Restrictions to Goldbach's conjecture using Bertrand's postulate, initiation to a proof

BY ANASS MASSOUDI

Abstract

The Goldbach conjecture, also named the binary Goldbach's conjecture, proposed by the Russian mathematician Christian Goldbach in 1742, states that for every even integer n bigger than 2, there is always two primes a and b such that $n = a + b$, and until now this conjecture remained unproven, in this paper, we use what's known as Bertrand's postulate to restrict the conditions for the two primes a and b that verify this conjecture for every even number $n = 2p$, namely proving the interesting fact that the Goldbach statement is valid if and only if we have $p < a < 2p - 2$ and $b < p$, leading to a clue to prove this conjecture in a simpler manner than attack it brutally without any knowledge about the properties of a and b and the inequality that we will prove, making at last an initiation for a proof of the Goldbach statement .

I. INTRODUCTION

The Goldbach conjecture appeared for the first time in the mathematical litterature in a letter [1] written by Russian mathematician and number theorist Christian Goldbach to the famous Swiss mathematician and physicist Leonhard Euler in the 7th of June, 1742 on the matter of a new mathematical statement proposed by Goldbach for which he couldn't find a proof, his statement now named after him. Although Euler regarded

Goldbach's conjecture as an entirely certain theorem and was convinced by its validity, he couldn't also prove it [2] and until now, Goldbach conjecture still an unproven conjecture, making it one of the oldest unsolved problems in number theory, figuring along with the Reimann hypothesis and the twin prime conjecture; two famous conjectured statements in number theory that are also still unproven until now; in the 8th problem on the list of Hilbert problems announced in 1900 by German mathematician David Hilbert [3], and In the first problem on the list of the four landau problems; Goldbach conjecture, twin prime conjecture, Legendre conjecture and the question of whether there is an infinite number of primes that can be written in the form n^2+1 or not; announced in 1912 by German mathematician and number theorist Edmund landau at the international congress of mathematicians at Cambridge, noticing that Edmund landau clammed that this conjecture along with Legendre conjecture and the twin prim conjecture in his famous list, is unattackable in the present state (in 1912) of mathematical knowledge, where the Bertrand's postulate was well known from 60 years ago [4].

In 20th century, Many interesting works on the subject of Goldbach conjecture have been made for different mathematical statements concerning writing integer as sums of prime numbers, to mention some examples, in 1930, a soviet mathematician named Lev Genrikhovich Shnirelman proved that any integer $n > 1$ can be written as the sum of a maximum of 20 prime numbers in his research of a proof for the Goldbach conjecture [5], in 1937 another soviet mathematician named Ivan Matveyevich Vinogradov proved that every "sufficiently large" (without stating exactly how large) odd integer can be written as the sum of a maximum of 3 prime numbers, it's what we now call Vinogradov's theorem [6], in 1995, the French mathematician Olivier Ramaré proved using Vinogradov's theorem that every even number can be written as the sum of 6 primes or more [7], all of those important contributions concerns statements that have the same general form like that of the Goldbach's statement, it all concerns theorems and conjectures about mathematical techniques to write numbers as sums of primes.

Many mathematicians starting with Euler and Goldbach are convinced by its validity including the author of this paper, especially because of statistical considerations, a notable one was an empirical verification of the validity of this conjecture up to $n = 4.10^{18}$ published in 2014 by Thomas Oliveira e silva, Siegfried Herzog and Silvio Pard [8]. However, Using Bertrand's postulate; a mathematical statement in number theory first proposed by French mathematician Joseph Bertrand in 1845 [9] and proven after by Russian mathematician Pafnuty Chebyshev in 1852 [10] so 110 years after the birth of Goldbach's conjecture, thus Bertrand's statement is often called Chebyshev's theorem or Bertrand-Chebyshev's theorem, without forgetting the contributions of the famous Indian mathematician Srinivasa Ramanujan who gave a simple proof of Bertrand's statement in a paper published in 1919 [11]; we restrict the Goldbach's conjecture, remarquing the interesting fact that by the Bertrand's postulate , this conjecture is valid only if the two primes a,b in the statement verify the condition that $p < a < 2p - 2$ and $b < p$ such that $n = 2p = a + b$.

A version of Bertrand's postulate is stated as following:
"For any integer $p > 3$, there is at least one prime a such that $p < a < 2p - 2$" There are of course other mathematically equivalent

versions, namely the less restrictive formulation of the main Bertrand's statement: "For any integer $p > 1$, there is at least one prime a such that $p < a < 2p$", but in the following analysis, we will use the first formulation for $p > 3$, and then study the $p = \{2,3\}$ corresponding to $n = 2p = \{4,6\}$, separately.

II. THE MAIN ANALYSIS FOR RESTRICTIONS

The Goldbach statement states that for every even number $n = 2p$ with $p > 1$, there is two primes a, b such that $n = a + b$, in the following we will consider that $p > 3$

By Bertrand's postulate, we know that for any integer $p > 3$, there is at least one prime a such that $p < a < 2p - 2$, therefore there are 5 cases of the values of the two primes a and b relative to the interval $]p, 2p - 2[$ in the set of natural numbers $\mathbb{N}$ by Bertrand's postulate, and here they are:

- Case 1: $p < a < 2p - 2$ and $p < b < 2p - 2$
- Case 2: $a < p$ and $b < p$
- Case 3: $p < a < 2p - 2$ and $b < p$
- Case 4: $2p - 2 < a$ and $2p - 2 < b$
- Case 5: $p < a < 2p - 2$ and $2p - 2 < b$

In the following, we will study each one of them separately to know in which cases the Goldbach statement is valid, which will make it more precise and simpler:

1) If $p < a < 2p - 2$ and $p < b < 2p - 2$ We obtain $2p < a + b < 4p - 4$ so in this case $a + b \neq 2p$, therefore the Goldbach statement is not valid for the case where both a and b are included in the interval $]p, 2p - 2[$

2) If $a < p$ and $b < p$ we obtain $a + b < 2p$, and so $a + b \neq 2p$, therefore Goldbach statement is not valid for $a, b < p$
3) If $2p - 2 < a$ and $2p - 2 < b$ we obtain $4p - 4 < a + b$, now if we suppose that $a + b = 2p$, we obtain $2p - 4 < 0$ which is equivalent to $p < 2$, but we know that $p > 3$, so $p < 2$ is impossible, which implies that the Goldbach statement isn't valid for $2p - 2 < a, b$
4) If $p < a < 2p - 2$ and $2p - 2 < b$ we obtain $3p - 2 < a + b$, now if we suppose that $a + b = 2p$, we obtain $p - 2 < 0$ which is equivalent to $p < 2$, but we know that $p > 3$, so $p < 2$ is impossible, which implies that the Goldbach statement isn't valid for $p < a < 2p - 2$
5) If $p < a < 2p - 2$ and $b < p$ we obtain $a + b < 3p - 2$, now if we suppose that $a + b = 2p$, we obtain $p - 2 > 0$ which is equivalent to $p > 2$, which is true, so if the Goldbach conjecture was right, the Goldbach statement will be valid for $p < a < 2p - 2$ and $b < p$

Since, we've proven that, if the Goldbach statement was right, meaning that effectively any even number $n = 2p$ bigger than 2 can be written as the sum of two primes a and b, therefore the two primes a and b must verify the following condition: $p < a < 2p - 2$ and $b < p$, because as we saw, it's the only case for a and b for which the Goldbach statement can be valid, so we can restrict the Goldbach conjecture to the following statement:
"Any even integer $n = 2p$ such that $p > 1$, can be written as the sum of two primes a and b verifying the following condition: $b < p$ and $p < a < 2p - 2$ "

III. INITIATION TO A PROOF

We have proven that the Goldbach statement can be true or valid only for p < a < 2p – 2 and b < p, we find that this important result leads to a much simpler and remarquable way to prove definitively the Goldbach conjecture, by a pure mathematical implication that we will show in the following:

Assume that the Goldbach statement is right, so we have for an even integer n = 2p two primes a and b such that n = a + b with a and b verifying the following condition $\begin{cases} p < a < 2p - 2 \\ b < p \end{cases}$ so we obtain p < a + b < 3p – 2 factoring the two bornes by 2p we obtain $\frac{2p}{2} < a + b < 2p \times (\frac{3}{2} - \frac{1}{p})$ equivalent to $\frac{1}{2} < \frac{a+b}{2p} < \frac{3}{2} - \frac{1}{p}$ by knowing the fact that $\frac{3}{2} - \frac{1}{p} < \frac{3}{2}$ we obtain the following inequality:

$$\frac{1}{2} < \frac{a+b}{2p} < \frac{3}{2}$$

The Goldbach statement states that 2p = a + b and so $\frac{a+b}{2p} = 1$, and since the only possible integer value for $\frac{a+b}{2p}$ in the inequality $\frac{1}{2} < \frac{a+b}{2p} < \frac{3}{2}$ is 1, so proving the Goldbach conjecture for p > 3 and therefore n = 2p > 6 is equivalent to proving that for any given integer p there is two primes a and b such that p < a < 2p – 2 and b < p verifying that the fraction $\frac{a+b}{2p}$ is an integer number, meaning $\frac{a+b}{2p} \in \mathbb{N}$, in other words the validity of the Goldbach statement could be proven by just proving that for any integer number p there are two primes p < a < 2p – 2 and b < p such that $\frac{a+b}{2p} \in N$, for that matter, only a proof of existence is required, because whatever this integer value is, if the Goldbach conjecture is true and mathematically valid, this value will be equal to 1 by the proven inequality $\frac{1}{2} < \frac{a+b}{2p} < \frac{3}{2}$ This result, makes therefore an important initiation for a possible proof of Goldbach's statement based on it, ending more than 250 years of a belief in its conjectured validity rather than an objective mathematical certainty, remembering that this way uses the version of Bertrand's postulate that is valid only for p > 3 for the Goldbach conjecture which is known to be valid for n > 2 and therefore p > 1 since 1 is excluded from the list of prime numbers by the fundamental theorem of arithmetic, so we must for completeness prove the validity of Goldbach statement for n = {4,6} separately, and in fact it's true because 4 = 2+2 and 6 = 3+3.

Email of the author: anassmassoudi6@gmail.com

IV. REFERENCES

[1] Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle, St. Pétersbourg 1843, pp. 125–129

[2] Caldwell, Chris (2008). "Goldbach's conjecture". Retrieved 2008-08-13

[3] Hilbert, David (1902). "Mathematical Problems". Bulletin of the American Mathematical Society. 8 (10) : 437–479. Doi :10.1090/S0002-9904-1902-00923-3.

[4] János pintz, Landau's problems on primes. Journal de Théorie des Nombres de Bordeaux. Vol. 21, No. 2 (2009), pp. 357-404, Société Arithmétique de Bordeaux

[5] Schnirelmann, L.G. (1930). "On the additive properties of numbers", first published in Proceedings of the Don Polytechnic

Institute in Novocherkassk (in Russian), vol XIV (1930), pp. 3–27, 1939, no. 6, 9–25

[6] I.M. Vinogradov (1954). The Method of Trigonometrical Sums in the Theory of Numbers. New York: Interscience

[7] Olivier Ramaré et Robert Rumely (de), « Primes in Arithmetic Progressions », Mathematics of Computation, vol. 65, no 213, Janvier 1996, p. 397-425

[8] Tomás Oliveira e Silva, Siegfried Herzog and Silvio Pard Empirical verification of the even Goldbach conjecture and computation of prime gaps up to 4.10^{18}, Mathematics of Computation, 83 (2014), 2033-2060

[9] Bertrand, Joseph (1845), "Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu'elle renferme.", Journal de l'École Royale Polytechnique, 18 (Cahier 30) : 123–140.

[10] Tchebychev, P. (1852), "Mémoire sur les nombres premiers", Journal de mathématiques pures et appliquées, Série 1 (in French) : 366–390. (Proof of the postulate : 371-382). Also see Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg, vol. 7, pp.15-33, 1854

[11] Ramanujan, S. (1919). "A proof of Bertrand's postulate". Journal of the Indian Mathematical Society. 11 : 181–182.