A Characterization of Complete Multipartite Graphs

Elmar Guseinov
elmarguseinov@yahoo.com
15 September 2021

Abstract

$G = (V, E)$ is a complete multipartite graph if and only if $\forall (a, b, c \in V): ab, bc \notin E \rightarrow ac \notin E$.

Keywords

computer science
discrete mathematics
graph theory
multipartite graphs
set theory
In this note, I’ll expand my answer [1] to a question posted on Mathematics Stack Exchange.

**Definition**

Consider a simple graph $G = (V, E)$. Suppose $V$ is a union of pairwise disjoint non-empty sets (parts) indexed by a set $S$:

$$V = \bigcup_{\sigma \in S} V_{\sigma}$$

Further, suppose the following condition holds:

$$\forall (ab \in E)\forall (\sigma_1, \sigma_2 \in S): a \in V_{\sigma_1}, b \in V_{\sigma_2} \rightarrow \sigma_1 \neq \sigma_2 \ (\mathcal{U}_1)$$

Then $G$ is called multipartite. If, moreover, the following condition holds, then $G$ is called a complete multipartite graph:

$$\forall (a, b)\forall (\sigma_1, \sigma_2 \in S): \sigma_1 \neq \sigma_2, a \in V_{\sigma_1}, b \in V_{\sigma_2} \rightarrow ab \in E \ (\mathcal{U}_2)$$

**Theorem**

$G = (V, E)$ is a complete multipartite graph if and only if the following condition holds:

$$\forall (a, b, c \in V): ab, bc \notin E \rightarrow ac \notin E \ (\mathcal{U}_3)$$
Proof

First, suppose \( G \) is a complete multipartite graph with \( a, b, c \in V \). If \( ab, bc \notin E \), then \( \mathfrak{I}_2 \) implies that \( a, b, c \) belong to the same part. Thus \( \mathfrak{I}_1 \) implies \( ac \notin E \). This proves the «only if» part of the claim.

To prove the «if» part, suppose \( \mathfrak{I}_3 \) holds and consider the relation \( \sim \) on \( V \) defined as follows:

\[
\forall (a, b \in V): a \sim b \iff ab \notin E \quad (\mathfrak{I}_4)
\]

The definition of an edge implies that \( \sim \) is symmetric. Moreover, \( \sim \) is reflexive since \( G \) is simple and thus contains no loops. Further, \( \mathfrak{I}_3 \) implies that \( \sim \) is transitive. Thus \( \sim \) is an equivalence relation. Thus, consider the quotient set \( V/\sim \) and distinct classes \( !, " \in V/\sim \). By \( \mathfrak{I}_4 \), we have

\[
\forall (a_1, a_2 \in A): a_1 a_2 \notin E \quad (\mathfrak{I}_5)
\]

\[
\forall (a \in A) \forall (b \in B): ab \in E \quad (\mathfrak{I}_6)
\]

Now, taking elements of \( V/\sim \) as the parts of a complete multipartite graph, we can see that \( \mathfrak{I}_5 \) and \( \mathfrak{I}_6 \) imply \( \mathfrak{I}_1 \) and \( \mathfrak{I}_2 \), respectively. \( \blacksquare \)
References

1. https://math.stackexchange.com/a/4250720/959258