * L$$

where

v is their current relative speed, according to the home-twin, at the given instant t in the traveler's life, with v taken as positive when the twins are moving apart,

L is the distance from the home-twin to the traveler, at the given instant t in the traveler's life, according to the home twin,

and

the asterisk denotes multiplication.

Strictly speaking, the quantity $L(t)$ is the position of the traveler, relative to the home-twin, according to the home-twin, when the traveler's age is t. The distinction will be clarified later (in Section 5), but for now, it's simplest to just think of it as a distance (a number either positive or zero).

The above equation gives the relationship between those four quantities (*CADO_T*, *CADO_H*, v, and L), at the given instant t of the traveler's life. I.e., although it is not shown explicitly, each of the four quantities in the equation are functions of t.

What makes the CADO equation especially useful is that it allows the quantity *CADO_T*, which is a quantity which is otherwise relatively difficult to determine, to be easily calculated from the other three quantities (*CADO_H*, L, and v), which are each very easy to determine.

In order to make the equation strictly correct, a factor of $c*c$ dividing the last term is required, where c is the constant speed of any light pulse, as determined by any perpetually-inertial observer. If the time and spatial units are chosen so that c has unity value, the factor in that case is required only for dimensional correctness. In this article, units of years and lightyears will be used exclusively (but often abbreviated as y and ly), and the factor of $c*c$ will be suppressed entirely, purely for simplicity and brevity.

It can be immediately seen from the CADO equation that if at any instant , either v or L is zero, then CADO_T is equal to CADO_H. I.e., if, at any instant in the traveler's life, he is stationary with respect to his twin, then he will agree with her about their respective ages, regardless of how far apart they are. Or, if at any instant they are co-located, they will agree about their respective ages, regardless of whether or not they have any relative motion at that instant. And it is equally clear from the CADO equation that at any instant t when v and L are non-zero, the two twins will *not* agree about their respective ages. (This last statement is true for the one-dimensional motion we have been considering so far, but the statement must be modified for motion in two or three spatial dimensions. The higher-dimensional case will be addressed in a later section.)

I originally derived the CADO equation, many years ago, using only the Minkowski diagram for the twin "paradox" scenario. I show explicitly show how to do that derivation near the the end of Section 11, on "The Graphical Interpretation of the CADO Frame".

Idealized Instantaneous Velocity Changes

In the idealized, limiting case of the instantaneous turnaround usually assumed in the twin paradox scenario, the quantities *CADO_H* and L that are needed in the CADO equation are very easy to obtain, and v is given in the statement of the scenario.

For example, suppose that immediately after the twins are born, the traveling twin moves away from the home-twin at a constant relative velocity of 0.866 lightyears/year for 20 years of his life. That complicated-looking value of the velocity was chosen for this example because it produces the very nice value of 2 for the gamma factor (the time-dilation factor):

gamma = 1 / sqrt(1 - v * v) ,

where "sqrt( )" denotes the square-root operation, and where, again for simplicity, the factor $c*c$ that should actually be dividing the $v*v$ term has been omitted.

The traveler then instantaneously reverses course, and spends the next 20 years of his life returning to his home-twin. The magnitude of his velocity is still 0.866 ly/y, but since he is now moving *toward* his twin, by convention his velocity is now negative, -0.866 ly/y. Since *gamma* depends only on the magnitude of the velocity, *gamma* is still equal to 2.

So, the traveler is 20 years old at his turnaround, and 40 years old when he is reunited with his twin. Since the home-twin is perpetually inertial, she is entitled to use the time dilation result for his entire trip. Since *gamma* = 2 for the entire trip, she concludes that the traveler ages half as fast as she herself does, so she concludes that she is 40 years old when he turns around, and 80 years old when they are reunited. (Of course, when they are reunited, they will each *know* both of their ages). So, just from the time-dilation result, we've been able to quickly determine that

CADO_H(20) = 40 years old.

Now, from the definition of the CADO frame, the MSIRF*(t)* for all t from 0 years up to, but not including, 20 years, is the *same* inertial frame ... it's the one which is moving at a velocity relative to the home-twin of 0.866 ly/y, and in which the traveler is located at the spatial origin. During that entire segment, $0 <= t < 20$, the traveler (by definition) agrees with that single MSIRF about the age of any distant inertial object or person, and thus he also agrees with that MSIRF about how fast or how slowly any distant person is ageing, compared to his own ageing. So, during that outbound leg (but not including the instant at $t = 20$), the traveler is entitled to use the time-dilation result, and he concludes that the home-twin is ageing half as fast as he himself is. So he concludes that, right at the end of his constant-velocity outbound leg (but before he does his instantaneous turnaround), that the home-twin is 10 years old. Therefore we've been able to determine that

CADO_T(immediately before turnaround) = 10 years old.

The fact that the traveler is entitled to use the time-dilation result, during his entire unaccelerated outbound segment, is also true of *any* unaccelerated segment, of finite duration, in his life. During any unaccelerated finite segment of his life, he is a *full-fledged* inertial observer during that entire segment, and he is entitled to use the Lorentz equations to determine simultaneity at a distance, and he is entitled to use the time-dilation and length-contraction results that follow from the Lorentz equations.

So, for the entire outbound leg, we didn't need to use the CADO equation at all ... the time-dilation result was all that we needed. But we *do* need the CADO equation in order to determine what happens during the turnaround, right at the instant $t = 20$ years. How do we do that?

To make use of the CADO equation during the turnaround, we need to know the values of the three quantities on the right-hand-side of the CADO equation (*CADO_H*, v, and L), immediately before and immediately after the instantaneous turnaround. *CADO_H* and L are quantities that are computed in the home-twin's inertial frame, and they are always continuous ... they never change discontinuously, even when v changes discontinuously. So *CADO_H* and L don't change during the turnaround, but v does change.

We can denote the instant in the traveler's life, immediately *before* the turnaround, as $t = 20$-, and the instant immediately *after* the turnaround as $t = 20$+. So, we have

v(20-) = 0.866 ly/y,

and

v(20+) = -0.866 ly/y.

We also already know that

CADO_H(20-) = *CADO_H*(20+) = *CADO_H*(20) = 40 years.

So all we still need to determine is L(20). How do we do that? We know that, in the home-twin's frame, the velocity of the traveler is 0.866 ly/y during the outbound frame, and we know that that outbound leg lasts for 40 years of the home-twin's life, so she will conclude that the traveler's distance from her at the turnaround is

$L = 0.866 * 40 = 34.64$ ly.

Since, in the CADO equation, all of the quantities need to be specified as functions of the variable t (the traveler's age), we therefore have

L(20-) = L(20+) = L(20) = 34.64 ly.

So, we've got all the quantities we need, to evaluate *CADO_T*(20-) and *CADO_T*(20+) using the CADO equation. We actually were already able to determine *CADO_T*(20-) using only the time-dilation result for the outbound leg ... we got the value 10 years. But it is instructive to use the CADO equation for the instants immediately before and immediately after the instantaneous turnaround, just to understand why the CADO frame concludes that the home-twin's age abruptly changes during the instantaneous turnaround. Immediately before the turnaround, we get

$CADO\_T(20-) = CADO\_H(20-) - v(20-) * L(20-) = 40 - 0.866 * 34.64$,

so

$CADO\_T(20-) = 40 - 30 = 10$ years.

And, immediately after the turnaround, we get

$CADO\_T(20+) = CADO\_H(20+) - v(20+) * L(20+) = 40 + 0.866 * 34.64$,

so

$CADO\_T(20+) = 40 + 30 = 70$ years.

So, the CADO equation says that, according to the traveler, the home-twin instantaneously get 60 years older during his instantaneous turnaround. And the CADO equation makes it clear *why* the traveler's abrupt velocity change causes (according to the traveler) the abrupt change in the home-twin's age: by definition, at any instant *t* of the traveler's life, he adopts as his own the conclusions of his MSIRF, at that instant, about simultaneity. The MSIRF at the instant immediately *before* the turnaround, *MSIRF*(20-), and the MSIRF at the instant immediately *after* the turnaround, *MSIRF*(20+), have very different conclusions about the current age and current position of the home-twin.

The change in the home-twin's age, before and after the instantaneous velocity change, is

$delta\_CADO\_T(20) = CADO\_T(20+) - CADO\_T(20-)$,

and since nothing on the right-hand-side of the CADO equation changes during the instantaneous turnaround except the velocity, we get the very simple equation

$delta\_CADO\_T(20) = - L(20) * (v(20+) - v(20-))$

or

$delta\_CADO\_T(20) = - L * \mathrm{delta\_v}(20)$.

So, getting the change in the home-twin's age during an instantaneous velocity change is very simple: you just multiply the negative of their separation by the change in the velocity.

Note that in this case (for the turnaround that occurs in the standard twin paradox scenario), the change in the velocity is *negative*:

$delta\_v(20) = v(20+) - v(20-) = (-0.866) - (0.866) = -1.732$,

and so the change in the home-twin's age is

$delta\_CADO\_T(20) = -34.64 * (-1.732) = 60$ years.

But note that, for other scenarios, the traveler *could* change his velocity from (say) -0.866 ly/y to +0.866 ly/y (corresponding to an acceleration *away from* the home twin), and in that case, his velocity change would be positive (+1.732), and so the home-twin's age change would be -60 years .... i.e., she would suddenly get 60 years *younger* (according to the traveler).

The fact, that the traveler concludes that the home-twin's age changes abruptly whenever he abruptly changes his velocity, certainly has no impact on the home-twin's own perception of the progression of her own age. Lots of additional accelerating observers would generally come to very different conclusions about the way her age changes while they accelerate in various ways, and it is really of no consequence to her *what* they conclude. But no one's conclusions are any more correct than any one else's conclusions. They are *all* correct ... in special relativity, different observers generally just have to agree to disagree.

To complete our application of the CADO frame to the standard twin paradox, we've still got to analyze the inbound leg. The analysis is essentially the same as for the outbound leg. Since the traveler is unaccelerated during the entire inbound leg, the CADO frame says that the traveler is a full-fledged inertial observer during that entire 20-year segment of his life. So he uses the time-dilation result, and concludes that the home-twin ages 10 years during the inbound leg. So, when they are reunited, she is 80 years old, and he is 40 years old. The home-twin and the traveler agree, about the correspondence between their two ages, when they are reunited (as of course they must), even though they generally disagreed about that correspondence, during the trip.

Instead of using the time-dilation result to determine CADO_T when the twins are reunited (as we did above), we can also easily get the answer from the CADO equation: since L is obviously zero when they are reunited, the CADO equation says that CADO_T = CADO_H there (and so the twins agree about their ages there).

Given the above results, it is easy (and very useful) to sketch an "age-correspondence graph" ... a plot of the home-twin's age (according to the traveler) as a function of the traveler's age. I.e, we want a graph, with the home-twin's age plotted vertically, and the traveler's age plotted horizontally. (The following description is most easily understood if the reader roughly sketches the graph as the description proceeds). What does that graph look like?

On the outbound leg, the traveler says that the home-twin's age increases half as fast as his own age. So the curve starts from the origin, and increases linearly along a straight line of slope 1/2, until his (the traveler's) age is 20, and her (the home-twin's) age is 10. At that point, the curve jumps vertically to 70 for her age (with no increase in his age). Finally, the curve increases linearly from there, along a straight line of slope 1/2, until she reaches 80 years old, and he reaches 40 years old. After that, as long as they remain together, they will age at the same rate, but she will always be 40 years older than he is.

The home-twin can do her own age-correspondence graph, again with her age plotted vertically, and his age plotted horizontally. I.e., both graphs show her age as a function of his age; the only difference is that the two graphs show the conclusions of two different observers.

Her graph will be quite different from his graph: hers will consist of a single, straight line of slope 2, because the time-dilation result tells her that, during his entire trip, *he* ages half as fast as *she* does, which means that *she* ages twice as fast as *he* does. But the two different graphs *do* start at the same point (the origin), and they *do* end at the same point (the point where she is 80, and he is 40). But in between those two points, the curves are very different.

In the standard paradox scenario (with a single instantaneous velocity change, and a reunion at the end of the trip), it is actually possible to avoid having to use the CADO equation to determine how the home-twin's age changes during the turnaround. That change can simply be *inferred* by determining the sum of the amount of her ageing (according to him) during the two unaccelerated segments of his life (10 + 10 = 20 years), and then using the fact that her age at the end of the trip *must* be 80 years. So we have to come up with an additional 60 years somewhere, and the turnaround is the *only* place that extra time could have occurred.

But for more complicated scenarios, where the traveler can instantaneously change his velocity *multiple* times during the trip (both positively and negatively), and in cases where there is never any reunion of the twins), then the CADO equation is indispensable in determining how much the distant perpetually-inertial person instantaneously ages (positively or negatively) during the traveler's instantaneous velocity changes. And even in the standard paradox scenario, the use of the CADO equation at the turnaround makes it clear *why* the home-twin's age (according to the traveler) instantaneously increases during the instantaneous turnaround. And the CADO equation also makes it clear why the traveler's *initial* instantaneous velocity change (when he begins his trip), and his *final* instantaneous velocity change (when they are reunited), does *not* cause any instantaneous change in her age (because L is zero then).

Subsection 8.3 Philosophical Considerations

I've never been able to adopt the "simultaneity at a distance is meaningless" view, mainly for philosophical reasons (which are supposed to be off-limits in physics, but I think everyone is influenced by philosophical thoughts to some extent). I don't believe that my home twin *ceases to exist* whenever we are separated. If she *does* still exist "right now", she must be *doing something specific* right now. And if she is doing something specific right now, she must be a *specific age* right now (because at each instant of a person's life, their brain at that instant is in a state that is uniquely consistent with their actions at that instant). So I believe she *must* have some specific current age. Her current age is *not* just one of a set of equally good "conventions" of simultaneity, as some physicists believe. Therefore there *must* be a single, correct simultaneity method. Since I believe that Dolby and Gull simultaneity, and Minguizzi simultaneity are both disqualified because they are non-causal, that means that the correct simultaneity is either the CMIF simultaneity method, or my simultaneity method, or perhaps some currently unknown method. I hope some day the correct simultaneity method will be known.

Subsection 8.4 A CADO Cartoon

Shortly after I first came up with the CADO equation (several decades ago), and after I started to realize some of its bizarre implications, I created a cartoon (only in my mind) that captures (in only a slightly exaggerated way) the essence of what makes those implications so shocking.

Imagine that a spaceship left Earth many years ago (maybe 20 years ago or so, in ship time), and that the spaceship (at some local date-and-time on the ship) is currently very far away from Earth (maybe 50 lightyears or so, as measured in the Earth frame). The passengers on that ship still remember well their previous lives on Earth, and they still often think about the people they cared about then (and still very much care about). They naturally would wonder if their loved-ones are still alive, and if they are OK. The passengers would probably often try to imagine, if they can figure out their loved-ones' current ages, what they might be currently doing, "right now".

In my imagined cartoon, the ship is having its annual New Year's Eve party. One of the passengers asks the captain, "What is the date right now, back on Earth?" The captain, with his hand on a HUGE throttle, answers, "What date would you LIKE it to be?".