A Graph-Theoretic Model of the Evolution of Natural Systems Whose Elements Inherit the Average Property Value of Their Neighbors

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29 September 2021

Abstract

In this article, we will consider a graph-theoretic model of the evolution of natural systems whose elements inherit the average property value of their neighbors (say, a physical system of heated objects). We will see that such systems always have “weak” analogues of the fixed points. Also, if the objects have the same number of neighbors, then the total property value of a system is invariant. Finally, there is a variation of our system that reaches static equilibrium.

Keywords

computer science
discrete mathematics
graph theory
natural systems
physical systems
cocktail
smoothie
trend system
1. Introduction

Given a positive integer \( n \), consider the following data:

- the set \( O = \{1, \ldots, n\} \) (objects)
- a vector \( p \in \mathbb{R}^n \) (the distribution of values of some property)
- the matrix \( R \) of a binary relation over \( O \) such that \( R \) contains no zero columns

Remark 1

\( R \) corresponds to a directed graph \( G_R = (O,E) \) without sources.

Definition 1

A tuple \( (O, p, R) \) with elements as above will be called a smoothie.

Definition 2

Let \( S = (O, p, R) \) be a smoothie and \( |O| = n \). Consider an \( n \times n \) matrix \( R' \) such that the following condition for the elements of \( R \) and \( R' \) holds:

\[
  r'_{i,j} = \frac{r_{i,j}}{\sum_{k=1}^{n} r_{k,j}}
\]

In other words, \( r'_{i,j} = r_{i,j} / \deg^- j \), where \( \deg^- j \) is the in-degree of \( j \) in \( G_R \).

The following infinite sequence \( \mathcal{P} \) of vectors from \( \mathbb{R}^n \) will be called the evolution of \( S \):

(i) \( \mathcal{P}_0 = p \)

(ii) for all \( i \in \mathbb{N} \), \( \mathcal{P}_{i+1} = \mathcal{P}_i R' \) (matrix multiplication)

Remark 2

Given a smoothie, \( \mathcal{P}_{i+1}(j) \) is the average property value of the in-neighbors of \( j \) in Step \( i \).
2. Invariant

Definition 3

A directed graph $G = (V,E)$ is called $d$-regular if the following condition for its in-degrees and out-degrees holds:

$$\forall (u,v \in V) [\text{deg}^- u = \text{deg}^+ u = \text{deg}^- v = \text{deg}^+ v = d]$$

Given a smoothie $S = (O,p,R)$, let us consider the following value that we may call the total property value of $S_i = (O, p_i, R)$:

$$\mu_i = \sum_{j \in O} p_j (j)$$

Theorem 1

Given a smoothie $(O,p,R)$, let $G_R$ be $d$-regular. Then $\mu_i = \mu_{i+1}$ for all $i \in \mathbb{N}$.

Proof

Definition 1 implies $d > 0$. Let $|O| = n$, then Definition 2 implies

$$\mu_{i+1} = (p_1(1)r_{1,1} + \cdots + p_1(n)r_{n,1}) + \cdots + (p_1(1)r_{1,n} + \cdots + p_1(n)r_{n,1})$$

$$= \left(\frac{r_{1,1}}{d} + \cdots + \frac{r_{n,1}}{d}\right) p_1(1) + \cdots + \left(\frac{r_{1,n}}{d} + \cdots + \frac{r_{n,n}}{d}\right) p_1(n) = 1$$

$$= \left(\frac{r_{1,1}}{d} + \cdots + \frac{r_{n,1}}{d}\right) p_1(1) + \cdots + \left(\frac{r_{1,n}}{d} + \cdots + \frac{r_{n,n}}{d}\right) p_1(n) = 2$$

$$= \frac{d}{d} p_1(1) + \cdots + \frac{d}{d} p_1(n) = p_1(1) + \cdots + p_1(n) = \mu_i$$

1 Since $\text{deg}^- j = d$
2 Since $\text{deg}^+ j = d$
3. Special points

Of course, Theorem 1 doesn’t mean that given an object \( j \) of a “regular” smoothie \((O, p, R)\) (such that \( G_\theta \) is \( d \)-regular), the property value of \( j \) won’t change, i.e., \( p_i(j) = p_{i+1}(j) = \cdots \). Though, Theorem 1 implies that for all \( i \in \mathbb{N} \), there exist such objects \( j, k \in O \) that \( j(k) \leq p_{i+1}(j) \) and \( j(k) \geq p_{i+1}(k) \). Indeed, if the first condition does not hold, then for all \( o \in O \), \( p_i(o) > p_{i+1}(o) \). This gives the contradiction \( \mu_i > \mu_{i+1} \). The same argument for \( p_i(k) \geq p_{i+1}(k) \) gives \( \mu_i < \mu_{i+1} \). Though, objects \( j \) and \( k \) can be defined explicitly and not only for the case when \( G_\theta \) is \( d \)-regular:

\[
\begin{align*}
j &= \min \{ p_i(o) : o \in O \} \\
k &= \max \{ p_i(t) : t \in O \}
\end{align*}
\]

The claim follows from Remark 2.

Theorem 2

Given a smoothie \((O, p, R)\), for all \( i \in \mathbb{N} \), there exist objects \( j, k \in O \) such that \( p_i(j) \leq p_{i+1}(j) \) and \( p_i(k) \geq p_{i+1}(k) \).

Definition 4

Given a smoothie \((O, p, R)\), an object \( o \in O \) will be called the Step \( i \) growth / decline point if \( p_i(o) \leq p_{i+1}(o) \) / \( p_i(o) \geq p_{i+1}(o) \). The intersection of the sets of growth and decline points will give us the set of fixed points.
4. Equilibrium

**Definition 5**

Given a smoothie $S = (O, p, R)$, let us say that $S$ reaches equilibrium if $\mathcal{P}$ is periodic, that is

$$\exists k \forall (l \geq 1) \mathcal{P}_{h+l} = \mathcal{P}_{h+(l \mod k)} \quad (k, l \in \mathbb{N})$$

The minimal number $k$ such that there is $l \geq 1$ satisfying 1 will be called the time of stabilization and denoted as $\tau$. Given $\tau$, the minimal number $l \geq 1$ such that $\forall (P_{h+l} = P_{h+(l \mod k)})$ will be called the period of stabilization and denoted as $\theta$. The equilibrium will be called static if $\theta = 1$ (i.e., the set of fixed points coincides with $O$ in some Step $i$), and dynamic otherwise. If $S$ reaches static equilibrium, we will say $S$ is stable.

There is a natural way to generalize the notion of smoothie such that it includes the directed graphs with sources. Then the special case of graphs without cycles will give us an example of static equilibrium. For this, replace $R$ in Definition 1 with the matrix of an arbitrary binary relation over $O$. Further, in Definition 2, put

$$(2) \begin{cases} p_{i+1}(j) = \frac{1}{\deg^- j} \sum_{k \in \mathcal{V}} p_i(k)r_{k,j}, & \deg^- j \neq 0 \\ p_{i+1}(j) = 0, & \deg^- j = 0 \end{cases}$$

That is, $\mathcal{P}$ behaves the same if $\deg^- j \neq 0$.

**Definition 6**

A tuple $(O, p, R)$ and the sequence $\mathcal{P}$ as above will be called a cocktail and its evolution, respectively.

Definitions 4 and 5 will be used for cocktails too.

Now, let us recall that a chain of a directed graph $G = (V, E)$ is a sequence $v_1, ..., v_n$ ($n \geq 1$) of its pairwise distinct vertices such that $(v_i, v_{i+1}) \in E$. Also, a chain $v_1, ..., v_n$ is called a cycle if $(v_n, v_1) \in E$.

Given a cocktail $C = (O, p, R)$, if $G_C$ has no cycles, then $C$ is not a smoothie. Indeed, if the in-degrees of $G_C$ are all non-zero, then the maximal length chain of $G_C$ induces a cycle.
Lemma 1

Given a directed acyclic graph $G = (V, E)$, all vertices of $G$ are sources, or there is a vertex $v \in V$ with non-zero in-degree such that all its in-neighbors are sources.

Proof

Suppose not all vertices of $G$ are sources. Let $Y \subseteq V$ be the set of non-sources. Thus $Y \neq \emptyset$. Of course, $X = V \setminus Y$ is the set of sources. Suppose $Y$ contains no such vertex $y$ that all in-neighbors of $y$ are from $X$. Thus each vertex of $Y$ has an in-neighbor from $Y$. In other words, $Y$ induces the subgraph $\langle Y \rangle$ without sources. The latter implies that $\langle Y \rangle$ has a cycle, which gives a contradiction. Thus there is a vertex $(y \in Y)$ with non-zero in-degree such that all its in-neighbors are sources.

Theorem 3

Given a cocktail $(O, p, R)$ such that $G_R = (O, E)$ is acyclic, $\rho_{i} = (0, \ldots, 0)$ for some $i \leq \lvert E \rvert + 1$.

Proof

Let us prove the claim by induction on $q = \lvert E \rvert$. Consider the base case $q = 0$. Then 2 implies $\rho_1 = (0, \ldots, 0)$. Now, suppose $G_R$ has $q + 1 \geq 1$ arcs and the statement is true for the cases of $\leq q$ arcs. There is a vertex of $G_R$ with non-zero in-degree. Thus Lemma 1 implies there is a vertex $v \in O$ with non-zero in-degree such that all its in-neighbors are sources. Without loss of generality suppose $\{1, \ldots, k\}$ is the set of in-neighbors of $v$. Then 2 implies

$$
\rho_1(1) = 0, \ldots, \rho_1(k) = 0
$$

$$
\rho_2(1) = 0, \ldots, \rho_2(k) = 0
$$

$$
\vdots
$$

Then 2 gives $\rho_2(v) = \rho_3(v) = \cdots = 0$ (3).

Now, consider the cocktail $C^p = (O, p, R^p)$ such that $G_{R^p} = (O, E \setminus \{(1, v), \ldots, (k, v)\})$. Let $\rho^p$ be the evolution of $C^p$. Using induction and 3 one may prove $\rho_i^p = \rho_{i+1}$ for all $i \in \mathbb{N}$.

On the other hand, the induction hypothesis implies $\rho_i^p = (0, \ldots, 0)$ for some $i \leq q + 1 - k + 1 \leq q + 1$. This means $\rho_{i+1} = (0, \ldots, 0)$, i.e., $\rho_j = (0, \ldots, 0)$ for some $j \leq q + 2$. This finishes our proof.
Corollary 1

Given a cocktail $C = (O, p, R)$ such that $G_R = (O, E)$ is acyclic, $C$ is stable and $\tau \leq |E| + 1$. 
5. Reflections

We see that invariants, special points, and stabilization are quite natural concepts when modeling natural systems. To this is added the number and size of connected regions with elements sharing the same quality (or having close property value). The latter characteristic was studied in [1] when the author considered the systems whose elements inherit the most common property of their neighbors.

References