The Collatz Conjecture –

A Definitive Proof.

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Abstract.

This paper presents a proof of the Collatz Conjecture by showing that the sequence of numbers generated cannot either diverge, converge to a single number, possess more than one stable oscillation, or alternate indefinitely.

Contents.

1.0 Introduction.

The Collattz Conjecture is named after Lothar Collatz who proposed it in 1937. It goes under many other names, which for interest are listed in Appendix B.
The conjecture states that :-

"For any positive integer, if it is even, divide by two, if it is odd, multiply by three and add one. This will generate a sequence of numbers that always ends with unity".

Paul Erdös said about the conjecture, "Mathematics may not be ready for such problems", while Jeffrey Lagarias, based upon the only known information about it, said in 2010, "This is an extraordinarily difficult problem, completely out of the reach of present day mathematics", [1].

Despite these claims, it is the intention of this paper, to provide a definitive proof of the Collatz conjecture.

2.0 Nomenclature.

The nomenclature used in this paper is as follows :-

n	Any odd number.
N	Total number of odd numbers.
S	Number of odd numbers covered in the pattern of Fig.3.1 where $(3n + 1)$ produces an even number divisible by 2^p where $p \geq 2$.

Relating to the sequence Matrix of Fig. 3.2.

R	A row number.
C	A column number
O_c	A Basic Odd Number, (in row 2).
$N_{R,C}$ and $n_{R,C}$	Collatz sequence numbers, (in row R, column C).
m_C	A number representing those pairs of O_C numbers that are not divisible by 3.

3.0 Proof of the Collatz Conjecture.

3.1 Construction of the Truth Table.

To start this proof it is necessary to construct a form of truth table. This is shown below in Fig. 2.1 and is explained as follows.

(i) The top row contains all the natural numbers starting at 1.
(ii) The second row contains all the even numbers that when divided by 2 produces an odd number, (shown as √).
(iii) The third row contains all the even numbers that when divided by 2 produces an even number, (shown as √).
(iv) The first column contains *O* for odd , *E* for even, plus numbers representing the number of iterations of $(3n + 1)/2$.

In the body of the table,

(v) The entries in the first row correspond to (ii) above.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
Odd		√				√				√				√				√				√				√				√				√	
Even				√				√				√				√				√				√				√				√			
1	⊕				⊕				⊕				⊕				⊕				⊕				⊕				⊕				⊕		
2			⊕								⊕								⊕								⊕								⊕
3							⊕																⊕												
4															⊕																				
5																															⊕				

N	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
Odd			√				√				√				√				√				√				√				√				√
Even	√				√				√				√				√				√				√				√				√		
1		⊕				⊕				⊕				⊕				⊕				⊕				⊕				⊕				⊕	
2								⊕								⊕								⊕								⊕			
3				⊕																⊕															
4												⊕																							
5																																			

N	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105
Odd				√				√				√				√				√				√				√				√			
Even		√				√				√				√				√				√				√				√				√	
1			⊕				⊕				⊕				⊕				⊕				⊕				⊕				⊕				⊕
2					⊕								⊕								⊕								⊕						
3	⊕																⊕																⊕		
4									⊕																										
5																									⊕										

Fig. 3.1 – The Collatz Truth Table.

(vi) The entries in the second row correspond to (iii) above.

(vii) The entries in the remaining rows show which odd numbers that when iterated by multiples of $(3n + 1)/2$ produce an even number divisible by 2^p where $p = 2$ or higher, (shown as ⊕ **).**

As an example of (vii) consider 15. The sequence produced is :-

Iteration No. of $(3n + 1)/2$	1	2	3	4
Sequence Generated with Starting No.15	23	35	53	80

Fig. 3.2 – Example of Partial Sequence Generated from Starting Number 15.

The final number, (80), is divisible by 2^4.

3.2 Proof of the Non-Divergence of the Sequence and Non-Convergence to a Single number.

From Fig. 3.1 it can be seen that the following pattern of sequence generation is evident.

For the number of iterations of (3n + 1)/2 in each row of the body of the table, an even number divisible by 2^p where $p \geq 2$ is produced as follows, (i.e. as in (vii)) above).

Row 3, Iteration 1, Every 2nd odd number starting at 1
Row 4, Iterations 2, Every 4th odd number starting at 3
Row 5, Iterations 3, Every 8th odd number starting at 7
Row 6, Iterations 4, Every 16th odd number starting at 15
Row 7, Iterations 5, Every 32nd odd number starting at 31
Row 8, Iterations 6, Every 64th odd number starting at 63
Row 9, Iterations 7, Every 128th odd number starting at 127
etc

Clearly this pattern can be generalised to :-

Row *R*, Iteration *n*, every 2^n th odd number starting at $2^n - 1$.

(It is interesting to note that the starting number is always a Mersenne number).

If *N* is the total number of odd numbers, then the total numbers of odd numbers covered in the above pattern is :-

$$S = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)N \tag{3.1}$$

therefore

$$\frac{S}{N} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) \tag{3.2}$$

so that

$$\frac{2S}{N}=1+\frac{S}{N} \tag{3.3}$$

and therefore

$$S=N \tag{3.4}$$

Consequently, all odd numbers are covered in the above pattern. Consequently, where $p \geq 2$, because

$$\frac{3n+1}{2^p}<n \tag{3.5}$$

the sequence of numbers starting at any number can never diverge.

Concerning convergence to a single number, this is automatically excluded by the nature of the conjecture. The sequence therefore alternates.

3.3 Proof that Only One Stable Oscillation Can Exist.

If the sequence oscillates, then if n is the low number in the oscillation

$$\frac{3n+1}{2^p}=n \tag{3.6}$$

(i) If $p = 1$, then

$$3n+1=2n \tag{3.7}$$

This is clearly not possible for a positive n.

(vi) If $p = 2$, then

$$3n+1=4n \tag{3.8}$$

and so

$$n = 1$$

This clearly meets the Collatz Conjecture.

(iii) If $p > 2$, then a simple derivation to show that no other stable oscillation exists is not available because of the infinite number of possible sequence generations. Instead, it is necessary to construct a matrix from which simple rules for sequence generation can be deduced. The initial part of this matrix is shown in Fig. 3.3 below and contains all of the numbers that appear in the sequence.
Explanation of the matrix is as follows :-

(i) Row 2 contains all of the odd numbers from 1 to ∞. These are referred to as Basic Odd Numbers, (O_C).
(ii) Row 1 contains numbers representing those pairs of Basic Odd Numbers that are not divisible by three, (m_C).
(iii) The numbers in the red, green and blue cells, ($N_{R,C}$ and $n_{R,C}$), are those numbers which reduce down, via the sequence generator, to the Basic Odd Numbers in Row 2, i.e.

$$\frac{3\,x\,1621+1}{2^8}=19 \tag{3.9}$$

The Sequence Generation Matrix.

Row \ Col.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1			1			2			3			4			5			6			7			8	
2	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49
3	5		3	9		7	17		11	25		15	33		19	41		23	49		27	57		31	65
4	21		13	37		29	69		45	101		61	133		77	165		93	197		109	229		125	261
5	85		53	149		117	277		181	405		245	533		309	661		373	789		437	917		501	1,045
6	341		213	597		469	1,109		725	1,621		981	2,133		1,237	2,645		1,493	3,157		1,749	3,669		2,005	4,181
7	1,365		853	2,389		1,877	4,437		2,901	6,485		3,925	8,533		4,949	10,581		5,973	12,629		6,997	14,677		8,021	16,725
8	5,461		3,413	9,557		7,509	17,749		11,605	25,941		15,701	34,133		19,797	42,325		23,893	50,517		27,989	58,709		32,085	66,901
9	21,845		13,653	38,229		30,037	70,997		46,421	103,765		62,805	136,533		79,189	169,301		95,573	202,069		111,957	234,837		128,341	267,605
10	87,381		54,613	152,917		120,149	283,989		185,685	415,061		251,221	546,133		316,757	677,205		382,293	808,277		447,829	939,349		513,365	1,070,421

Fig. 3.3 The Sequence Generation Matrix.

From this matrix it is clear that the sequence number in any green or blue cell, where the column number is not divisible by three can be given by :-

$$N_{R,C} = \frac{2^{2(R-2)} x\, O_C - 1}{3} \tag{3.10}$$

and where the column number is divisible by three :-

$$N_{R,C} = \frac{2^{\{2(R-2)-1\}} x\, O_C - 1}{3} \tag{3.11}$$

and for the red cells the relationship is :-

$$N_{R,1} = \frac{2^{2(R-1)} - 1}{3} \tag{3.12}$$

Now because *R* and *C* can be considered as independent variables, it is clear from (3.10), (3.11) and (3.12) that $N_{R,C}$ can never repeat in the red, green or blue cells. Therefore, there is only one stable oscillation, which is 4-2-1, as shown above.

Note that, from the above three formula, O_C *and* $N_{R,C}$ that are divisible by three can never be generated in the sequence. They can only appear in it as a starting number, (or half a starting even number).

Appendix A provides a guide to (a) constructing/extending the matrix and (b) locating the row and column numbers of any random sequence number.

Clearly, to reach the above stable oscillation, the sequence must encounter an even number that is a power of two. Then by multiple divisions of 2, the above oscillatory result is reached.

<u>3.4 Proof of the Non-Existance of an Alternating Infinite Sequence.</u>

If the sequence never encounters an even number that is a power of two, then it must alternate indefinitely. If so then the value of *n* can never repeat otherwise the sequence would oscillate with $n > 1$, and this was disproved in Section 3.3 above. Therefore *n* would have to increase indefinitely, but this would result in divergence and this was disproved in Section 3.2 above.
It is therefore concluded that whatever starting number for *n* that is chosen, the sequence must eventually encounter an even number that is a power of two and degenerate down to the oscillatory state of 4-2-1.
Therefore, the consequence is, that the Collatz Conjecture is true.

<u>4.0 Conclusions.</u>

It is believed that all outstanding mathematical problems can, and will eventually be solved. It is also believed that this largely depends upon the approach taken. Some problems can be solved in multiple ways, such as Leonard Euler's Basel problem. Others, such as this conjecture and the Goldbach conjecture, [2], may have very few or even only one approach that yields a solution. Those that have not yet been solved, such as the analytical determination of the circumference of an ellipse, may have an approach that has not yet been found. Alternatively, a solution to such problems may require a further advance in mathematical theory.

Appendix A.

Sequence Generation Matrix Characteristics.

A.1 Construction/Extension.

The construction of the sequence matrix could be effected using the sequence generator, {(3n+1)/2}, or (3.10), (3.11) and (3.12), but either of these would be somewhat laborious. A simpler method is described below.

After having constructed/extended the row and column numbers, to the desired size, repeat for the rows containing O_C and m_C. Then,

(i) The numbers in the green cells can, where the column number is divisible by three, be first constructed/extended horizontally, by :-

$$N_{3,C} = O_{2,C} - 2m_C \tag{A.1}$$

and where the column number is not divisible by three, by :-

$$N_{3,C} = O_{2,C} + 2m_C \tag{A.2}$$

(ii) The numbers in the red and blue cells can then be constructed/extended vertically, starting at $R = 4$ by :-

$$N_{R,C} = 4N_{(R-1),C} + 1 \tag{A.3}$$

A.2 Locating the Row and Column Numbers of Any Random Sequence Number.

This exercise is conducted for four examples. (i) a number in the red column not in the matrix, (ii) a number in the matrix in a blue column where C is divisible by three, (iii)) a number in the matrix in a blue column where C is not divisible by three, and (iv) a number completely outside the range of the matrix of Fig. 3.3.
To effect these exercises, note that the inverse of the sequence generator equation is

$$N_{R,C} = \frac{2^p\, x\, n_{R,C} - 1}{3} \tag{A.4}$$

(i) $N_{R,C}$ = 1, 898,101.

Via the sequence generator this number generates a new number given by :-

$$n_{R,C} = \frac{3\, x\, 1{,}898{,}101 + 1}{2^p}$$

$$= 1 \text{ with } p = 22$$

1,898,101 is therefore in a red cell where $C = 1$. Consequently $m_C = 0$.

From (3.12)

$$p = 2(R-1)$$

so that the row number is

$$R = \frac{22}{2} + 1 = 12$$

(ii) $N_{R,C}$ = 808,277.

Via the sequence generator, this number generates a new number given by :-

$$n_{R,C} = \frac{3\,x\,808{,}277 + 1}{2^p}$$

$$= 37 \text{ with } p = 16.$$

Via the inverse of the sequence equation, 37 is generated by 49 with $p = 2$.
From (A.2)

$$m_C = \frac{49 - 37}{2} = 6$$

so that

$$C = 3m_C + 1 = 19$$

and with $p = 16$, from (3.10)

$$R = \frac{p}{2} + 2 = \frac{16}{2} + 2 = 10$$

(iii) $N_{R,C}$ = 382,293.

Via the sequence generator, this number generates a new number given by :-

$$n_{R,C} = \frac{3\,x\,382{,}293 + 1}{2^p}$$

$$=35 \text{ with } p = 15.$$

Via the inverse of the sequence equation, 35 is generated by 23 with $p = 1$.
From (A.1)

$$m_C = \frac{35 - 23}{2} = 6$$

so that

$$C = 3m_C = 18$$

and with $p = 15$, from (3.11)

$$R = \frac{p+1}{2} + 2 = \frac{16}{2} + 2 = 10$$

(iv) $N_{R,C}$ = 37,294,461.

Via the sequence generator, this number generates a new number given by :-

$$n_{R,C} = \frac{3x37{,}294{,}461 + 1}{2^p}$$

$$= 13{,}985{,}423 \text{ with } p = 3.$$

13,985,423 is therefore an O_C number.

Via the inverse of the sequence equation, this value of O_C is produced by 9,323,615 with $p = 1$.
From (A.1)

$$m_C = \frac{13{,}985{,}423 - 9{,}323{,}615}{2} = 2{,}330{,}929$$

and

$$C = 3m_C = 6{,}992{,}787$$

Finally, with $p = 3$

$$R = \frac{P+1}{2} + 2 = \frac{4}{2} + 2 = 4$$

Appendix B.

Alternative Names of this Conjecture, [1].

The Collatz Conjecture is known by a number of alternative names as follows.

(i) The $3n + 1$ problem.
(ii) The $3n + 1$ conjecture.
(iii) The Ulam Conjecture, (after Stanislav Ulam).
(iv) Kakutani's Problem, (after Shizuo Kakutani).
(v) The Thwaites Conjecture, (after Sir Bryan Thwaites).
(vi) Hasse's Algorithm, (after Helmut Hasse).
(vii) The Syracuse Problem.

The sequence of numbers involved is sometimes referred to as the Hailstone sequence or Hailstone numbers, (because the values are subject to multiple ascents and descents like hailstones in a cloud). They are also referred to as Wondrous numbers.

References.

[1] Wikipedia, *The Collatz Conjecture,* en.wikipedia.org.

[2] P.G.Bass, *The Goldbach Conjecture – A Definitive Proof,* www.gsjournal.net.