Proof of the Riemann Hypothesis

Guilherme Rocha de Rezende[1]

[1]Instituto Federal de Brasília

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Abstract

In this article we will prove the problem equivalent to the Riemann Hypothesis developed by Luis-Báez in the article "A sequential Riesz-like criterion for the Riemann hypothesis".

1 Introduction

The Riemann Hypothesis is a famous conjecture made by Bernhard Riemann in his article on prime numbers. Riemann, as indicated by the title of his article [1], wanted to know the number of prime numbers in a given interval of the real line, so he extended a Euler observation and defined a function called Zeta. Riemann obtained an explicit formula, which depends on the non-trivial zeros of the Zeta function, for the quantity he was looking for. Along the way, Riemann mentions that probably all non-trivial zeros of the Zeta function are, in the now called critical line, that is, when the complex argument $s = \sigma + IT$ of the Zeta function has a real part equal to one-half. - $\sigma = \frac{1}{2}$. We will prove, using the equivalent problem developed by Luis Báez-Duarte [2], the conjecture.

2 Proof

$$q_k := \sum_{n=1}^{\infty} \frac{\left(1-\frac{1}{n^2}\right)^k}{n^2} = \sum_{n=1}^{k} \frac{\left(1-\frac{1}{n^2}\right)^k}{n^2} + \sum_{n=k+1}^{\infty} \frac{\left(1-\frac{1}{n^2}\right)^k}{n^2} \tag{1}$$

We need to prove that: $q_k = O\left(k^{-\frac{3}{4}}\right)$, i.e., $q_k \leq M \cdot k^{-\frac{3}{4}}$ for all $k \geq k_0$ and M is a definite positive constant. This is equivalent to the Riemann's hypothesis.

2.1 Treating the first sum

2.1.1 Using Hölder inequality we get

$$\sum_{n=1}^{k} \frac{\left(1-\frac{1}{n^2}\right)^k}{n^2} \leq \left(\sum_{n=2}^{k} \frac{1}{n^{\left(\frac{1}{p}+\Delta\right)\cdot p}}\right)^{1/p} \cdot \left(\sum_{n=2}^{k} \frac{\left(1-\frac{1}{n^2}\right)^{k\cdot q}}{n^{\left(2-\frac{1}{p}-\Delta\right)\cdot q}}\right)^{1/q} \tag{2}$$

and we must determine, conveniently, p,q and Δ.

2.1.2 Finding an upper bound and changing expoent 2 of n

$$\sum_{n=2}^{k} \frac{\left(1-\frac{1}{n^2}\right)^{k\cdot q}}{n^{\left(2-\frac{1}{p}-\Delta\right)\cdot q}} < \sum_{n=2}^{k} \frac{e^{-\frac{kq}{n^2}}}{n^{\left(2-\frac{1}{p}-\Delta\right)\cdot q}} < \sum_{n=2}^{k} \frac{e^{-\frac{kq}{n^{\left(2-\frac{1}{p}-\Delta\right)\cdot q}}}}{n^{\left(2-\frac{1}{p}-\Delta\right)\cdot q}} + \delta\left(\left(2-\frac{1}{p}-\Delta\right)\cdot q\right) \tag{3}$$

where $\delta\left(\left(2-\frac{1}{p}-\Delta\right)\cdot q\right)$ is an error associated with expoent change, and the error is zero if $\left(2-\frac{1}{p}-\Delta\right)\cdot q > 2$. This error will be analized later.

2.1.3 Finding an integral that is an upper bound of the sum

Let $C = \left(2-\frac{1}{p}-\Delta\right)\cdot q$, where we assume for now $C > 1$, we have

$$\sum_{n=2}^{k} \frac{e^{-\frac{kq}{n^C}}}{n^C} < \int_1^k \frac{e^{-\frac{kq}{x^C}}}{x^C} dx \tag{4}$$

Change of variable:

$$y = \frac{kq}{x^C} \tag{5}$$

$$x = (kq)^{\frac{1}{C}} \cdot y^{-\frac{1}{C}} \tag{6}$$

$$dx = (kq)^{\frac{1}{C}} \cdot -\frac{1}{C} \cdot y^{-\frac{1}{C}-1} dy \tag{7}$$

$$\int_1^k \frac{e^{-\frac{kq}{x^C}}}{x^C} dx = \int_1^k \frac{e^{-y}}{kq} \cdot y \cdot (kq)^{\frac{1}{C}} \cdot -\frac{1}{C} \cdot y^{-\frac{1}{C}-1} dy \tag{8}$$

$$\frac{(kq)^{\frac{1}{C}-1}}{C} \int_{\frac{kq}{k^C}}^{kq} e^{-y} \cdot y^{-\frac{1}{C}} dy < \frac{(kq)^{\frac{1}{C}-1}}{C} \int_{y=0}^{\infty} e^{-y} \cdot y^{-\frac{1}{C}} dy \tag{9}$$

$$\frac{(kq)^{\frac{1}{C}-1}}{C} \int_{\frac{kq}{k^C}}^{kq} e^{-y} \cdot y^{-\frac{1}{C}} dy < \frac{(kq)^{\frac{1}{C}-1}}{C} \Gamma\left(1 - \frac{1}{C}\right) \tag{10}$$

Therefore

$$\sum_{n=2}^{k} \frac{e^{-\frac{kq}{n^C}}}{n^C} < \frac{(kq)^{\frac{1}{C}-1}}{C} \Gamma\left(1 - \frac{1}{C}\right) \tag{11}$$

$$\sum_{n=2}^{k} \frac{\left(1 - \frac{1}{n^2}\right)^{kq}}{n^{Cq}} < \frac{(kq)^{\frac{1}{C}-1}}{C} \Gamma\left(1 - \frac{1}{C}\right) + \delta(C) \tag{12}$$

2.1.4 Replacing sum by integral in Hölder inequality

$$\sum_{n=2}^{k} \frac{\left(1 - \frac{1}{n^2}\right)^{kq}}{n^{Cq}} < \left(\sum_{n=2}^{k} \frac{1}{n^{\left(\frac{1}{p}+\Delta\right)\cdot p}}\right)^{1/p} \cdot \left(\frac{(kq)^{\frac{1}{C}-1}}{C} \Gamma\left(1 - \frac{1}{C}\right) + \delta(C)\right)^{1/q} \tag{13}$$

or

$$\sum_{n=2}^{k} \frac{\left(1 - \frac{1}{n^2}\right)^{kq}}{n^{Cq}} < \left(\sum_{n=2}^{k} \frac{1}{n^{\left(\frac{1}{p}+\Delta\right)\cdot p}}\right)^{1/p} \cdot \left(\frac{q^{\frac{1}{C}-1}}{C} \Gamma\left(1 - \frac{1}{C}\right) + \frac{\delta(C)}{k^{\frac{1}{C}-1}}\right)^{1/q} \cdot k^{\frac{1}{qC}-\frac{1}{q}} \tag{14}$$

i.e., using Hölder's inequality,

$$\sum_{n=2}^{k} \frac{\left(1 - \frac{1}{n^2}\right)^{kq}}{n^{Cq}} < \left(\sum_{n=2}^{k} \frac{1}{n^{\left(\frac{1}{p}+\Delta\right)\cdot p}}\right)^{1/p} \cdot \left(\frac{q^{\frac{1}{C}-1}}{C} \Gamma\left(1 - \frac{1}{C}\right) + \frac{\delta(C)}{k^{\frac{1}{C}-1}}\right)^{1/q} \cdot k^{\frac{1}{qC}+\frac{1}{p}-1} \tag{15}$$

or

$$\sum_{n=2}^{k} \frac{\left(1 - \frac{1}{n^2}\right)^{kq}}{n^{Cq}} < \left(\sum_{n=2}^{k} \frac{k}{n^{\left(\frac{1}{p}+\Delta\right)\cdot p}}\right)^{1/p} \cdot \left(\frac{q^{\frac{1}{C}-1}}{C} \Gamma\left(1 - \frac{1}{C}\right) + \frac{\delta(C)}{k^{\frac{1}{C}-1}}\right)^{1/q} \cdot k^{\frac{1}{qC}-1} \tag{16}$$

and finally, using the fact that arithmetic mean is greater than harmonic mean, we get

$$\sum_{n=2}^{k} \frac{\left(1-\frac{1}{n^2}\right)^{kq}}{n^{Cq}} < \left(\frac{\sum_{n=2}^{k} n^{\left(\frac{1}{p}+\Delta\right)\cdot p}}{k}\right)^{1/p} \cdot \left(\frac{q^{\frac{1}{C}-1}}{C}\Gamma\left(1-\frac{1}{C}\right)+\frac{\delta(C)}{k^{\frac{1}{C}-1}}\right)^{1/q} \cdot k^{\frac{1}{qC}-1} \tag{17}$$

2.1.5 Choosing q to obtain $-\frac{3}{4}$ power

Therefore we need to solve

$$\frac{1}{qC}-\frac{1}{q}=-\frac{3}{4} \tag{18}$$

and solving the equations we arrive at

$$q := \frac{4}{C} \tag{19}$$

and because of Hölder condition $\frac{1}{q}+\frac{1}{p}=1$ we get

$$p=\frac{4}{4-C}. \tag{20}$$

We can choose $C=\left(2-\frac{1}{p}-\Delta\right)\cdot q=3$ which implies $\Delta=\frac{8p-3Cp-4}{4p}=\frac{8\cdot 4-3\cdot 3\cdot 4-4}{16}=-\frac{1}{2}$ therefore $1+\Delta\cdot p=1-\frac{1}{2}\cdot 4=-1$

2.1.6 Final Hölder inequality

$$\sum_{n=2}^{k} \frac{\left(1-\frac{1}{n^2}\right)^{kq}}{n^{Cq}} < \left(\frac{\sum_{n=2}^{k}\frac{1}{n}}{k}\right)^{1/4} \cdot \left(\frac{\left(\frac{3}{4}\right)^{\frac{3}{4}}}{3}\Gamma\left(\frac{2}{3}\right)\right)^{3/4} \cdot k^{-\frac{3}{4}} \tag{21}$$

2.2 Treating the second sum

We must find an upper bound to the sum

$$\sum_{n=k+1}^{\infty} \frac{\left(1-\frac{1}{n^2}\right)^{k}}{n^2}. \tag{22}$$

We can write

$$k^{\frac{3}{4}} \sum_{n=k+1}^{\infty} \frac{\left(1-\frac{1}{n^2}\right)^{k}}{n^2} = \sum_{n=k+1}^{\infty} \frac{k^{\frac{3}{4}}}{n^{\frac{3}{4}}} \frac{\left(1-\frac{1}{n^2}\right)^{k}}{n^{\frac{5}{4}}} \tag{23}$$

but

$$\sum_{n=k+1}^{\infty} \frac{k^{\frac{3}{4}}}{n^{\frac{3}{4}}} \frac{\left(1-\frac{1}{n^2}\right)^k}{n^{\frac{5}{4}}} < \sum_{n=k+1}^{\infty} \frac{\left(1-\frac{1}{n^2}\right)^k}{n^{\frac{5}{4}}} < \zeta\left(\frac{5}{4}\right) \tag{24}$$

where ζ is the Riemann Zeta function. Therefore

$$\sum_{n=k+1}^{\infty} \frac{\left(1-\frac{1}{n^2}\right)^k}{n^2} < \zeta\left(\frac{5}{4}\right) \cdot k^{-\frac{3}{4}}. \tag{25}$$

2.3 Putting the two results together

$$q_k < \left[\left(\frac{\sum_{n=2}^{k} \frac{1}{n}}{k}\right)^{1/4} \cdot \left(\frac{\left(\frac{3}{4}\right)^{\frac{3}{4}}}{3}\Gamma\left(\frac{2}{3}\right)\right)^{3/4} + \zeta\left(\frac{5}{4}\right)\right] \cdot k^{-\frac{3}{4}} \tag{26}$$

or

$$q_k < \left[\left(\frac{\left(\frac{3}{4}\right)^{\frac{3}{4}}}{3}\Gamma\left(\frac{2}{3}\right)\right)^{3/4} + \zeta\left(\frac{5}{4}\right)\right] \cdot k^{-\frac{3}{4}} \tag{27}$$

where $\delta(c) = 0$ for $C = 3$. Consequently $q_k = O(k^{-\frac{3}{4}})$ or in alternative notation $q_k << k^{-\frac{3}{4}}$. By Báez theorem RH is true and the zeroes are simple.

3 Conclusion

After the efforts of several mathematicians and scientific disseminators [3], the problem has reached maturity and can be solved.

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References

[1] B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Berliner Akademie, 1859. Translation: R. Wilkins, David. On the Number of Prime Numbers less than a Given Quantitiy, 1998.

[2] Luiz Báez-Duarte. A sequential Riesz-like criterion for the Riemann hypothesis, International Journal of Mathematics and Mathematical Sciences 2005:21 (2005) 3527-3537.

[3] John Derbishire. Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics, Joseph Henry Press (1 april 2003).