A NESTED INFINITE RADICAL
EXPRESSION FOR ODD NUMBERS

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Abstract

I am going to provide a Nested Infinite Radical Expression for all Odd Numbers greater than 1. Along with that, I will also prove the Proposition mentioned by various methods.

1 Introduction

Srinivasa Ramanujan proposed a nested infinite radical problem in JIMS (Journal of the Indian Mathematical Society). The problem (Q.289) says Find the value of:

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \ldots}}}$$

(1)

Ramanujan gave the solution of the above problem as 3. I have tried to create some similar series like this and later generalized it for all odd numbers greater than 1.

Proposition 1. For all odd numbers (in the form 2k-1) greater than 1 we have

$$2k - 1 = \sqrt{1 + 4\sqrt{1 + (k^2 - k - 1)\sqrt{1 + (k^2 - k)\sqrt{1 + \ldots}}}$$

(2)

Later in the paper, I will also provide some strong proof using the generalization of nested expression given by Srinivasa Ramanujan, along with that I will also show that how any odd numbers can be transformed into a nested infinite expression and vice versa.
2 Generalization

I begin by proving Proposition 1,

$$2k - 1 = \sqrt{1 + 4\sqrt{1 + (k^2 - k - 1)\sqrt{1 + (k^2 - k)\sqrt{1 + ...}}}}$$

**Proof.** From the generalization of (1) given by Srinivasa Ramanujan, i.e.

$$\sqrt{ax + (n + a)^2 + x\sqrt{a(x + n) + (n + a)^2 + (x + n)\sqrt{...}}} = x + n + a \quad (3)$$

if we set a=0, x = $k^2 - k - 1$, and n=1 we get,

$$\sqrt{1 + (k^2 - k - 1)\sqrt{1 + (k^2 - k)\sqrt{1 + ...}}} = k^2 - k$$

After some algebric manipuluation we get,

$$\Rightarrow \sqrt{1 + 4\sqrt{1 + (k^2 - k - 1)\sqrt{1 + (k^2 - k)\sqrt{1 + ...}}}} = \sqrt{1 + 4(k^2 - k)}$$

$$\Rightarrow \sqrt{1 + 4\sqrt{1 + (k^2 - k - 1)\sqrt{1 + (k^2 - k)\sqrt{1 + ...}}}} = \sqrt{(2k - 1)^2}$$

and this completes our proof. Now I will provide some examples which would help us understand the need for the formula (2).

**Example 2.1** Find a nested radical expression for: i) 131 ii) 729

**Solution.** i) To find the required expression we first set

$$2k - 1 = 131 \Rightarrow k = 66$$

Now, using (2) we get,

$$131 = \sqrt{1 + 4\sqrt{1 + 4289\sqrt{1 + 4290\sqrt{1 + ...}}}}$$

ii) Again, to find the required expression we first set

$$2k - 1 = 729 \Rightarrow k = 365$$

Now, using (2) we get,

$$729 = \sqrt{1 + 4\sqrt{1 + 132859\sqrt{1 + 132860\sqrt{1 + ...}}}}$$
Example 2.2  Find the value of :

$$\sqrt{1 + 4 \sqrt{1 + 65279 \sqrt{1 + 65280 \sqrt{1 + ...}}} }$$

Solution. Now to solve this we need to compare this with (2),

$$\sqrt{1 + 4 \sqrt{\frac{1 + 65279 \sqrt{1 + 65280 \sqrt{1 + ...}}} {k^2 - k - 1} } }$$

So, $$k^2 - k - 1 = 65279$$

$$\Rightarrow k = 256$$

or

$$\Rightarrow k = -255$$

Since k is positive so our final value for k is 256 and thus our final answer is 511.

3  The Famous Problem

Now I will show how we can use the above-mentioned formula to solve the problem given by Srinivasa Ramanujan, i.e.

Let, $$S = \sqrt{1 + 2 \sqrt{1 + 3 \sqrt{1 + 4 \sqrt{1 + 5 \sqrt{1 + ...}}}}}$$

On comparing with (2) we get,

$$S = \sqrt{1 + 2 \sqrt{\frac{1 + 3 \sqrt{1 + 4 \sqrt{1 + 5 \sqrt{1 + ...}}} }{k^2 - k - 1} } }$$

$$\Rightarrow k^2 - k - 1 = 5$$

$$\Rightarrow k = 3$$

or

$$\Rightarrow k = -2$$
Since $k$ is positive so our final value for $k$ is 3, Now on moving back to our original equation we get,

$$\Rightarrow S = \sqrt{1 + 2} \sqrt{1 + 3} \sqrt{1 + 4} \sqrt{1 + 5} \sqrt{1 + \ldots} = 2(3) - 1$$

$$\Rightarrow S = \sqrt{1 + 2} \sqrt{1 + 3} \cdot 5 = \sqrt{1 + 2} \sqrt{16} = \sqrt{1 + 2} \cdot 4 = \sqrt{9} = 3 \quad \square$$

4 Acknowledgements

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References

[1] Nested radical - Wikipedia.