Weak-measurement induced Quantum discord and Monogamy of X states

Asutosh Kumar

P. G. Department of Physics, Gaya College, Magadh University, Rampur, Gaya 823001, India
Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Prayagraj 211 019, India
Vaidic and Modern Physics Research Centre, Bhagal Bhim, Bhinmal, Jalore 343029, India

Weak-measurement induced quantum discord or super quantum discord (SQD) is a generalization of the normal quantum discord and is defined as the difference between quantum mutual information and classical correlation obtained by weak measurements in a given quantum system. This correlation is an information-theoretic measure and is, in general, different from entanglement-separability measures such as entanglement. Super quantum discord may be nonzero even for certain separable states. So far, SQD has been calculated explicitly only for a limited set of two-qubit quantum states and expressions for more general quantum states are not known. In this article, we derive explicit expressions for SQD for X states, a seven real-parameter family of two-qubit states and investigate its monogamy properties. The monogamy behaviour of SQD depends on the measurement strength. The formalism can be easily extended to N-qubit X states.

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I. INTRODUCTION

Quantum correlations have an apparent advantage over classical correlations in compution and information processing tasks which were rather hard to conceive without them [1–4]. Surprisingly, some separable states may also speed up certain tasks over their classical counterparts [5–8], Therefore, it is important to know whether a given quantum state has non-classical correlations or not.

For a composite bipartite system ρ^{AB}, the quantum mutual information is defined as

$$\mathcal{I}(\rho^{AB}) = S(\rho^{A}) + S(\rho^{B}) - S(\rho^{AB}), \tag{1}$$

where ρ^{A} (ρ^{B}) are the reduced density operators of part A (B) respectively and $S(\rho) = -\mathrm{tr}\,(\rho \log_2 \rho)$ is the von Neumann entropy. Quantum mutual information may be written as a sum of classical correlation $\mathcal{C}(\rho^{AB})$ and quantum correlation $\mathcal{Q}(\rho^{AB})$, that is, $\mathcal{I}(\rho^{AB}) = \mathcal{C}(\rho^{AB}) + \mathcal{Q}(\rho^{AB})$ [9–11]. It is an information-theoretic measure of the total correlation in a bipartite quantum state [12]. The quantum part $\mathcal{Q}$ has been called quantum discord [9]. Quantum discord may be greater than, equal to or less than classical correlation in a given state [11, 13]. It is a different type of quantum correlation than entanglement because separable mixed states (that is, with no entanglement) can have non-zero quantum discord. Quantum discord is a measure of nonclassical correlations that may include entanglement but is an independent measure.

This article is organized as follows. In Sec. II, we review the concept of quantum discord and super quantum discord. In Sec. III, we compute classical correlation and super quantum discord for two-qubit X states. In Sec. IV, we investigate the monogamy properties of SQD for X states and finally conclude our work in Sec. V.

II. QD AND SQD

In this Sec. we briefly review quantum discord and super quantum discord.

Quantum discord [9–11, 13, 14].– Let the projection operators $\{\Pi_k^A\}$, ($k = 0, 1$), describe a von Neumann measurement for subsystem A only, then the conditional density operator ρ_k associated with the measurement result k is

$$\rho_k^B = \frac{\mathrm{Tr}_A[(\Pi_k^A \otimes I)\rho_{AB}(\Pi_k^A \otimes I)]}{p_k} \tag{2}$$

where the probability $p_k = \mathrm{Tr}_{AB}[(\Pi_k^A \otimes I)\rho_{AB}(\Pi_k^A \otimes I)]$. The quantum conditional entropy with respect to this measurement is given by

$$S(\rho|\{\Pi_k^A\}) := \sum_k p_k\, S(\rho_k^B), \tag{3}$$

and the associated quantum mutual information of this measurement is defined as

$$\mathcal{I}(\rho|\{\Pi_k^A\}) := S(\rho^B) - S(\rho|\{\Pi_k^A\}). \tag{4}$$

A measure of the resulting classical correlations is provided by

$$\mathcal{C}(\rho) := \sup_{\{\Pi_k^A\}} \mathcal{I}(\rho|\{\Pi_k^A\}). \tag{5}$$

Analytic computation of quantum discord for a class of two-qubit X states has been studied in [11, 13, 15–17]. This class includes the maximally entangled Bell states, 'Werner' states [18] which include both separable and nonseparable states, as well as others. Later, a prescription for analytic computation of quantum discord in an N-qubit extended X states was provided in [19].

Weak measurements and the super quantum discord.– The concept of weak measurements can be formulated using the measurement operator formalism [20]. The weak measurement operators are given by

$$\begin{aligned} P(x) &= a(x)\Pi_0 + a(-x)\Pi_1, \\ P(-x) &= a(-x)\Pi_0 + a(x)\Pi_1, \end{aligned} \tag{6}$$

where $a(\pm x) = \sqrt{\frac{(1\pm \tanh x)}{2}}$, x is a parameter that denotes the strength of the measurement process, Π_0 and Π_1 are two orthogonal projectors with $\Pi_0 + \Pi_1 = I$. The weak measurement operators satisfy $P^\dagger(x)P(x) + P^\dagger(-x)P(-x) = I$. These operators have the following properties: (i) $P(0) = \frac{I}{\sqrt{2}}$ resulting in no state change, (ii) in the strong measurement limit we have the projective measurement operators, i.e., $\lim_{x\to\infty} P(x) = \Pi_0$ and $\lim_{x\to\infty} P(-x) = \Pi_1$, (iii) $P(x)P(y) \propto P(x+y)$, and (iv) $[P(x), P(-x)] = 0$. Now consider a bipartite state ρ_{AB}. After we perform weak measurement on the subsystem A by the weak operators $\{P^A(x), P^A(-x)\}$, the post-measurement state for the subsystem B is given by

$$\begin{aligned} \rho_{B|P^A(\pm x)} &= \frac{\mathrm{Tr}_A[(P^A(\pm x)\otimes I)\rho_{AB}(P^A(\pm x)\otimes I)]}{p(\pm x)} \\ &= \frac{a(\pm x)^2 p_0 \rho_0^B + a(\mp x)^2 p_1 \rho_1^B}{p(\pm x)}\,. \end{aligned} \tag{7}$$

where $p(\pm x) = \mathrm{Tr}_{AB}[(P^A(\pm x)\otimes I))\rho_{AB}(P^A(\pm x)\otimes I)] = a(\pm x)^2 p_0 + a(\mp x)^2 p_1$. The "weak quantum conditional entropy" after we perform a weak measurement on the subsystem A, is given by

$$S_w(B|\{P^A(x)\}) = p(x) S(\rho_{B|P^A(x)}) + p(-x) S(\rho_{B|P^A(-x)}). \tag{8}$$

The SQD in the state ρ_{AB}, denoted by $D_w(\rho_{AB})$, is defined as [21]

$$D_w(\rho_{AB}) := \min_{\{\Pi_i^A\}} S_w(B|\{P^A(x)\}) - S(B|A), \tag{9}$$

where $S(B|A) = S(\rho_{AB}) - S(\rho_A)$. The SQD satisfies $I(\rho_{AB}) \geq D_w(\rho_{AB}) \geq D_s(\rho_{AB})$ [21]. It is a monotonic function of the measurement strength x. The extra quantum correlation captured by the weak measurement is defined as $\Delta(\rho_{AB}) = D_w(\rho_{AB}) - D_s(\rho_{AB})$. In the strong measurement limit, the extra quantum correlation becomes zero, i.e., $\lim_{x\to\infty}\Delta(\rho_{AB}) = 0$. Very recently, it was shown in [22] that weak measurement induced super discord is capable of resurrecting the lost quantumness, due to the projective measurement, of a given quantum state.

III. TWO-QUBIT X STATES AND SQD

The density matrix of a two-qubit X state [23] in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is of the general form

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{10}$$

The symmetry structure of two-qubit X states and generalized X states of N-qubits is examined in [24, 25].

Eq. (10) describes a quantum state provided the unit trace and positivity conditions $\sum_{i=1}^4 \rho_{ii} = 1$, $\rho_{22}\rho_{33} \geq |\rho_{23}|^2$, and $\rho_{11}\rho_{44} \geq |\rho_{14}|^2$ are fulfilled. X states are entangled if and only if either $\rho_{22}\rho_{33} < |\rho_{14}|^2$ or $\rho_{11}\rho_{44} < |\rho_{23}|^2$ [13]. Both conditions cannot hold simultaneously [26]. Eq. (10) is a 7-real parameter state with three real parameters along the main diagonal and two complex (or four real) parameters at off-diagonal positions.

The eigenvalues of the density matrix ρ_X in Eq. (10) are given by

$$\begin{aligned} \lambda_{0,1} &= \frac{1}{2}\left[(\rho_{11}+\rho_{44}) \pm \sqrt{(\rho_{11}-\rho_{44})^2 + 4\,|\rho_{14}|^2}\right], \\ \lambda_{2,3} &= \frac{1}{2}\left[(\rho_{22}+\rho_{33}) \pm \sqrt{(\rho_{22}-\rho_{33})^2 + 4\,|\rho_{23}|^2}\right]. \end{aligned} \tag{11}$$

The quantum mutual information is given as

$$\mathcal{I}(\rho_X) = S(\rho_X^A) + S(\rho_X^B) + \sum_{j=0}^{3} \lambda_j \log_2 \lambda_j\,, \tag{12}$$

where ρ_X^A and ρ_X^B are the marginal states of ρ_X, and

$$\begin{aligned} S(\rho_X^A) &= -\Big[(\rho_{11}+\rho_{22})\log_2(\rho_{11}+\rho_{22}) + \\ &\qquad (\rho_{33}+\rho_{44})\log_2(\rho_{33}+\rho_{44})\Big], \\ S(\rho_X^B) &= -\Big[(\rho_{11}+\rho_{33})\log_2(\rho_{11}+\rho_{33}) + \\ &\qquad (\rho_{22}+\rho_{44})\log_2(\rho_{22}+\rho_{44})\Big]. \end{aligned} \tag{13}$$

To compute the classical correlation $\mathcal{C}(\rho_X)$ we consider the weak measurement operators $\{P^A(x), P^A(-x)\}$ for subsystem A. It is known that any von Neumann measurement for subsystem A can be written as Ref. [11]

$$\Pi_i = V\,A_i\,V^\dagger : \quad i = 0, 1\,, \tag{14}$$

where $A_i = |i\rangle\langle i|$ is the projector for subsystem A along the computational base $|i\rangle$ and $V \in SU(2)$ is a unitary operator with unit determinant. We may write any $V \in SU(2)$ as

$$V = t\,I + \mathrm{i}\,\vec{y}\cdot\vec{\sigma}\,, \tag{15}$$

with $t, y_1, y_2, y_3 \in \mathbb{R}$ and $t^2 + y_1^2 + y_2^2 + y_3^2 = 1$. This implies that these parameters, three among them independent, assume their values in the interval $[-1, 1]$, i.e. $t\,,\, y_i \in$

$[-1,1]$ for $i=1,2,3$. After the measurement, the state ρ_X will change to the ensemble $\{\rho^B(\pm x), p(\pm x)\}$, where

$$\rho^B(\pm x) := \frac{1}{p(\pm x)}\left(P^A(\pm x)\otimes I\right)\rho_X\left(P^A(\pm x)\otimes I\right), \quad (16)$$

and $p(\pm x) = \mathrm{tr}\left[\left(P^A(\pm x)\otimes I\right)\rho_X\left(P^A(\pm x)\otimes I\right)\right]$. The ensemble $\{\rho^B(\pm x), p(\pm x)\}$ are of subsystem B and thus 2×2 density matrices.

The probabilities of weak measurements are given as

$$p(\pm x) = \frac{1}{2}\left[\,1 \pm \alpha\beta z_3(\rho_{11}+\rho_{22}-\rho_{33}-\rho_{44})\right]. \quad (17)$$

where $\alpha = a(x)+a(-x)$ and $\beta = a(x)-a(-x)$. The probabilities can be re-written as

$$\begin{aligned} p(x) &= \left[\,(\rho_{11}+\rho_{22})\,k + (\rho_{33}+\rho_{44})\,l\,\right], \\ p(-x) &= \left[\,(\rho_{11}+\rho_{22})\,l + (\rho_{33}+\rho_{44})\,k\,\right]. \end{aligned} \quad (18)$$

where

$$k = \tfrac{1}{2}(1+\alpha\beta z_3)\,,\ \ l = \tfrac{1}{2}(1-\alpha\beta z_3)\,. \quad (19)$$

The two eigenvalues each of $\rho^B(x)$ and $\rho^B(-x)$ are given as

$$\begin{aligned} v_{\pm}(\rho^B(x)) &= \frac{1}{2}(1\pm\theta)\,, \\ w_{\pm}(\rho^B(-x)) &= \frac{1}{2}(1\pm\theta')\,. \end{aligned} \quad (20)$$

The θ and θ' are given as below

$$\theta = \sqrt{\frac{\left[\,(\rho_{11}-\rho_{22})\,k + (\rho_{33}-\rho_{44})\,l\,\right]^2 + \Theta}{\left[\,(\rho_{11}+\rho_{22})\,k + (\rho_{33}+\rho_{44})\,l\,\right]^2}}\,, \quad (21)$$

$$\theta' = \sqrt{\frac{\left[\,(\rho_{11}-\rho_{22})\,l + (\rho_{33}-\rho_{44})\,k\,\right]^2 + \Theta}{\left[\,(\rho_{11}+\rho_{22})\,l + (\rho_{33}+\rho_{44})\,k\,\right]^2}}\,, \quad (22)$$

where $\Theta = \alpha^2\beta^2[\,(\rho_{14}+\rho_{32})\,z_1 + i(\rho_{14}-\rho_{32})\,z_2\,]\,[\,(\rho_{41}+\rho_{23})\,z_1 - i(\rho_{41}-\rho_{23})\,z_2\,]$ or, $\Theta = \alpha^2\beta^2[\,(1-z_3^2)(|\rho_{14}|^2 + |\rho_{23}|^2 + 2\,\Re(\rho_{14}\rho_{23})) - 4\,z_2^2\,\Re(\rho_{14}\rho_{23}) - 4\,z_1\,z_2\Im(\rho_{14}\rho_{23})\,]$ or ,$\Theta = (4\,k\,l + \alpha^2\beta^2 - 1)\,[\,|\rho_{14}|^2 + |\rho_{23}|^2 + 2\,\Re(\rho_{14}\rho_{23})\,] + 16\,\alpha^2\beta^2\,[\,n\,\Im(\rho_{14}\rho_{23}) - m\,\Re(\rho_{14}\rho_{23})\,]$. The parameters, m, n are defined as

$$m = \ (t\,y_1 + y_2\,y_3)^2\,,\ \ n = (t\,y_2 - y_1\,y_3)(t\,y_1 + y_2\,y_3) \quad (23)$$

The parameters k, l, m and n satisfy the following two constraints

$$\begin{aligned} k + l &= 1\,, \\ 4(m^2+n^2) &= m[\,(4kl-1)\coth^2 x + 1\,]\,. \end{aligned} \quad (24)$$

Therefore, Eqs. (21) and (22) for a given density matrix depend on only two independent real parameters. These parameters are related to the three independent parameters in Eq. (15) through $4m = z_2^2, 4n = -z_1 z_2, k-l = \alpha\beta z_3$ where $z_1 = 2(-ty_2 + y_1y_3)$, $z_2 = 2(ty_1 + y_2y_3)$ and $z_3 = t^2 + y_3^2 - y_1^2 - y_2^2$ [11]. Alternatively, since $z_1^2 + z_2^2 + z_3^2 = 1$, Eqs. (21) and (22) for a given density matrix depend on two real parameters z_1 and z_2.

The entropies of the ensemble $\{\rho^B(\pm x), p(\pm x)\}$ are given as

$$S(\rho^B(x)) = -\frac{1-\theta}{2}\log_2\frac{1-\theta}{2} - \frac{1+\theta}{2}\log_2\frac{1+\theta}{2}\,, \quad (25)$$

$$S(\rho^B(-x)) = -\frac{1-\theta'}{2}\log_2\frac{1-\theta'}{2} - \frac{1+\theta'}{2}\log_2\frac{1+\theta'}{2} \quad (26)$$

The quantum conditional entropy in Eq. (3) is given as

$$S(\rho_X|\{P^A(\pm x)\}) = p(x)\,S(\rho^B(x)) + p(-x)\,S(\rho^B(-x)) \quad (27)$$

As per Eq. (5), the classical correlation is obtained as

$$\begin{aligned} \mathcal{C}(\rho_X) &= \sup_{\{P^A(\pm x)\}}\left[\mathcal{I}(\rho_X|\{P^A(\pm x)\})\right] \\ &= S(\rho_X^B) - \min_{\{P^A(\pm x)\}}\left[\,S(\rho_X|\{P^A(\pm x)\})\,\right]. \end{aligned} \quad (28)$$

Therefore, to calculate the classical correlation and consequently quantum discord, we have to minimize the quantity $S(\rho_X|\{P^A(\pm x)\})$ (Eq. (27)) with respect to the weak measurements which, in turn, implies the minimization with respect to any two independent parameters.

In the strong measurement limit, the weak measurement operators $\{P^A(x), P^A(-x)\}$ will reduce to the von Neumann projective operators $\{\Pi_0^A, \Pi_1^A\}$. In [13] authors obtained analytic values of $k = 0, \frac{1}{2}, 1$, $m = 0, \frac{1}{4}$ and $n = 0, \pm\frac{1}{8}$ by setting equal to zero the partial derivatives of Eq. (27) (of course, in the strong measurement limit) with respect to k, m and n. The same values will be obtained if one takes partial derivatives with respect to z_1 and z_2 and equates them to zero. The results obtained in Ref. [13], however, has been shown to be reliable only for a more restricted class of states identified in [27]. An effective algorithm for the computation of quantum discord of general two-qubit states has been presented in Ref. [28]. The optimization problem for the conditional entropy, and hence for the discord, has been recast in terms of the eigenvalues of the post-measurement states obtained after the local measuremet process on one qubit. The derived transcendental constraints are shown to be amenable to direct numerical solution. We argue (and can be proved after some complicated algebra!) that optimization under the weak operator formalism, that is, the minimization of Eq. (27) will yield the same values of z_1 and z_2 (consequently, of k and m) as in the strong measurement case. This is expected because from the continuity of the weak conditional entropy with measurement strength x, the measurement basis which minimizes the weak conditional entropy is same as the basis that minimizes the strong conditional entropy [22].

The above formalism can be straightforward extended to N-qubit X states according to the prescription given in [19].

IV. MONOGAMY OF X STATES

It can be easily shown that N-qubit X states satisfy the following monogamy relations (when normal quantum discord is considered):

$$\begin{aligned}\mathcal{D}^{\rightarrow}(\rho^X_{A_1:A_2\cdots A_N}) &\geq \mathcal{D}^{\rightarrow}(\rho^X_{A_1:A_2}) + \cdots + \mathcal{D}^{\rightarrow}(\rho^X_{A_1:A_N}) \\ \mathcal{D}^{\leftarrow}(\rho^X_{A_1\cdots A_{N-1}:A_N}) &\geq \mathcal{D}^{\leftarrow}(\rho^X_{A_1:A_N}) + \cdots + \mathcal{D}^{\leftarrow}(\rho^X_{A_{N-1}:A_N})\,.\end{aligned}$$

This is because all the two qubit reduced density matrices of N-qubit X states are diagonal and the normal quantum discord of any 4×4 diagonal matrix always vanishes when measured either from left or from right. In the case of weak-operator formalism, SQD depends on the measurement strength x. However, for large x, it reduces to normal quantum discord. Below we illustrate the SQD-score for two typical states. Consider the three qubit X state

$$\rho = p|GHZ\rangle\langle GHZ| + \frac{1-p}{8} I_8\,. \qquad (29)$$

where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $0 \leq p \leq 1$. The optimal measurement for X state in Eq. (29) and its two-qubit reduced density matrices is the Pauli matrix σ_z (see [27] for the optimization conditions of two-qubit X state). The analytic expression of SQD-score is, then, given by

$$\delta^{\rightarrow}_{\mathcal{D}} = (1 - S(\rho) + S_a) - 2(1 - S_b + S_c) \qquad (30)$$

where

$$\begin{aligned} S_a &= S\Big(\Big\{\frac{1-p}{4} + a^2(x)p, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4} + a^2(-x)p\Big\}\Big) \\ S_b &= S\Big(\Big\{\frac{1+p}{4}, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1+p}{4}\Big\}\Big) \\ S_c &= S\Big(\Big\{\frac{1\pm p\tanh x}{2}\Big\}\Big) \end{aligned} \qquad (31)$$

where $S(\{\lambda_i\})$ is the Shannon entropy of λ_is.

Now consider the Dicke state [29], which is symmetric with respect to interchange of qubits, given by

$$|W^r_N\rangle = \frac{1}{\sqrt{\binom{N}{r}}}\sum_{permuts}|\underbrace{00...0}\underbrace{11...1}\rangle \qquad (32)$$

where $\sum_{permuts}$ represents sum over all $\binom{N}{r}$ permutations of $N-r$ $|0\rangle$s and r $|1\rangle$s. The Dicke state itself, in Eq. (32), is not a X state but its all two-qubit reduced density matrices are same and has X structure. We find that for these X states, the Pauli matrix σ_x turns out to be the optimal measurement (see Ref. [27]). The closed expression of SQD-score for the Dicke states is obtained as

$$\delta_{\mathcal{D}} = S_1 - (N-1)(S_2 - S_{12} + S(\{\lambda_\pm\})) \qquad (33)$$

where

$$\begin{aligned} S_1 &= -\frac{r}{N}\log_2\frac{r}{N} - (1-\frac{r}{N})\log_2(1-\frac{r}{N}) \\ S_2 &= -(a+b)\log_2(a+b) - (b+c)\log_2(b+c) \\ S_{12} &= -a\log_2 a - 2b\log_2 2b - c\log_2 c \\ S(\{\lambda_\pm\}) &= -\lambda_+\log_2\lambda_+ - \lambda_-\log_2\lambda_- \end{aligned} \qquad (34)$$

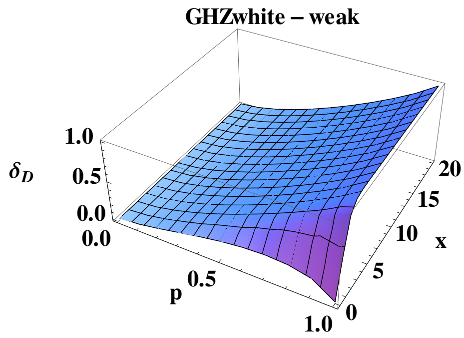

FIG. 1: (Color online) Discord-score vs. the measurement strength x for super quantum discord (SQD) of three qubit X state in Eq. (29).

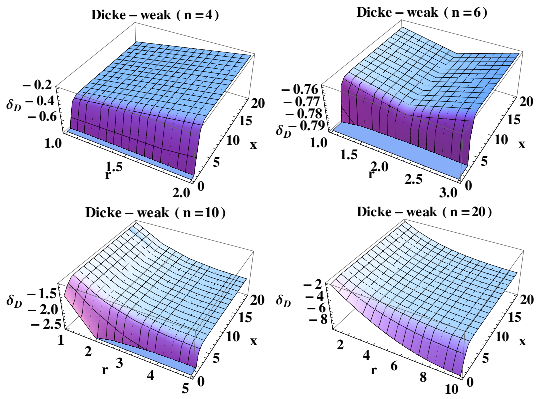

FIG. 2: (Color online) Non-monogamy of Dicke states in Eq. (32) for super quantum discord (SQD). Discord-score vs. the measurement strength x for $N = 4$ (top-left panel), $N = 6$ (top-right panel), $N = 10$ (bottom-left panel) and $N = 20$ (bottom-right panel) respectively for $1 \leq r \leq [\frac{N}{2}]$.

where $a = \frac{(N-r)(N-r-1)}{N(N-1)}$, $b = \frac{r(N-r)}{N(N-1)}$ and $c = \frac{r(r-1)}{N(N-1)}$ and $\lambda_\pm$ are the solutions of the equation

$$\lambda^2 - \lambda + ab + bc + ca + b^2(1 - \tanh^2 x) = 0. \qquad (35)$$

Very recently, it was shown in Ref. [30] that Dicke state is always non-monogamous with respect to normal quantum discord and a Dicke state with more number of parties is more non-monogamous to that with a smaller number of parties. Here we observe the similar features (see Fig. 2).

V. SUMMARY

We have derived analytical expressions for the classical correlation and super quantum discord in X states, a seven-parameter family of states of two qubits. A large class of two-qubit states that includes maximally or partially entangled states, and mixed states that are separable or non-separable can now be examined for classical correlations and super quantum discord. The monogamy behaviour of SQD depends on the measurement strength.

Note.– After we finished this work, we came to know about another similar work entitled *Role of weak measurements on states ordering and monogamy of quantum correlation* [31].

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