Golden Ratios and Golden Angles

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Abstract. In a p-sequence, every term is the sum of p *previous* terms given p initial values called *seeds*. It is an extension of the Fibonacci sequence. In this article, we investigate the p-golden ratio of p-sequences. We express a positive integer power of the p-golden ratio as a polynomial of degree $p-1$, and obtain values of golden angles for different p-golden ratios. We also consider further generalizations of the golden ratio.

1 Introduction

The Fibonacci sequence is a series of numbers, starting from 0 and 1, where every number is the sum of two previous numbers. It is named after the Italian mathematician Fibonacci who introduced it to the Western world in his book *Liber Abaci* in 1202. The ratio of two consecutive Fibonacci numbers approaches the *golden ratio* $\Phi = 1.618$. The Fibonacci numbers and the golden ratio are central concepts in modern mathematics. The golden ratio together with the Fibonacci numbers is often called the nature's code because it is observed in several natural phenomena. See, e.g., [1–13], and the references given therein for the theory of Fibonacci numbers and the golden ratio.

In this article, we present golden ratio (Φ_p) and golden angle ($\theta_g(p)$) associated with p-sequences, and consider other generalizations of golden ratio.

2 p-golden ratio

The *golden ratio* is one of the most famous numbers. Given a and $b(< a)$ two positive numbers, the golden ratio is defined as [1]

$$\frac{a}{b} = \frac{a+b}{a}. \tag{1}$$

Figure 1: Division of a line into p segments.

Taking $\frac{a}{b} = \Phi$, Eq. (1) reduces to the quadratic equation $\Phi^2 = \Phi + 1$ whose positive solution is $\Phi = \frac{\sqrt{5}+1}{2} = 1.61803$. This value corresponds to the limiting ratio value of the Fibonacci sequence.

For the golden ratios associated with p-sequences, we first ask a couple of questions: *(i) Does there exist a ratio, like golden ratio [Eq. (1)], for given $p \geq 3$ positive real numbers? (ii) What is the value of this ratio? Is this value unique? (ii) Is this value of ratio equal to the limiting ratio value of p-sequences?* Surprisingly enough, the answer is in affirmative.

Suppose $a_1 < a_2 < \cdots < a_p$ are $p \geq 2$ positive real numbers (see Fig. 1). We define the p-golden ratio as [1]

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{\sum_{k=1}^{p} a_k}{a_p} (= \Phi_p). \tag{2}$$

Note that Eq. (1) is a special case of Eq. (2) for $p = 2$.

2.1 Characteristic equation for Φ_p

We find that from Eq. (2) follows naturally the p-degree algebraic equation whose positive solution gives the value of Φ_p:

$$X_p(x) \equiv x^p - \sum_{k=0}^{p-1} x^k = 0. \tag{3}$$

We call this *golden equation*. Note that $X_p(0) = -1$ for all p and $X_p(1) = -(p-1)$. This equation has been obtained recently in an interesting physical problem concerning center of masses in two and higher dimensions [14].

[1] We will see later that actually $\frac{t_{n+1}}{t_n}$ is the golden ratio for large n. The relation of limiting ratio value of p-sequence with the Euclid's problem, Eq. (2), is accidental.

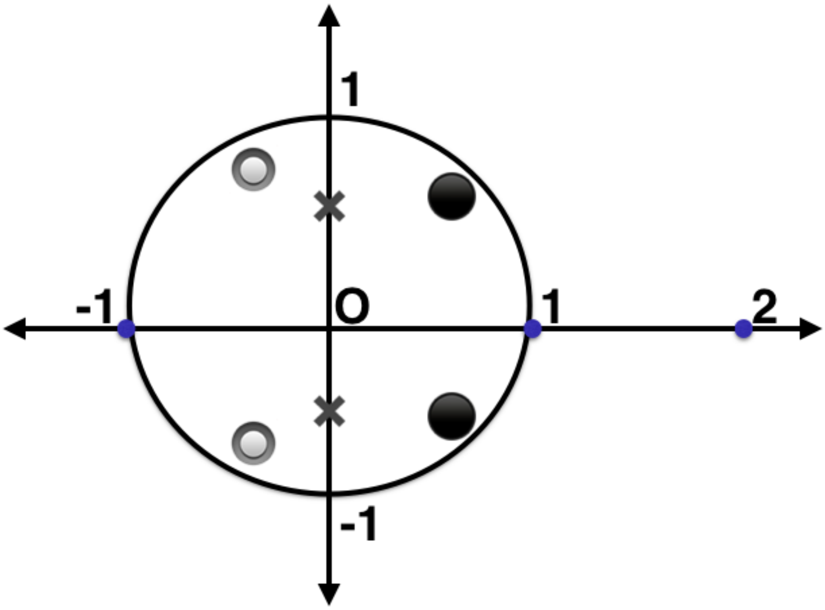


Figure 2: Schematic representation of roots of the *golden equation*: $x^p = \sum_{k=0}^{p-1} x^k$.

2.2 Less radical characteristic equations

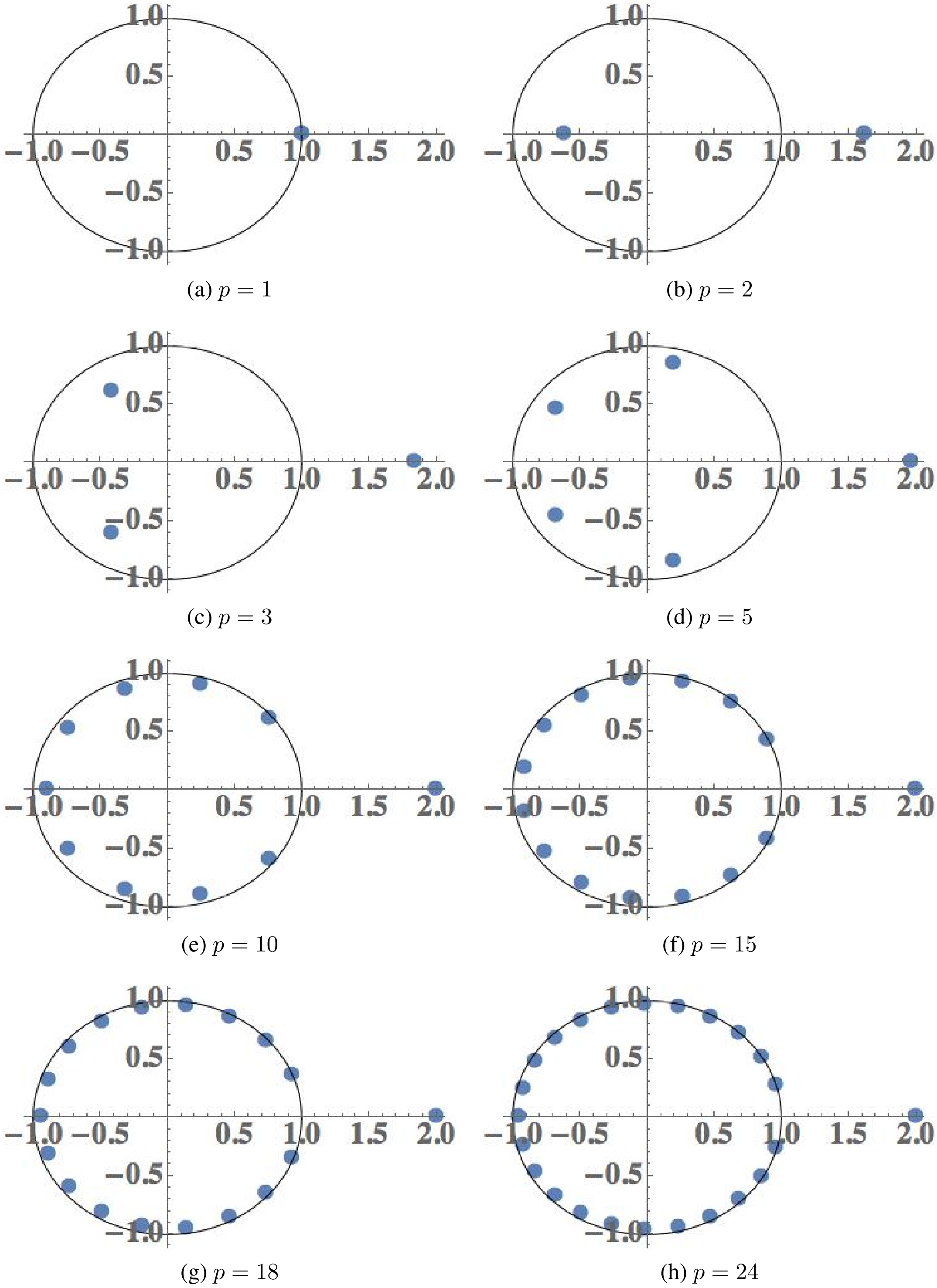
For fixed p and positive integers $\{k_i\}$, one can choose recurrence relations with m terms ($2 \leq m, k_i < p$) to obtain the following less radical characteristic equations,

$$\begin{aligned} x^p &= 1 + x^{k_1}, \\ x^p &= 1 + x^{k_1} + x^{k_2}, \\ x^p &= 1 + x^{k_1} + x^{k_2} + x^{k_3}, \end{aligned}$$

and so on, each with its own convergence. Wilson's *Meru 1* through *Meru 9* are particular examples of the above 2-term characteristic equation for $p = 2, 3, 4, 5$.

2.3 Roots of the golden equation

Here we look at the nature of roots of Eq. (3). Roots can be positive, negative and complex. Complex roots obviously occur in pairs and lie within a unit circle and approaches towards the boundary of the circle with increasing p. The only negative root approaches -1 for large p. The only positive root lies between 1 and 2, and tends to 2 for large p. See Figs. 2 and 3.



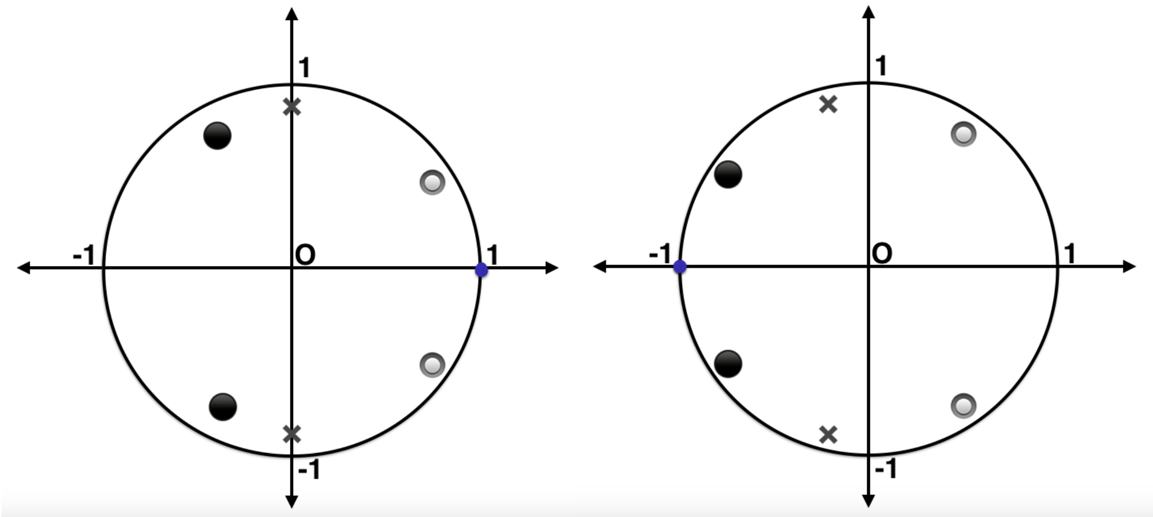
(a) $p = 1$ (b) $p = 2$

(c) $p = 3$ (d) $p = 5$

(e) $p = 10$ (f) $p = 15$

(g) $p = 18$ (h) $p = 24$

Figure 3: Roots of the *golden equation* for different values of p.



(a) $x^p = x^{p-1} - x^{p-2} + \cdots \pm 1$ (b) $x^p = -\sum_{k=0}^{p-1} x^k$

Figure 4: Schematic representation of roots of the near cousins of the golden equation.

2.4 Near cousins of the golden equation

The equations $(a)\ x^p = x^{p-1} - x^{p-2} + \cdots \pm 1$ and $(b)\ x^p = -\sum_{k=0}^{p-1} x^k$ are two immediate near cousins of the golden equation $x^p = \sum_{k=0}^{p-1} x^k$. Roots of both (a) and (b) are complex and real. Complex roots obviously occur in pairs and lie within a unit circle and approaches towards the boundary of the circle with increasing p. The only real positive root of (a) is $+1$, and the only real negative root of (b) is -1. See Fig. 4.

3 Recursion relation for Φ_p

Because Φ_p is a solution of Eq. (3), we have

$$\Phi_p^p = \Phi_p^{p-1} + \Phi_p^{p-2} + \cdots + \Phi_p + 1 = \sum_{k=0}^{p-1} \Phi_p^k, \tag{4}$$

$$\begin{aligned} \Phi_p^{p+1} &= \Phi_p^p + \Phi_p^{p-1} + \cdots + \Phi_p^2 + \Phi_p, \\ &= 2\Phi_p^p - 1. \end{aligned} \tag{5}$$

Eq. (4) can be equivalently rewritten as

$$\Phi_p = 1 + \frac{\sum_{k=0}^{p-2} \Phi_p^k}{\Phi_p^{p-1}}, \tag{6}$$

$$= 1 + \frac{1}{\Phi_p - 1 + \frac{1}{\sum_{k=0}^{p-2} \Phi_p^k}}. \tag{7}$$

Also, Eq. (4) implies a recursion relation

$$\Phi_p^n = \Phi_p^{n-1} + \Phi_p^{n-2} + \cdots + \Phi_p^{n-p} = \sum_{k=n-p}^{n-1} \Phi_p^k. \tag{8}$$

4 Φ_1 of 1-sequence

We have seen above that Eq. (4) is the basic equation for $\Phi_{p\geq 2}$. If we consider this sacred golden equation for $p = 1$, we have

$$\Phi_1 = \Phi_1^0 = 1. \tag{9}$$

We remark that Φ_1 is related with the limiting ratio value of 1-sequences. We construct a 1-sequence by choosing a seed $s_0 \geq 0$ and a constant $a \geq 0$ such that $t_0 = s_0$, and for $n \geq 1$

$$t_n = t_{n-1} + a = s_0 + na. \tag{10}$$

The limiting ratio value for this 1-sequence is then

$$\lim_{n\to\infty} \frac{t_{n+1}(1)}{t_n(1)} = \lim_{n\to\infty} \frac{s_0 + (n+1)a}{s_0 + na} = \lim_{n\to\infty} \left(1 + \frac{1}{n + \frac{s_0}{a}}\right) = 1 = \Phi_1. \tag{11}$$

We note that Eq. (9) provides the *lower limit* on Φ_p's. That is,

$$\Phi_{p\geq 1} \geq 1. \tag{12}$$

p	Φ_p	$\theta = \arcsin[\frac{\Phi_p - 1}{2}]$	$\theta = \arcsin[\Phi_p/2]$
1	1.0	0.0	30.0
2	1.61803	18.0	54.0
3	1.83929	24.8122	66.8742
4	1.92756	27.6313	74.5321
5	1.96595	28.8799	79.4124
6	1.98358	29.4583	82.6531
7	1.99196	29.7344	84.8608
8	1.99603	29.8688	86.3893
9	1.99803	29.9349	87.4567
10	1.99902	29.9676	88.2063
11	1.99951	29.9838	88.7317
12	1.99976	29.9921	89.1124
13	1.99988	29.996	89.3724
14	1.99994	29.998	89.5562
15	1.99997	29.999	89.6862
16	1.99998	29.9993	89.7438
17	1.99999	29.9997	89.8188
18	2.0	30.0	90.0
19	2.0	30.0	90.0
20	2.0	30.0	90.0

Table 1: Values of Φ_p, and the trigonometric angles such that $\Phi_p = 1 + 2\sin\theta = 2\sin\theta$.

5 Φ_p^n as a polynomial of degree $p-1$

We have seen earlier the following relations:

$$\Phi_p^p = \Phi_p^{p-1} + \Phi_p^{p-2} + \cdots + \Phi_p + 1,$$

and

$$\Phi_p^n = \Phi_p^{n-1} + \Phi_p^{n-2} + \cdots + \Phi_p^{n-p}.$$

Here the question we want to address is: *is it possible to reduce Φ_p^n to a polynomial of degree $p-1$?* Put differently, can we express Φ_p^n in terms of $\{\Phi_p^k\}_{k=0}^{p-1}$? It is very illuminating to see that it is possible to express Φ_p^n $(n \geq 0)$ in terms of $\{\Phi_p^k\}_{k=0}^{p-1}$ as follows:

$$\begin{aligned} \Phi_p^n &= t_n[S_{p-1}(p)]\Phi_p^{p-1} + \cdots + t_n[S_1(p)]\Phi_p + t_n[S_0(p)] \\ &= \sum_{k=0}^{p-1} t_n[S_k(p)]\Phi_p^k, \end{aligned} \tag{13}$$

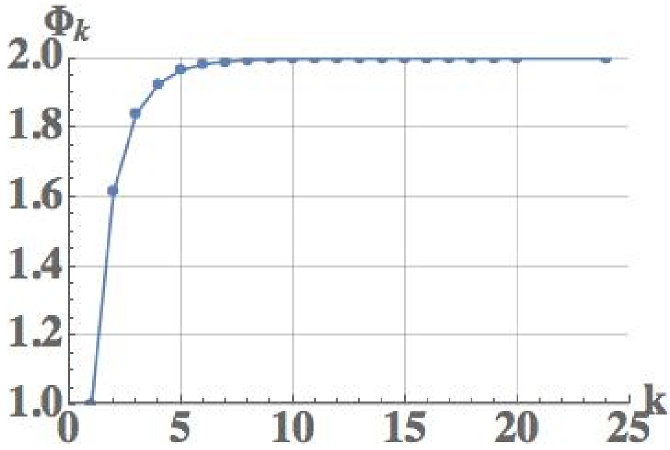


Figure 5: Plot of p-golden ratios.

where $t_n[S_k(p)] = \sum_{j=n-p}^{n-1} t_j[S_k(p)]$. Eq. (13) can be easily verified from Tables 2, 3, 4 and 5 [1].

In particular, for $p = 2$ and 3, the explicit expressiosn are

$$\Phi_2^n = \begin{cases} t_n[S_1(2)]\Phi_2 + t_n[S_0(2)] & (n \geq 0), \\ t_{n+1}[S_0(2)]\Phi_2 + t_n[S_0(2)] & (n \geq 2), \\ t_n[S_X(2)]\Phi_2 + t_{n-1}[S_X(2)] & (n \geq 2), \end{cases} \tag{14}$$

and

$$\Phi_3^n = \begin{cases} t_n[S_2(3)]\Phi_3^2 + t_n[S_1(3)]\Phi_3 + t_n[S_0(3)] & (n \geq 0), \\ t_{n-3}[S_S(3)]\Phi_3^2 + t_{n-2}[S_X(3)]\Phi_3 + t_{n-4}[S_S(3)] & (n \geq 4). \end{cases} \tag{15}$$

6 Applications of p-golden ratios

We have seen earlier that the golden ratio and the related Fibonacci sequence are present in abundance in our everyday life. We also learnt the skeptical view on this, and that not all objects exhibit the golden ratio in the sense that convergent limits do not settle down to the numerical value 1.618. This is now evident with the introduction of p-sequences and the associated p-golden ratios why it is not the case. In fact, $\Phi_2 = 1.618$ is only one member of several families of golden ratios (such as those of Stakhov, Spinadel, Krcadinac, etc. including the present work). Therefore, it is natural to expect that $\Phi_{p>2}$ will have many interesting applications as well.

n	$S_1(2)$	$S_0(2)$	$S_C(2)$	$S_S(2)$	$S_G(2)$
0	0	1	1	1	2
1	1	0	1	2	21
2	1	1	2	3	23
3	2	1	3	5	44
4	3	2	5	8	67
5	5	3	8	13	111
6	8	5	13	21	178
7	13	8	21	34	289
8	21	13	34	55	467
9	34	21	55	89	756
10	55	34	89	144	1223
11	89	55	144	233	1979
12	144	89	233	377	3202
13	233	144	377	610	5181
14	377	233	610	987	8383
15	610	377	987	1597	13564
16	987	610	1597	2584	21947
17	1597	987	2584	4181	35511
18	2584	1597	4181	6765	57458
19	4181	2584	6765	10946	92969
20	6765	4181	10946	17711	150427
21	10946	6765	17711	28657	243396
22	17711	10946	28657	46368	393823
23	28657	17711	46368	75025	637219
24	46368	28657	75025	121393	1031042
25	75025	46368	121393	196418	1668261

Table 2: 2-sequences. (i) $S_C \equiv S_1 + S_0$. (ii) $S_X = S_1$. (iii) $S_1 \sim S_0 \sim S_C \sim S_S$. (iv) S_G is a general 2-sequence with seeds $s_0 = 2,\ s_1 = 21$. (v) For each of these 2-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.61803$.

n	$S_2(3)$	$S_1(3)$	$S_0(3)$	$S_C(3)$	$S_X(3)$	$S_S(3)$
0	0	0	0	1	0	1
1	0	1	0	1	1	2
2	1	0	1	1	2	4
3	1	1	1	3	3	7
4	2	2	2	5	6	13
5	4	3	4	9	11	24
6	7	6	7	17	20	44
7	13	11	13	31	37	81
8	24	20	24	57	68	149
9	44	37	44	105	125	274
10	81	68	81	193	230	504
11	149	125	149	355	423	927
12	274	230	274	653	778	1705
13	504	423	504	1201	1431	3136
14	927	778	927	2209	2632	5768
15	1705	1431	1705	4063	4841	10609
16	3136	2632	3136	7473	8904	19513
17	5768	4841	5768	13745	16377	35890
18	10609	8904	10609	25281	30122	66012
19	19513	16377	19513	46499	55403	121415
20	35890	30122	35890	85525	101902	223317
21	66012	55403	66012	157305	187427	410744
22	121415	101902	121415	289329	344732	755476
23	223317	187427	223317	532159	634061	1389537
24	410744	344732	410744	978793	1166220	2555757
25	755476	634061	755476	1800281	2145013	4700770

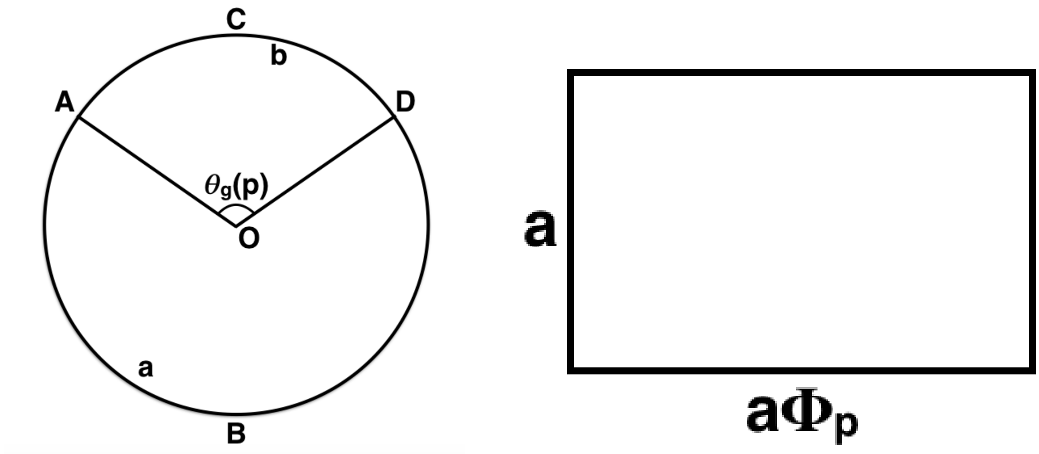
Table 3: 3-sequences. (i) $S_C \equiv S_2 + S_1 + S_0$. (ii) $S_2 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 3-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.83929$.

n	$S_3(4)$	$S_2(4)$	$S_1(4)$	$S_0(4)$	$S_C(4)$	$S_X(4)$	$S_S(4)$
0	0	0	0	1	1	0	1
1	0	0	1	0	1	1	2
2	0	1	0	0	1	2	4
3	1	0	0	0	1	3	8
4	1	1	1	1	4	6	15
5	2	2	2	1	7	12	29
6	4	4	3	2	13	23	56
7	8	7	6	4	25	44	108
8	15	14	12	8	49	85	208
9	29	27	23	15	94	164	401
10	56	52	44	29	181	316	773
11	108	100	85	56	349	609	1490
12	208	193	164	108	673	1174	2872
13	401	372	316	208	1297	2263	5536
14	773	717	609	401	2500	4362	10671
15	1490	1382	1174	773	4819	8408	20569
16	2872	2664	2263	1490	9289	16207	39648
17	5536	5135	4362	2872	17905	31240	76424
18	10671	9898	8408	5536	34513	60217	147312
19	20569	19079	16207	10671	66526	116072	283953
20	39648	36776	31240	20569	128233	223736	547337
21	76424	70888	60217	39648	247177	431265	1055026
22	147312	136641	116072	76424	476449	831290	2033628
23	283953	263384	223736	147312	918385	1592363	3919944
24	547337	507689	431265	283953	1770244	3068654	7555935
25	1055026	978602	831290	547337	3412255	5623572	14564533

Table 4: 4-sequences. (i) $S_C \equiv S_3 + S_2 + S_1 + S_0$. (ii) $S_3 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 4-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.92756$.

n	$S_4(5)$	$S_3(5)$	$S_2(5)$	$S_1(5)$	$S_0(5)$	$S_C(5)$	$S_X(5)$	$S_S(5)$
0	0	0	0	0	1	1	0	1
1	0	0	0	1	0	1	1	2
2	0	0	1	0	0	1	2	4
3	0	1	0	0	0	1	3	8
4	1	0	0	0	0	1	4	16
5	1	1	1	1	1	5	10	31
6	2	2	2	2	1	9	20	61
7	4	4	4	3	2	17	39	120
8	8	8	7	6	4	33	76	236
9	16	15	14	12	8	65	149	464
10	31	30	28	24	16	129	294	912
11	61	59	55	47	31	253	578	1793
12	120	116	108	92	61	497	1136	3525
13	236	228	212	181	120	977	2233	6930
14	464	448	417	356	236	1921	4390	13624
15	912	881	820	700	464	3777	8631	26784
16	1793	1732	1612	1376	912	7425	16968	52656
17	3525	3405	3169	2705	1793	14597	33358	103519
18	6930	6694	6230	5318	3525	28697	65580	203513
19	13624	13160	12248	10455	6930	56417	128927	400096
20	26784	25872	24079	20554	13624	110913	253464	786568
21	52656	50863	47338	40408	26784	218049	498297	1546352
22	103519	99994	93064	79440	52656	428673	979626	3040048
23	203513	196583	182959	156175	103519	842749	1925894	5976577
24	400096	386472	359688	307032	203513	1656801	3786208	11749641
25	786568	754784	707128	603609	400096	3257185	7443489	23099186

Table 5: 5-sequences. (i) $S_C \equiv S_4 + S_3 + S_2 + S_1 + S_0$. (ii) $S_4 \sim S_0 \sim S_S$. (iii) For each of these 5-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.96595$.



(a) (b)

Figure 6: The golden angle θ_g is determined by using $a/b = \Phi_p$.

p	1	2	3	18
Φ_p	1.0	1.61803	1.83929	2.0
$\theta_g(p)$	180°	137.5°	126.8°	120°

Table 6: The golden angles for p-sequences, $p = 1,\ 2,\ 3,\ 18$.

7 Golden geometry

7.1 Golden angles

The golden angle is defined as the acute angle θ_g that divides the circumference of a circle into two arcs ABD and ACD with lengths in the golden ratio. See Fig. 6(a). The golden ratio here satisfies $\Phi_p = \frac{a}{b}$. We then determine the golden angle by $\frac{\theta_g(p)}{2\pi} = \frac{b}{a+b} = \frac{1}{1+\frac{a}{b}} = \frac{1}{1+\Phi_p}$. Hence,

$$\theta_g(p) = \frac{2\pi}{1 + \Phi_p}. \tag{16}$$

From Table 6 we see that $\frac{2\pi}{3} \leq \theta_g(p) \leq \pi$.

7.2 Golden shapes

We can construct geometrical objects such as polygons (rectangle, pentagon, etc.) and spirals which have properties characterizing the golden p-ratio or certain p-sequences. Note that *a square is a golden rectangle with golden ratio* $\Phi_1 = 1$.

8 Further generalizations of golden ratio

The trouble with the notion of golden ratio is that it can be extended in many ways such that the original golden ratio Φ_2 is a particular case. In an earlier section, we have seen that the recurrence relation $t_n(p) = \sum_{k=1}^{p} t_{n-k}(p)$ and the golden ratio $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{\sum_{k=1}^{p} a_k}{a_p}$ correspond to the characteristic equation $x^p = \sum_{k=0}^{p-1} x^k$. A straight-forward generalization of these yield

$$t_n(p) = \sum_{k=1}^{p} c_k t_{n-k}(p), \tag{17}$$

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{\sum_{k=1}^{p} c_k a_k}{a_p}, \tag{18}$$

$$x^p = \sum_{k=0}^{p-1} c_k x^k. \tag{19}$$

That is, for a sequence of numbers whose terms are given by the (weighted) sum of its *consecutive p-previous terms*, the characteristic polynomial equation can be obtained by using the golden ratio. However, how do we obtain the characteristic polynomial equation for an arbitrary recurrence relation [2],

$$t_n = c_1 t_{n-m_1} + c_2 t_{n-m_2} + \cdots + c_p t_{n-m_p}, \tag{20}$$

otherwise? In this case also, we can project a ratio like the golden one, Eq. (18), as given below

$$x = \frac{t_{n-m+1}}{t_{n-m}} = \frac{t_{n-m+2}}{t_{n-m+1}} = \cdots = \frac{t_n}{t_{n-1}}, \tag{21}$$

where $m = \max\{m_1,\ m_2,\ \cdots,\ m_p\}$ so that

$$\begin{aligned} t_{n-m_k} &= x^{m-m_k} t_{n-m}, \quad (1 \le k \le p) \\ t_{n-1} &= x^{m-1} t_{n-m}. \end{aligned} \tag{22}$$

Then, the characteristic polynomial equation is [3]

$$x^m = c_1 x^{m-m_1} + c_2 x^{m-m_2} + \cdots + c_p x^{m-m_p}. \tag{23}$$

[2] Wilson's *Meru 1* through *Meru 9* with their limiting ratios (see [1]) are particular examples of Eq. (20).

[3] Proof of Eq. (23).

$$\begin{aligned} x t_{n-1} = t_n &= c_1 t_{n-m_1} + c_2 t_{n-m_2} + \cdots + c_p t_{n-m_p}, \\ \Rightarrow x(x^{m-1} t_{n-m}) &= (c_1 x^{m-m_1} + c_2 x^{m-m_2} + \cdots + c_p x^{m-m_p}) t_{n-m}, \\ \Rightarrow x^m &= c_1 x^{m-m_1} + c_2 x^{m-m_2} + \cdots + c_p x^{m-m_p}. \end{aligned}$$

We state a proposition below which gives us a straightforward general rule to obtain the characteristic polynomial equation for an arbitrary recurrence relation.
Proposition. The polynomial equation characteristic to a given recurrence relation is obtained by requiring $x^{u-v} := \lim_{n\to\infty} \frac{t_{n+u}}{t_{n+v}}$*, where* u *and* v *are integers. The characteristic equation is the minimal polynomial which gives the value of the limiting ratio of the sequence, and from which all its algebraic properties follow. For the generalized recurrence relation,* $t_n = c_1 t_{n-m_1} + c_2 t_{n-m_2} + \cdots + c_p t_{n-m_p}$*, the characteristic polynomial equation is given by* $x^m = c_1 x^{m-m_1} + c_2 x^{m-m_2} + \cdots + c_p x^{m-m_p}$*, where* $m = max\{m_1,\ m_2,\ \cdots,\ m_p\}$ [4] .

Moving a step further, we consider the relation

$$\big(u_1\frac{a_2}{a_1}\big)^{v_1} = \big(u_2\frac{a_3}{a_2}\big)^{v_2} = \cdots = \big(u_{p-1}\frac{a_p}{a_{p-1}}\big)^{v_{p-1}} = \big(u_p\frac{\sum_{k=1}^{p} c_k a_k}{a_p}\big)^{v_p}, \tag{24}$$

where $\{(u_i,\ v_i)\}$ and $\{c_k\}$ are given. Goal is to find values of the ratios $\{\frac{a_{k+1}}{a_k}\}$ and $\frac{\sum_{k=1}^{p} c_k a_k}{a_p}$ such that Eq. (24) holds. Does a solution exist? This problem is rather hard to solve in general.

Next, one can choose any pair of ratios at a time. Say, $\big(u_1\frac{a_2}{a_1}\big)^{v_1} = \big(u_2\frac{a_3}{a_2}\big)^{v_2}$. There are two cases here. (i) Assume that $\frac{a_2}{a_1} = x$ and $\frac{a_3}{a_2} = f_{23}(x)$. Then the characteristic equation is $\big(u_1 x\big)^{v_1} = \big(u_2 f_{23}(x)\big)^{v_2}$ and the positive solution is $x = \frac{1}{u_1}\big(u_2 f_{23}(x)\big)^{\frac{v_2}{v_1}}$. (ii) For $\frac{a_3}{a_2} = x$ and $\frac{a_2}{a_1} = f_{12}(x)$, the characteristic equation is $\big(u_1 f_{12}(x)\big)^{v_1} = \big(u_2 x\big)^{v_2}$ and the positive solution is $x = \frac{1}{u_2}\big(u_1 f_{12}(x)\big)^{\frac{v_1}{v_2}}$. Thus, equating two ratios at a time, we will have $2(p-1)!$ characteristic polynomial equations and consequently as many roots of them for given $\{(u_i,\ v_i)\}$ and $\{c_k\}$. To the best of our knowledge, most generalizations of the Fibonacci sequence and the golden ratio (see [1] and the references therein) can be seen as special cases of Eqs. (20), (23) and (24).

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[4] Another proof of Eq. (23).

$$\begin{aligned}
x &= \lim_{n\to\infty}\frac{t_{n+1}}{t_n}, \\
&= \lim_{n\to\infty}\frac{c_1 t_{n-(m_1-1)} + c_2 t_{n-(m_2-1)} + \cdots + c_p t_{n-(m_p-1)}}{t_n}, \\
&= c_1 \lim_{n\to\infty}\frac{t_{n-(m_1-1)}}{t_n} + c_2 \lim_{n\to\infty}\frac{t_{n-(m_2-1)}}{t_n} + \cdots + c_p \lim_{n\to\infty}\frac{t_{n-(m_p-1)}}{t_n}, \\
&= c_1 x^{-(m_1-1)} + c_2 x^{-(m_2-1)} + \cdots + c_p x^{-(m_p-1)}, \\
&= \frac{c_1 x^{m-m_1} + c_2 x^{m-m_2} + \cdots + c_p x^{m-m_p}}{x^{m-1}}, \\
\Rightarrow x^m &= c_1 x^{m-m_1} + c_2 x^{m-m_2} + \cdots + c_p x^{m-m_p}.
\end{aligned}$$

Thus, $\lim_{n\to\infty}\frac{t_{n+1}}{t_n}$ is the golden ratio in general.

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