Hamiltonian Flow of the Riemann $\xi$-Function

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Abstract

The Riemann $\xi$-Function can be expressed as $\xi(s) = u(x, y) + iv(x, y)$ where $s = x + iy$. The structure of a Hamiltonian flow in the critical strip, $0 \leq x \leq 1$, $0 \leq y \leq \infty$ of $\dot{x} = u(x, y)$, $\dot{y} = -v(x, y)$ is determined by its behavior near zeros of $\xi(s)$. Phase portraits are considered and proved that all zeros of the Riemann $\xi$-Function on the critical line are saddle points.

1. Introduction

This paper is dependent on papers from [1-3]. The Riemann Zeta function $\zeta(s)$ is a function of the complex variable $s = x + iy$, defined in the half plane $x > 1$ by the absolutely convergent series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$  \hspace{1cm} (1)

$\zeta(s)$ can be extended by analytical continuation to the whole complex plane, with only a simple pole at $s = 1$ and trivial zeros at the negative even integers that is, when $s$ is one of $-2$, $-4$, $-6$, $-8$ ....... $\zeta(s)$ has an infinity of zeros on the critical line, $x = \frac{1}{2}$. The Riemann hypothesis is stated that all the non-trivial zeros of the Riemann Zeta function must lie on the critical line, $x = \frac{1}{2}$.

In order to eliminate pole at $s = 0, 1$ and all trivial zeros, the $\xi$-function is formulated as

$$\xi(s) = \frac{1}{2} s(s-1) \frac{\Gamma(\frac{s}{2}) \zeta(s)}{\pi^{s/2}} ,$$  \hspace{1cm} (2)

which satisfies the functional equation

$$\xi(s) = \xi(1-s) ,$$  \hspace{1cm} (3)

and has the same zeros as $\zeta(s)$ in the critical strip, $0 < x < 1$. 

\(\xi(s)\) is an entire function with real and imaginary parts \(u(x, y)\) and \(v(x, y)\), thus

\[
\xi(x + iy) = u(x, y) + iv(x, y),
\]

where \(s = x + iy\).

From Eq. (3), relationship of \(u(x, y), v(x, y)\) in the critical strip can be stated as:

\[
\begin{align*}
u(x, y) &= u(1 - x, y), \\
v(x, y) &= -v(1 - x, y).
\end{align*}
\]

(5)

From these symmetries, the following results applying along \(x = \frac{1}{2}\), such as

\[
\begin{align*}
v\left(\frac{1}{2}, y\right) &= 0, \\
\frac{\partial u}{\partial x}(1/2, y) &= 0
\end{align*}
\]

(6)

Since \(\xi(s)\) is an analytical function of \(s\), it satisfies the Cauchy-Riemann equations:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

(7)

2. Phase Portraits of Hamiltonian Systems

The Jacobian matrix of \(\dot{x} = u(x, y), \dot{y} = -v(x, y)\) is defined as

\[
J = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
\]

(8)

Let \(\alpha = \frac{\partial u}{\partial x}\) and \(\beta = \frac{\partial u}{\partial y}\). By using relationship from Eq. (7), \(J\) can be represented as

\[
J = \begin{bmatrix}
\alpha & \beta \\
\beta & -\alpha
\end{bmatrix}
\]

(9)
At zeros of $\zeta(s)$ on the critical line, $\alpha = 0$ and $\beta \neq 0$, then Eigen values of $J$ at zeros of $\zeta(s)$ on the critical line are $\pm \beta$ and its Eigen vectors are $\left[ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]^T$ and $\left[ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]^T$, respectively. Thus zeros of $\zeta(s)$ on the critical line are saddle points as shown in Fig. 1 and Fig. 2 for the first zeros and the second zero at $\rho = \frac{1}{2} + i14.1347$ and $\rho_2 = \frac{1}{2} + i21.0220$, respectively. As shown in [2], the vorticity of Riemann zero on the critical line alternate in sign as one move along it. The first and second Riemann zero has vorticity $-$ and $+$, respectively.

Figure 1. The phase portrait of $\dot{x} = u(x,y)$, $\dot{y} = -v(x,y)$ near $\rho_1 = \frac{1}{2} + i14.1347$
Figure 2. The phase portrait of $\dot{x} = u(x,y)$, $\dot{y} = -v(x,y)$

near $\rho_2 = \frac{1}{2} + i 21.0220$

3. Index Theory of Dynamical Systems and Application to the Critical Strip

Consider a dynamical system in the plane represented by

$$\begin{align*}
\dot{x} &= f(x, y), \\
\dot{y} &= g(x, y)
\end{align*}$$

(10)

Index theory provides global information as compared with local information from linearization about fixed points. To find an index of a closed curve, pick some curve $C$ that does not have a fixed point on it. Let $\emptyset$ be the angle that the flow vector on $C$ make w.r.t x-axis and $[\emptyset]_C$ be a net change in $\emptyset$ over one counterclockwise of $C$ (in radians). Then the index of the closed $C$, $I_C$, defined as

$$I_C = \frac{1}{2\pi} [\emptyset]_C$$

(11)
As shown in [4], the index of a closed curve $C$ encloses a saddle point is -1. By the index theory, the index of a closed curve is additive, that is, when $C$ is sub-divided as

$$C = C_1 + C_2,$$

then

$$I_C = I_{C_1} + I_{C_2} \quad (12)$$

Let consider the Hamiltonian system $\dot{x} = u(x, y), \dot{y} = -v(x, y)$ in the critical strip, $0 \leq x \leq 1, 0 \leq y \leq \infty$. This critical strip can be sub-divided into $R_{i,i+1}$, $i = 1, 2, \ldots \infty$ that index theory can be applied to each subdivision separately.

The first region $R_{1,2}$ is defined as a rectangle with four corners at $(1,0)$, $(1,y_{12})$, $(0,y_{12})$ and $(0,0)$, $\text{Im}(\rho_1) < y_{12} < \text{Im}(\rho_2)$. A path from $(1,y_{12})$ to $(0,y_{12})$ does not pass through any zeros of $\zeta(s)$.

All other regions $R_{i,i+1}$, $i = 2, 3, \ldots$ are defined as a rectangle with four corners at $(1,y_{i-1,i})$, $(1,y_{i,i+1})$, $(0,y_{i,i+1})$, and $(0,y_{i-1,i})$, $\text{Im}(\rho_{i-1}) < y_{i-1,i} < \text{Im}(\rho_i)$ and $\text{Im}(\rho_i) < y_{i,i+1} < \text{Im}(\rho_{i+1})$.

Paths from $(1,y_{i,i+1})$ to $(0,y_{i,i+1})$ and from $(0,y_{i-1,i})$ to $(1,y_{i-1,i})$ do not pass through any zeros of $\zeta(s)$.

Let $C_{i,i+1}$ be a closed path along the perimeter of $R_{i,i+1}$ in the counter clockwise direction, $(1,y_{i-1,i}) \rightarrow (1,y_{i,i+1}) \rightarrow (0,y_{i,i+1}) \rightarrow (0,y_{i-1,i}) \rightarrow (1,y_{i-1,i})$. As shown by [2], angles along $C_{i,i+1}$ from $(1,y_{i-1,i})$ to $(1,y_{i,i+1})$ and along $C_{i,i+1}$ from $(0,y_{i,i+1})$ to $(0,y_{i-1,i})$ rotate in the clockwise direction. With clockwise direction of these angles and condition from Eq. (3), the index of $C_{i,i+1}$ must be -1.

For purposes of illustration, the region $R_{2,3}$ is considered. Let $y_{1,2} = 16$ and $y_{2,3} = 22$. A $C_{2,3}$ is a closed path, $(1,y_{1,2}) \rightarrow (1,y_{2,3}) \rightarrow (0,y_{2,3}) \rightarrow (0,y_{1,2}) \rightarrow (1,y_{1,2})$.

Define $\varphi_1, \varphi_2$ as angles at $(1,y_{1,2})$ and $(1,y_{2,3})$, respectively, one can find that $\varphi_1, \varphi_2$ are 3.041 radians and 0.018 radians, respectively.
A net angle changed from \((1, y_{1,2}) \rightarrow (1, y_{2,3}) = -(\emptyset_1 - \emptyset_2),\)

A net angle changed from \((1, y_{2,3}) \rightarrow (0, y_{2,3}) = -2\emptyset_2,\)

A net angle changed from \((0, y_{2,3}) \rightarrow (0, y_{1,2}) = -(\emptyset_1 - \emptyset_2),\)

A net angle changed from \((0, y_{1,2}) \rightarrow (1, y_{1,2}) = -2(\pi - \emptyset_1).\)

Thus, the angle changed = \(-(\emptyset_1 - \emptyset_2) - 2\emptyset_2 - (\emptyset_1 - \emptyset_2) - 2(\pi - \emptyset_1) = -2\pi.\)

Clearly, a net change of angle is \(-2\pi.\) Thus, the index of \(C_{2,3}\) is \(-1.\)

**Conclusions**

The Hamiltonian flow of \(\dot{x} = u(x, y), \quad \dot{y} = -v(x, y)\) near its critical points is analyzed. Phase portraits are considered and proved that all zeros of the Riemann \(\zeta\)-Function on the critical line are saddle points. Also by sub-divide the critical strip, index theory can be applied to each subdivision separately. Results indicate that the index of a closed curve around each subdivision is \(-1.\)

**References**


