Evaluating the Alignment of the Polarized Radio Waves from 13 QSOs in Ursa Major

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Abstract

A sample of 13 quasi-stellar objects, QSOs, with polarized radio emissions and located in the Southern part of Ursa Major is shown by the Hub Test to have significantly aligned polarization directions. The QSOs are taken from the JVAS1450 subset of the JVAS/-CLASS 8.4-GHz surveys. The Hub Test evaluates alignment indirectly by extending the sources’ polarization directions around the Celestial Sphere and quantifying the degree of convergence of these geodesics, i.e. great circles, at points on the Celestial Sphere. The hub of best convergence is found to be close to the sources. About one in 50,000 randomly directed samples would be better aligned than the polarization directions of these 13 QSOs. Some underlying calculations are presented in a Mathematica-coded Appendix. Access to a ready-to-run version is provided.

Keywords: Polarized Radio Sources; Alignment; Quasi-stellar objects

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0. Preface

The pdf version of this notebook is available online from the viXra archive. To find the ready-to-run notebook follow the link in Ref. 1. The notebooks in this series were created using Wolfram Mathematica, Version Number: 12.1, Ref. 2.

Note(s):

(1) Random numbers should be reliable. Thus, numerical quantities in the pdf version should differ from the live ready-to-run version in Ref. 1. Different sets of random runs, for a sufficiently large number of runs, should provide numerical values that differ only slightly.

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1. Introduction to Part I

Quasi-Stellar Objects, QSOs or quasars, can be polarized, making them candidates for studying correlations of polarization alignment. Large scale alignments are found for both optical and radio quasi stellar objects (QSOs), Refs. 3,4,5. In some studies, the tests that determine significant alignment compare the polarization direction of the electromagnetic radiation from one of the QSOs with one or more of its neighbors. An example of the potential value of such research is the finding of correlations between polarization directions and the local large scale structure, Refs. 6,7.

The Hub Test does not compare polarizations directly with each other, but indirectly, by finding points of convergence of the great circle geodesics obtained by extending polarization directions around the Celestial Sphere. Places where the geodesics are most dense are called “hubs” much as International Travel Hubs are places where the paths of passenger jets converge. Some other studies, Refs. 8,9, employ the Hub Test that is used here.

All tests, direct or indirect, serve to add to the information defining the behavior of QSOs. The tests inform Large Scale Structure, as noted above, as well as possibly intergalactic magnetic fields, Ref. 10, the properties of these objects, and other topics of interest.
2. Sample selection and the Hub Test

The sample of 13 QSOs in this report are taken from the JVAS1450, Ref. 11,12, a catalog of 1450 QSOs that was kindly communicated to me by one of the authors of Ref. 11. Details of the dataset can be found in Ref. 11. As explained in Ref. 11, the JVAS1450 catalog builds on data from the earlier large JVAS/CLASS 8.4-GHz catalog, Ref. 13.

To find candidate samples in the JVAS1450 to study, a survey was conducted. The QSO sources were binned, assigned to $5^\circ$ radius circular regions centered on the grid points of a $2^\circ$ mesh. A minimum of seven sources was enforced. The regions were sorted by the significance of their alignments according to the Hub Test. A previous report, Ref. 8, evaluated a clump of 27 QSOs, Clump 1 in Fig. 1, found in the overlap of eight of the $5^\circ$ regions.

In this report we investigate a second clump, ‘Clump 2’, of QSOs inhabiting the overlap of three significantly aligned regions somewhat North of Clump 1. The 13 QSOs are all the sources in the JVAS1450 catalog that have RA and dec in the ranges $161.86^\circ < RA < 179.62^\circ$ and $44.34^\circ < \text{dec} < 53.60^\circ$ and are located within $6.494^\circ$ from the sample center at $(\text{RA,dec}) = (171.445^\circ, 48.678^\circ)$.

The alignment of these 13 QSOs is evaluated with the Hub Test.

![Equatorial Coordinate System with Clumps 1 and 2](image)

Figure 1. Survey of some polarized radio QSOs. (Equatorial Coordinates, centered at $(\alpha,\delta) = (180^\circ, 0^\circ)$, East to the right.) The 1450 QSOs were grouped into $5^\circ$ radius regions centered on grid points. Those regions having at least 7 QSOs are plotted as gray dots. Just 35 regions showed very significant alignment, i.e. $S \leq 0.01 = 10^{-2}$, or, equivalently, $-\log_{10} S \geq 2.0$, and these are shaded in color. Clump 1 has 14 regions containing 27 QSOs and is analyzed elsewhere, Ref. 8. Clump 2 has 3 regions containing 13 QSOs and is selected for analysis here. Clump 3 remains unidentified.

The Hub Test is discussed more fully in Ref. 14. The basic idea is analogous to a well-known prescription for finding Polaris, the North Star. Assume one can find the stars Merak and Dubhe which are two stars in the constellation Ursa Major. Then the direction from Merak to Dubhe aligns with the direction from Merak to Polaris. In analogy with Fig. 2, let the source $S$ be the star Merak, take the direction from Merak to Dubhe to be the direction of polarization $\hat{v}_\psi$, and let Polaris be the point $H$. Then the alignment of the Merak-to-Dubhe direction $\hat{v}_\psi$ with the direction toward Polaris, the point $H$, illustrates the concept of alignment in the Hub Test. The alignment angle $\eta$ would be about $\eta = 3.47^\circ$ and the blue great circle would almost coincide with the purple great circle.
Figure 2: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source $S$. The linear polarization direction $\mathbf{v}_\psi$ lies in the tangent plane and determines the purple great circle on the sphere. A point $H$ on the sphere together with the point $S$ determine a second great circle, the blue circle drawn on the sphere. Clearly, $H$ and $S$ must be distinct in order to determine a great circle. The angle $\eta$ measures the alignment of the polarization direction $\psi$ with the point $H$.

In Fig. 2, the “alignment angle” $\eta$ is the acute angle $\eta$ between two great circles at $S$, $0^\circ \leq \eta \leq 90^\circ$. The alignment angle $\eta$ measures how well the polarization direction $\mathbf{v}_\psi$ matches the direction $\mathbf{v}_H$ toward the point $H$. Perfect alignment occurs when $\eta = 0^\circ$ and the two great circles overlap. Perpendicular great circles, $\eta = 90^\circ$, indicates maximum “avoidance” of the polarization direction $\mathbf{v}_\psi$ with the point $H$ on the sphere. The halfway value, $\eta = 45^\circ$, favors neither alignment nor avoidance.

With $N$ sources $S_i$, $i = 1, \ldots, N$, there are $N$ alignment angles $\eta_{ih}$ at each point $H$. One can calculate an average alignment angle $\bar{\eta}$ at $H$,

$$\bar{\eta}(H) = \frac{1}{N} \sum_{i=1}^{N} \eta_{ih},$$

where

$$\cos(\eta_{ih}) = |\mathbf{v}_\psi \cdot \mathbf{v}_H|.$$

Each angle $\eta_{ih}$ is taken to be the acute angle solving (2). Then the average alignment angle $\bar{\eta}(H)$ at the point $H$ must also be acute.

The alignment angle $\bar{\eta}(H)$ is a function of position $H$ on the sphere. It is symmetric across diameters, $\bar{\eta}(H) = \bar{\eta}(-H)$, because great circles are symmetric across diameters. The function $\bar{\eta}(H)$ measures convergence and divergence of the great circles determined by the polarization directions. For random polarization directions, the average $\bar{\eta}(H)$ should be near $45^\circ$, since each alignment angle $\eta_{ih}$ is acute, $0^\circ \leq \eta_{ih} \leq 90^\circ$, and random polarization directions should not favor any one value. Points $H$ where the alignment angle $\bar{\eta}(H)$ is smaller than $45^\circ$, the great circles tend to converge, where $\bar{\eta}(H)$ is larger than $45^\circ$, the great circles can be said to diverge.

In this article and notebook, we often use “min” to label the smallest alignment angle $\eta_{\text{min}}$ and the associated points on the sphere, the “hubs” $H_{\text{min}}$ and $-H_{\text{min}}$. Thus “min” is associated with convergence of the polarization directions. For divergence, the hubs $H_{\text{max}}$ and $-H_{\text{max}}$ locate places where the polarization directions avoid, as indicated by the largest alignment angle $\eta_{\text{max}}$. Thus, we very often label an avoidance related quantity with “max”.

3. The alignment of the polarization directions for the 13 QSOs

For the 13 sources considered in this report, the alignment angle function $\bar{\eta}(H)$ makes the following contour map. The global and local maps are computed in the Mathematica program below in Part II, Secs. 5b,c.
Equatorial Coordinate System

Figure 3: The alignment angle function $\eta(H)$ mapped on the Celestial Sphere (Aitoff plot, centered on $(\alpha, \delta) = (180^\circ, 0)$, East to the right). The QSOs are shaded green. To guide the eye, two Great Circles are plotted in gray, one through the sources’ center point and the avoidance hubs $H_{\text{max}}$ and $-H_{\text{max}}$ while the other Great Circle runs through the sources’ center and the alignment hubs $H_{\text{min}}$ and $-H_{\text{min}}$. The circles cross at an angle of $105^\circ$. The smallest alignment angle, $\eta_{\text{min}} = 10.86^\circ$, is located at the hubs $H_{\text{min}}$ and $-H_{\text{min}}$, where the polarization directions converge best. One alignment hub $H_{\text{min}}$ is located very close to the QSOs.

Figure 4: The region near the QSOs. The QSOs are located at the green dots. The short black lines through the QSOs indicate the polarization directions. Two of the QSOs are so close to the hub $H_{\text{min}}$ that it is difficult to distinguish the “X” at the hub from the polarization direction markers. Measuring polarization directions $\psi$ clockwise from North, one sees that the angles $\psi$ range from above $\psi = 90^\circ$ for the northern-most QSOs to $45^\circ$ or so for the more southerly QSOs. The QSOs display parallax: all are in the general direction of the alignment hub $H_{\text{min}}$, but their directions depend on where they are located.

4. Experimental uncertainty

All experimental results include uncertainty. The maps above were drawn based on the values reported in the JVAS1450 catalog. The catalog also reports uncertainties in the polarization directions. In Part II Sec. 6, below, the uncertainties are carried through the calculations yielding the uncertainties in the results.

The uncertainties reported with the observed polarization directions are assumed to make normal distributions, i.e. Gaussians that integrate to unity. For example, one of the QSOs, the sixth one, has a measured polarization position angle of $\psi_{\text{obs}} \pm \sigma = 115.1^\circ \pm 7.6^\circ$. We take this to mean that the probability that the actual value of $\psi$ was not $\psi_{\text{obs}} = 115.1^\circ$, but some other value $\psi_1$, is given by the Gaussian

$$P(\psi_1) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\psi_1 - \psi_{\text{obs}}}{\sigma} \right)^2 \right].$$

(3)
The Mathematica software has a special command, “RandomVariate”, that produces random values of $\psi_1$ with respect to the probability distribution in Eq. (3). Thus, an “uncertainty run” begins by selecting a set of polarization directions for the 13 QSOs conforming to the uncertainty distributions like the one in Eq. (3). The alignment angle function $\eta(H)$ in Eq. (1) is evaluated to find the smallest alignment angle $\eta_{\text{min}}$. As expected, the small changes to the observed polarization directions make small changes to the resulting angle $\eta_{\text{min}}$. By repeating the process many times, one obtains a distribution of values for the smallest alignment angle $\eta_{\text{min}}$.

The many uncertainty run values for the smallest alignment angle $\eta_{\text{min}}$ produce a distribution of the smallest alignment angle $\eta_{\text{min}}$, as well as the locations of alignment hubs. These distributions have corresponding mean values and distribution widths. See Fig. 5. The distribution of the uncertainty run values for the smallest alignment angle $\eta_{\text{min}}$ in Fig. 5 can be summarized by $\eta_{\text{min}} = 11.39^\circ \pm 1.07^\circ$. As noted previously, the recorded polarization directions $\psi_{\text{obs}}$ give the observed value, $\eta_{\text{min}} = 10.86^\circ$, and that value is in the range, $\eta_{\text{min}} = 11.39^\circ \pm 1.07^\circ$, determined by experimental uncertainty.

![Histogram of the smallest alignment angle $\eta_{\text{min}}$ for R = 10,000 uncertainty runs. The height $\Delta R$ is the number of uncertainty runs with a value of $\eta_{\text{min}}$ in the ‘bin’, the range covered by each bar. This Gaussian distribution peaks at a mean value of $\eta_{\text{min}}$ of 0.1988 radians = 11.39° and has a half-width of $\sigma = 0.0187055 = 1.07^\circ$ where the distribution is down from the peak by a fraction $e^{-1/2} = 0.607 = 60.7\%$. One writes the result as $\eta_{\text{min}} = 0.1988 \pm 0.0187$ radians = 11.39° ± 1.07°.](image)

Besides the uncertainty in the smallest alignment angle $\eta_{\text{min}}$, the uncertainty runs yield uncertainty ranges for other quantities such as the largest avoidance angle $\eta_{\text{max}}$. Each uncertainty run has its own set of alignment and avoidance hubs, $H_{\text{min}}$ and $H_{\text{max}}$, respectively. A plot of the polarization directions with their uncertainties and the locations of the uncertainty run hubs is displayed in Fig. 6.
5. Significance

Finally, we need to determine the significance of the alignment found for the polarization directions of these 13 QSOs. ‘Significance’ means how likely it is that randomly directed polarization vectors would give the same or better alignments than the observed polarization directions give.

To determine significance, we repeatedly find the smallest alignment angle function $\eta(H)$ many times, but with random $\psi$ for the 13 QSOs. The process is similar to the process that determines uncertainties in the previous section. Instead of experimental values of $\psi_{\text{obs}}$, one substitutes random $\psi$ for the 13 QSOs. The only experimental data used in this process is the location of the 13 QSO sources. The goal is to see what fraction of random runs yield a value with a lower $\eta_{\text{min}}$ than the value $\eta_{\text{min}} = 10.86^\circ$ obtained with the observed data.

Below, we deal with 10,000 random runs. By sorting those 10,000 runs by the value of $\eta_{\text{min}}$, smaller $\eta_{\text{min}}$ before larger $\eta_{\text{min}}$, one can find how many of those 10,000 runs gives a smaller alignment angle $\eta_{\text{min}}$ than the observed value of $\eta_{\text{min}}$, i.e. $\eta_{\text{min}} = 10.86^\circ$ using the recorded polarization directions $\psi_{\text{obs}}$ from the catalog. One and only one of the 10,000 runs is better. So the significance of $\eta_{\text{min}} = 10.86^\circ$ is about one in 10,000 or 0.0001, more or less. Clearly, we would need many more sets of 10,000 random runs for such considerations to produce a value of significance that we could assign a plus/minus, an uncertainty.

Rather than expending a large amount of computer time generating more random runs, we follow conventional practice and make do with the 10,000 random runs. We start by finding a function that fits the distribution of the 10,000 $\eta_{\text{min}}$, one smallest alignment angle $\eta_{\text{min}}$ per random run. Having found a function that fits the distribution, we make the assumption that the function...
accurately describes the distribution far down on the “tail” of the function where our well-aligned QSOs have their $\eta_{\text{min}}$.

A histogram of the resulting smallest alignment angles $\bar{\eta}_{\text{min}}$ from 10,000 runs is displayed in Fig. 7. Look closely at the distribution in Fig. 7. The right side, the side toward $\eta_{\text{min}} \rightarrow \pi/4 \sim 0.79$, has a steeper slope than the left side, the side toward $\eta_{\text{min}} \rightarrow 0$. Thus, the low $\eta_{\text{min}}$ side is favored; probability is pushed from the right side to the left side. A simple, symmetrical Gaussian would not fit the data well. The fitting curve shown combines a Gaussian with a unit step-function, that is unity to the left, and zero to the right, of the peak. Since the 13 QSOs have an alignment angle $\eta_{\text{min}}$ that is about 0.2 radians, it occurs far down the tail of the curve on the side where the step-function is unity and the curve is a Gaussian.

It is important for the application here to notice that the step-function is unity along the tail of the distribution on the left, $\eta_{\text{min}} \rightarrow 0$, side. The well-aligned sample of 13 QSOs has a smallest alignment angle around $\eta_{\text{min}} = 0.2$ radians, which is far down the tail, see the blue arrow in Fig. 7. The net effect of the steep right side of the distribution is to raise the probability of the observed $\eta_{\text{min}} = 0.2$ radians result by about 20%. Since random runs are thereby more likely in the region of the observed result, that makes the observed result somewhat less significant than if the distribution were symmetric.

Figure 7. The distribution of the smallest alignment angle $\bar{\eta}_{\text{min}}$ for $R = 10,000$ random runs. Each run assigns a random polarization direction to each of the 13 QSOs. The height $\Delta R$ is the number of runs with $\eta_{\text{min}}$ in the designated range of each bin. The fraction $\Delta R/R$ represents the likelihood that a random run result $\eta_{\text{min}}$ is in the bin. Thus the histogram approximates the shape of the probability distribution, aside from a normalizing scale factor. The observed polarization directions determine a value of $\eta_{\text{min}}$ at the blue arrow far down the tail.

To find the significance of the observed smallest alignment angle $\bar{\eta}_{\text{min}} = 10.86^\circ$, we integrate the probability distribution to find the likelihood that a random run would produce a smaller value. The significance is found to be $1.99 (30) \times 10^{-5}$ or about one in fifty thousand random runs would be better aligned than is experimentally observed for these QSOs. The alignment of the polarization directions with the hub $H_{\text{min}}$ is, therefore, very significant.

6. Conclusions

The polarization directions of these 13 QSOs are well-aligned with a point on the Celestial Sphere, the hub $H_{\text{min}}$, that is very close to the sample. Finding a correlation among polarization directions that display parallax is a property that distinguishes the Hub Test from other tests. Thus, the 13 QSOs offer a satisfying illustration of the Hub Test.

It is unlikely that the alignment is a consequence of selection bias. These 13 QSOs, Clump 2 in Fig. 1, are not alone; a sample of 27 QSOs, Clump 1, has been evaluated by the Hub Test. Clump 1 is better aligned than one in 80,000 random runs, while Clump 2 is
better aligned than one in 50,000 random runs. Since the survey of 5°-radius regions, Fig. 1, involves 1863 regions, it seems that the alignments are not due to selection bias. And the survey finds other locations with significant alignment, so there may be a Clump 3 waiting to be investigated.

While the article, Ref. 15, relating alignments to Large Scale Structure constrains the QSOs to have like-redshifts, one might argue that the alignment found in this article is due to a subset of the 13 QSOs with more-or-less equal redshifts. Then the alignment would speak to Large Scale Structures, as in Ref. 15.

Astronomical data is being acquired at fantastic rates, so there may be new catalogs of many more QSOs with linear polarization directions to analyze. Such an investigation would be intriguing.

The main motivation for this study is to illustrate an application of the Hub Test. Interpreting the results is deemed beyond the scope of this study, which is intended to be a simple application of a test of alignment. One hopes the results are of interest and potentially useful.

7. References

1. Shurtleff, R., the ready-to-run Mathematica version of this notebook is available at the following URL:
   https://www.dropbox.com/s/6zr8aneysic1zlv/20211009Clump2PaperFirst.nb?dl=0
12. The JVSA1450 catalog was emailed to me with kind thanks from V. Pelgrims.
17. The NASA/IPAC Extragalactic Database (NED), https://ned.ipac.caltech.edu/
Part II  Computer Program

1. Introduction to Part II

In[2] = Print["The computer time expended so far is ", TimeUsed[], " seconds."]

The computer time expended so far is 1.078 seconds.

The following computer program, a Mathematica notebook, performs the calculations made to evaluate the alignment of the sources in the sample under consideration.

Since Mathematica encodes the instructions, it is inconvenient to try to run the computer program from the pdf version of this work. A viable .nb version that runs on Mathematica is available by following the link in Ref. 1.

2. Coordinates, utility functions, derivation of basic formula

2a. Coordinates, utility functions

Consider the “Celestial Sphere”, a sphere with unit radius in 3 dimensional Euclidean space. See Figs. 1, 2, 3 in the article, Part 1 above. The sphere is also called the “sphere” or sometimes “the sky”. Picture the dome of a planetarium viewed from the outside. The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates (x, y, z). The direction of the positive z-axis is due “North”. Equatorial longitude is the Right Ascension $\alpha$ and latitude is the declination $\delta$.

Definitions:

homeDirectory directory containing the notebook and data files

Utilities:

er, eN, eE unit vectors in a 3D Cartesian coordinate system

($\alpha$, $\delta$) equatorial coordinates longitude and latitude

er($\alpha$, $\delta$) radial unit vectors from Origin

eN($\alpha$, $\delta$) local North at a point on the Celestial Sphere

eE($\alpha$, $\delta$) local East at a point on the Celestial Sphere

$\alpha$FROMer er $\alpha$ determined by a radial unit vector er

$\delta$FROMer er $\delta$ determined by a radial unit vector er

Aitoff Plot Functions:

$\alpha$HA($\alpha$, $\delta$), xH($\alpha$, $\delta$), yH($\alpha$, $\delta$), where xH is centered on $\alpha = 0$ and $\alpha$ increases from left-to-right, with $\alpha = -180^\circ$ on the left and $+180^\circ$ on the right

xH180($\alpha$, $\delta$), yH180($\alpha$, $\delta$), where xH is centered on $\alpha = 180^\circ$ and $\alpha$ increases from left-to-right, with $\alpha = 0^\circ$ on the left and $360^\circ$ on the right

mean the arithmetic average of a set of numbers, $\frac{1}{N} \sum_{i=1}^{N} n_i$
The quantity \( \alpha H \) is the RA coordinate of a point H on the Celestial Sphere. Thus, we use \( \alpha HA \) for Aitoff function.

For these formulas the angles \( \alpha \) and \( \delta \) should be in degrees.

They give an Aitoff Plot that is centered on \((0^\circ, 0^\circ)\).
Using the following functions produces an Aitoff Plot that is centered on (180°,0°)

\begin{align*}
\text{mean}_\text{data} & := \frac{1}{\text{Length}[\text{data}]} \sum_{i4} \text{data}[[i4]], \{i4, \text{Length}[\text{data}]\} \\
\text{stdDev}_\text{data} & := \left(\frac{1}{\text{Length}[\text{data}]} \sum_{i5} (\text{data}[[i5]] - \text{mean}[\text{data}])^2, \{i5, \text{Length}[\text{data}]\}\right)^{1/2} \\
\end{align*}

(\text{standard deviation})

2b. Derivation of a formula for the alignment angle \(\eta_{ih}\) given the position \(r_S\) of the \(i\)th source, the location \(r_H\) of point \(H\), and the polarization direction \(\psi\) for the \(i\)th source

From Fig 2b, we see that \(\cos\psi = \psi \cdot r_H\), Eq. 2.

\[
v_H = \frac{d_H - (r_H \cdot r_S)}{||d_H - (r_H \cdot r_S)||^2} : \text{unit vector in the 2D tangent plane at } S, \text{ in the direction of } H \text{ from } S, \psi \cdot r_S = 0, \text{ where} \\
\text{er}[\alpha H, \delta H].\text{er}[\alpha S, \delta S] = r_H \cdot r_S \text{ is the inner product of the radial unit vectors } r_H \text{ and } r_S \text{ to point } H \text{ and source } S
\]

Since \(\psi\) is also perpendicular to \(r_S\), it follows that \(\psi \cdot r_S = 0\), and we have \(\frac{d_H}{||d_H - (r_H \cdot r_S)||^2} = \frac{r_H}{r_H \cdot r_S}\) as the part of \(v_H\) that contributes to the dot product \(\cos\psi = \psi \cdot r_H\). Therefore, define

\[
v_{H\perp S} = \frac{d_H}{||d_H - (r_H \cdot r_S)||^2}
\]

Simplify the denominator,

\[
\text{denoSquared1} = \text{FullSimplify}\left[ (\text{er}[\alpha H, \delta H] - (\text{er}[\alpha H, \delta H] \cdot \text{er}[\alpha S, \delta S]) \cdot \text{er}[\alpha S, \delta S]) \cdot (\text{er}[\alpha H, \delta H] - (\text{er}[\alpha H, \delta H] \cdot \text{er}[\alpha S, \delta S]) \cdot \text{er}[\alpha S, \delta S]) \right];
\]

\[
\begin{align*}
\text{denoSquared} & = (r_H - (r_H \cdot r_S) \cdot r_S) \cdot (r_H - (r_H \cdot r_S) \cdot r_S) \\
& = r_H \cdot r_H - 2(r_H \cdot r_S) \cdot r_S \cdot r_S \\
& = 1 - 2(r_H \cdot r_S)^2 + (r_H \cdot r_S)^2 = 1 - (r_H \cdot r_S)^2
\end{align*}
\]

\[
\text{FullSimplify}\left[\text{denoSquared1} - \left(1 - (\text{er}[\alpha H, \delta H] \cdot \text{er}[\alpha S, \delta S])^2\right)\right]/\text{denoSquared} = 0
\]

Write the formula for the vector \(v_{H\perp S}\), with a denominator of \((1 - (r_H \cdot r_S)^2)^{1/2}\).
\( \text{Out[25]} = \text{vHperpS}[\text{aS}_-, \text{dS}_-, \text{aH}_-, \text{dH}_-] := \text{e}[\text{aH}, \text{dH}] / \left(1 - (\text{e}[\text{aH}, \text{dH}] \cdot \text{e}[\text{aS}, \text{dS}])^2\right)^{1/2} \)

\( \text{Out[26]} = \text{Simplify[vHperpS[\text{aH}, \text{dH}, \text{aH}, \text{dH}]]; } \star \text{ BANG, BOOM!! when } H = S \text{. See Fig. 2 for why this happens.} \star \)

... Simplify: Expression
\[ \frac{\text{Cos}[\text{aH}] \text{Cos}[\text{dH}]}{\sqrt{1 - (\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2}\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2}\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2]}} \]

... Simplify: Expression
\[ \frac{\text{Cos}[\text{dH}] \text{Sin}[\text{aH}]}{\sqrt{1 - (\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2}\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2}\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2]}} \]

... Simplify: Expression
\[ \frac{\text{Sin}[\text{dH}]}{\sqrt{1 - (\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2]\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2]\text{Power}[\text{e}[\text{aH}, \text{dH}]], \text{Power}[\text{e}[\text{aS}, \text{dS}])^2]}} \]

... General: Further output of Simplify::infd will be suppressed during this calculation.

The other vector we need is \( \psi \), the unit vector in the 2D tangent plane at \( S \) pointing in the direction of the polarization position angle \( \psi \). By Fig. 2b, one sees that

\[ \psi = \cos(\psi) \text{ N } + \sin(\psi) \text{ E}, \]

where \( N \) and \( E \) are local north and east unit vectors in the 2D tangent plane at \( S \).

\( \text{Out[27]} = \text{v}[\text{aS}_-, \text{dS}_-, \text{aH}_-, \text{dH}_-, \psi] := \text{Cos}[\psi] \text{ e}[\text{e}[\text{aS}, \text{dS}]) + \text{Sin}[\psi] \text{ e}[\text{e}[\text{aS}, \text{dS}]) \)

\( \star \text{v}[\text{aS}, \text{dS}, \text{aH}, \text{dH}, \psi] \star \)

The alignment angle \( \eta \) is the acute angle between \( \psi \) and \( \text{vH} \) in the 2D tangent plane at \( S \). By Eq. 2,

\( \text{Out[28]} = \text{etaH}[\text{aS}_-, \text{dS}_-, \text{aH}_-, \text{dH}_-, \psi] := \text{ArcCos[Abs}[\text{v}[\text{aS}, \text{dS}, \text{aH}, \text{dH}, \psi] \cdot \text{vHperpS}[\text{aS}, \text{dS}, \text{aH}, \text{dH}]) ] \]

\( \star \text{etaH}[\text{aS}, \text{dS}, \text{aH}, \text{dH}, \psi] \star \)

\( \text{FullSimplify[etaH}[\text{aS}, \text{dS}, \text{aH}, \text{dH}, \psi]] \)

\( \text{Out[29]} = \text{ArcCos[} \frac{\text{Cos}[\text{dS}] \text{Cos}[\psi] \text{Sin}[\text{dH}] + \text{Cos}[\text{dH}] (\text{Cos}[\text{aH} - \text{aS}] \text{Cos}[\psi] \text{Sin}[\text{dS}] + \text{Sin}[\text{aH} - \text{aS}] \text{Sin}[\psi])}{\sqrt{1 - (\text{Cos}[\text{aH} - \text{aS}] \text{Cos}[\text{dH}] \text{Cos}[\text{dS}] + \text{Sin}[\text{dH}] \text{Sin}[\text{dS}])^2}}} \]

\( \text{Out[30]} = \star \text{The following function is well-behaved everywhere except where } \pm \text{H coincides with } \pm \text{S.} \star \)

\( \text{etaHWithIndeterminate}[\text{aS}_-, \text{dS}_-, \text{aH}_-, \text{dH}_-, \psi] := \text{ArcCos[} \text{Abs}[\frac{\text{Cos}[\text{dS}] \text{Cos}[\psi] \text{Sin}[\text{dH}] + \text{Cos}[\text{dH}] (\text{Cos}[\text{aH} - \text{aS}] \text{Cos}[\psi] \text{Sin}[\text{dS}] + \text{Sin}[\text{aH} - \text{aS}] \text{Sin}[\psi])}{\sqrt{1 - (\text{Cos}[\text{aH} - \text{aS}] \text{Cos}[\text{dH}] \text{Cos}[\text{dS}] + \text{Sin}[\text{dH}] \text{Sin}[\text{dS}])^2}} \text{]} ] \)
(*Since \( \eta \) is an acute angle, let us take the halfway value,
\( \eta = \pi/4 \) in the neighborhood where \( H \approx S \).*)

\[
\eta_{iH}[aS_-, \delta S_-, aH_-, \delta H_-, \psi_-] := \\
\eta_{iH} with Indeterminate[aS, \delta S, aH, \delta H, \psi] \Big/ \left( \left( 1 - (er[aH, \delta H].er[aS, \delta S])^2 \right) \geq 0.000001 \right)
\]

\[
\eta_{iH}[aS_-, \delta S_-, aH_-, \delta H_-, \psi_-] := \pi/4 \Big/ \left( \left( 1 - (er[aH, \delta H].er[aS, \delta S])^2 \right) < 0.000001 \right)
\]

Print[
  "Thus \( \eta_{iH} = \pi/4 \) wherever \( \pm H \) is 'close' to \( \pm S \), with 'close' meaning within an angle of ",
  ArcSin[0.000001^{1/2}], " radians, or ", ArcSin[0.000001^{1/2}] \( \frac{360.}{2.\pi} \), "°."
]

Thus \( \eta_{iH} = \pi/4 \) wherever \( \pm H \) is 'close' to \( \pm S \), with 'close' meaning within an angle of
0.001 radians, or 0.0572958°.

3. Polarization and Position Data

3a. Source Data

The JVAS1450 catalog incorporates data from the large JVAS/CLASS 8.4 Ghz catalog Jackson 2007, Refs. 11,12,13. The JVAS1450 catalog sources were filtered from Jackson 2007 sources by identification as QSOs. Filters: for percent polarization, \( p > 0.6\% \), for the largest fractional uncertainty in percent polarization, \( \sigma_p/p < 0.6\% \), and for uncertainty in the polarization position angle \( \sigma_\psi < 16^\circ \).

Definitions:

data00 the catalog data, JVAS1450
secondClumpQsoIDimData001450 - record numbers in the catalog of the QSOs in the sample
nSrc number of sources
\( \alpha \) Src right ascension of the sources, longitude (radians)
\( \delta \) Src declination of the sources, latitude (radians)
\( \psi \) Src polarization position angle of the sources: clockwise from North with East to the right.
\( \sigma_\psi \) Src uncertainty in \( \psi \)
percentPol percentage of linear polarization of the sources
redshift redshift, no uncertainty reported
rSrc unit vectors from the Origin to Sources on Celestial Sphere
eNSrc Local North at each Source
eESrc Local East at each Source
\( \eta \) BarAtHwithAny\( \psi \) alignment angle function \( \eta(H) \), Eqn. 1, obtained using the location of the sources
sourceCenter unit radial vector to the arithmetic center of the sources
\( \alpha \) SourceCenter Right Ascension at the sourceCenter
\( \delta \) SourceCenter Declination at the sourceCenter
angleSourceToCenter angle from each Source to the sourceCenter
\( \rho \) RgnRadius angle to the furthest QSO from the sourceCenter
\( \rho \) RMS root-mean-square angular distance to the sources from the sourceCenter
Alternate names:
A position search of the NASA/IPAC Extragalactic Database (NED)*, Ref. 17, returned the following names of 13 QSOs whose position is coincident with those reported in the JVAS1450 catalog:

1. WISEA J104732.27+483531.1
2. B3 1048+470B (Redshift = 1.4194 JVAS1450, 1.8x10^-4 NED Sloan Digital Sky Survey)
3. WISE J105840.84+533543.1
4. B3 1108+454
5. WISEA J111740.33+525936.4
6. WISEA J112152.33+493225.5
7. SDSS J112337.12+504531.8, SDSS J112337.11+504531.8
8. B3 1124+455
9. B3 1140+466
10. B3 1143+446A
11. SBS 1149+499
12. SBS 1150+497
13. WISEA J115826.77+482516.1

Note the disagreement in the redshift values for object 2, B3 1048+470B. The other redshifts were nearly the same in both NED and JVAS1450.

These identifications are FYI, for your information. No data from the NED search is used in this notebook.

*The NASA/IPAC Extragalactic Database (NED) is funded by the National Aeronautics and Space Administration and operated by the California Institute of Technology.

```
In[36]:= (*right ascension in radians*)
αSrc = 10^-6*{2.825418, 2.841679, 2.874039, 2.928868, 2.956905, 2.975226,
            2.982850, 2.997432, 3.068298, 3.078944, 3.109077, 3.112830, 3.134812};

In[36]:= nSrc = Length[αSrc]
Out[36]= 13
```

```
In[37]:= (*declination in radians*)
δSrc = 10^-6*{848.089, 815.735, 935.416, 787.604, 924.911,
            864.641, 885.908, 790.083, 809.670, 773.869, 866.744, 864.271, 845.109};
```

```
In[38]:= (*position angle in radians*)
ψSrc = 10^-4*{1.293289, 1.328196, 1.930334, 0.925025, 1.759292,
           2.008874, 2.021091, 0.994838, 0.975639, 0.945968, 2.426008, 1.987930, 3.082251};
```
In[39]:= Histogram[\[psi\]Src \[\frac{360.}{2.\pi}\], (20), PlotLabel -> "PPA \[psi\], number \[DeltaR\] per bin",
AxesLabel -> "\[psi\]", "\[DeltaR\]"], PlotRange -> {{0, 200}, Automatic}]
Print["Figure 8: Distribution of position angles for the 13 polarization directions in the sample. Note the wide distribution over a hundred degrees or so, \[psi\] = 40° to \[psi\] = 150°, in two groupings."]

Figure 8: Distribution of position angles for the 13 polarization directions in the sample. Note the wide distribution over a hundred degrees or so, \[psi\] = 40° to \[psi\] = 150°, in two groupings.

In[41]:= (* uncertainty in \[psi\] in radians*)
\[alpha\]Src = \[10.\-6\]. {39697, 48409, 72563, 55071, 86756, 131967, 87055, 3977, 21712, 20791, 74085, 24677, 16969};

In[42]:= (* % polarization*)
percentPol = \[10.\-6\]. {2142363, 575196, 12801608, 4141751, 3722694, 2159228, 3458875, 1323236, 3206987, 2150994, 471406, 904146, 1224728};

In[43]:= (* uncertainty in % polarization*)
\[sigma\]percentPol = \[10.\-6\]. {170077, 55686, 1857703, 456149, 645884, 569853, 602180, 10524, 139250, 89438, 69845, 44620, 41561};

In[44]:= (*Redshift*)
redshift = \[10.\-6\]. {867000, 1419400, 1535100, 1492000, 1373300, 1875000, 2277500, 1819200, 1321800, 299800, 1094100, 333700, 2028000};

In[45]:= rSrc = Table[\[alpha\]Src[[i]], \[delta\]Src[[i]]], {i, nSrc}];(*calculated from Input.*)
eNSrc = Table[\[alpha\]Src[[i]], \[delta\]Src[[i]]], {i, nSrc}];(*calculated from Input.*)
eESrc = Table[\[alpha\]Src[[i]], \[delta\]Src[[i]]], {i, nSrc}];(*calculated from Input.*)

In[48]:= \[eta\]BarAtHwithAny\[psi\][\[alpha\]_, \[delta\]_, \[psi\]_] :=
\[\frac{1}{nSrc}\] Sum[\[eta\]H[\[alpha\]Src[[i]], \[delta\]Src[[i]], \[alpha\]H, \[delta\]H, \[psi\][[i]]], {i, nSrc}]
(*\[eta\]BarAtHwithAny\[psi\][3.5,0.6,\[psi\]Src]*)(* An example with a selected \[alpha\]H and \[delta\]H and with the observed polarization directions for \[psi\]*)
sourceCenter0 = \sum_{i=1}^{nSrc} rSrc[[i]], \{i, nSrc\};
sourceCenter = \frac{sourceCenter0}{(sourceCenter0 . sourceCenter0)^{1/2}};

(* unit radial vector to the arithmetic average center of the sources. *)
αSourceCenter = αFROMr[sourceCenter];
δSourceCenter = δFROMr[sourceCenter];
angleSourceToCenter = Table[ArcCos[rSrc[[i]].sourceCenter], \{i, nSrc\}];
ρRgnRadius = Sort[angleSourceToCenter][[-1]]; (* Furthest source from center *)
ρRMS = \left(\frac{1}{nSrc} \sum_{i=1}^{nSrc} (\text{angleSourceToCenter}[[i]]^2)^{1/2}\right);

3b. Section Summary

We consider Quasi-Stellar Objects, QSOs. From the data in JVAS1450, 5° radius regions are constructed, one centered at each of the 10518 grid points of a 2°x2° mesh. The 1450 QSOs were assigned to the regions based on location and we calculated the significance of the alignment of the polarization directions for the sources in each region.

The three such QSO regions selected for this notebook satisfied many requirements: (i) have 7 or more sources in order to use the significance formulas in Sec. 4 accurately, (ii) have longitude RA 160° ≤ α ≤ 180°, (iii) have latitude dec 40° ≤ δ ≤ 55°, (iv) whose QSOs are very significantly aligned, S ≤ 10^{-2}. There are 3 regions satisfying (i) - (iv) containing a total of 13 sources. See Fig. 1 and the discussion there.

Print["There are ", nSrc, " sources in the sample.""]
Print["Check that the Sample obeys the data cuts:"]
Print[
  "Check that the smallest % polarization \( p \) in the sample is 0.5% or more. Smallest: ",
  Sort[percentPol][[1]], ",% ."]
Print["Check that the largest fractional uncertainty in % polarization, \( \sigma_p/p \),
  is less than 0.6 . Largest: ",Sort[σpercentPol/percentPol][[-1]], ", ."]
Print["Check that the largest PPA \( \psi \) uncertainty \( \sigma_\psi \) is less than 16°. Largest: ",
  Sort[σψSrc][[-1]] \left(\frac{360.}{2.\pi}\right), ",° ."]

There are 13 sources in the sample.
Check that the Sample obeys the data cuts:
Check that the smallest % polarization \( p \) in the sample is 0.5% or more. Smallest: 0.471406% .
Check that the largest fractional uncertainty
  in % polarization, \( \sigma_p/p \), is less than 0.6 . Largest: 0.263915 .
Check that the largest PPA \( \psi \) uncertainty \( \sigma_\psi \) is less than 16°. Largest: 7.56115° .
```
In[61]= ListPlot[Table[αSrc[[j]], δSrc[[j]]] \[2\pi]/360., {j, nSrc}],
     PlotRange -> {{0, 360}, {-90, 90}},
     Ticks -> {Table[{i, i}, {i, 0, 360, 60}], Table[{j, j}, {j, -90, 90, 30}]},
     PlotLabel -> "Sources", AxesLabel -> {"\(\alpha\), degrees", "\(\delta\), degrees"},
     PlotStyle -> Green]
Print["Figure 9: The locations of the \(\), nSrc, \(\) QSOs in the sample. The center of the sample has (RA,Dec) = ",
   {αSourceCenter \[2\pi]/360., δSourceCenter \[2\pi]/360.}, ", in \{\text{hours, degrees}\}. 
   The angular separation of the furthest QSO from the sample center is 
   \text{Sort[angleSourceToCenter][-1]} \[2\pi]/360.\,\text{°}. The RMS radius is 
   \text{\rhoRMS} \[2\pi]/360.\,\text{°}."
```

```
	Sources

Figure 9: The locations of the 13 QSOs in the sample. The center of the sample has (RA,Dec) = {11.4297, 48.6782} , in \{\text{hours, degrees}\}. The angular separation of the furthest QSO from the sample center is 6.49406°. The RMS radius is 4.72813°.
```

4. Grid

While we have a formula \(\eta(H)\) for the alignment angle at a point \(H\) on the Celestial Sphere, there are occasions when it is better not to use it and, instead, construct a discrete table of values. To locate the values \(\eta(H)\) at a finite number of points \(H\) on the sphere, we create a grid, or mesh, of grid points.

When building the grid, we avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle \(d\phi\).

We grid one hemisphere. Symmetry across diameters gives the other hemisphere. The grid is conveniently developed centered at the North pole and then rotated to be centered on the sample of sources. For detailed work near the sources a 30° finely spaced grid cap is produced to supplement the more coarsely spaced grid. The fine and coarse grids are offset so that no grid points are common to the two grids.

4a. Construct the grid

Definitions:
gridSpacing, coarseGridSpacing - fine, coarse grid separation in degrees between grid points on and between constant latitude circles
fineCapRadius - radius of the fine grid cap in radians
dθ₁, dθ₂ - fine, coarse grid spacing in radians
idN, ai, ji, δj - dummy indices
αpointH, δpointH - α and δ of the grid points H_j
fineGrid, coarseGrid, gridN, grid - tables of data associated with grid points, record descriptions below
rotzToSample - rotation matrix from North pole to sourceCenter
lpgrid - plot of the radial unit vectors to the grid points
nGrid - number of grid points
αGrid, δGrid - longitudes at the grid points (-π ≤ α ≤ π)
δGrid - latitudes at the grid points (-π/2 ≤ α ≤ π/2)
rGrid - radial unit vectors from origin to grid points, in 3D Cartesian coordinates

```
in[62]:= gridSpacing = 0.6(*degrees*);
fineCapRadius = 0.5;
in[65] := (*KEEP this cell - DO NOT DELETE*)
(*The Northern Grid "gridN".*)
dθ₁ = 2.π/360.
gridSpacing (*Convert gridSpacing to radians*)
fineGrid = {};
idN = 1;
For[δj = 0., δj < fineCapRadius, δj++, δpointH = π - δj dθ₁ - dθ₁/2.1/2;
   (*Print["(δj,δpointH) = ", {δj,δpointH}];*)
   For[ai = 0., ai < Ceiling[2.π/dθ₁ (Cos[δpointH] + 0.01)], ai++, αpointH = ai dθ₁/(Cos[δpointH] + 0.01);
       (*Print["(ai,αpointH) = ", {ai,αpointH}];*)
       AppendTo[fineGrid, {idN, ai, δj, αpointH, δpointH, er[αpointH, δpointH]}];
   idN = idN + 1
   ]]
Length[fineGrid];
lpFine = ListPointPlot3D[Table[fineGrid[[1, 6]], {1, 1, Length[fineGrid], 10}], PlotRange → {{-1.2, 1.2}, {-1.2, 1.2}, {-1.2, 1.2}}, AxesLabel → {"x", "y", "z"}, BoxRatios → {1, 1, 1}];

Coarse Grid band runs from latitude (-π/2 - fineGridMAX) to latitude (-π/2 - southOfEquator)
in[69]:= coarseStart = fineCapRadius; coarseEnd = 1.65;(*radians*)
coarseGridSpacing = 2.0(*degrees*)
```
In[71]:= (*KEEP this cell - DO NOT DELETE*)
(* The coarse grid band. *)

d\theta^2 = \frac{2\pi}{360}.
coarseGridSpacing = \{\};

idB = 1 + Length[fineGrid];(* ID for the coarse band grid points*)

For[\[\delta j] = 0., \[\delta j] < (coarseEnd - coarseStart), \[\delta j]++, 
\[\delta pointH] = \frac{\pi}{2} - \text{coarseStart} - \[\delta j] \text{d}\theta^2 - \frac{\text{d}\theta^2}{3^{1/2}};

(*Print["{\[\delta j],\[\delta pointH]} = ",{\[\delta j],\[\delta pointH]}];*)

For[\[\alpha i] = 0., \[\alpha i] < \text{Ceiling}[\frac{2\pi \text{d}\theta^2}{\text{Cos}[\[\delta pointH] + 0.01]}], \[\alpha i]++, 
\[\alpha pointH] = \[\alpha i] \text{d}\theta^2 - \text{Cos}[\[\delta pointH] + 0.01];

(*Print["{\[\alpha i],\[\alpha pointH]} = ",{\[\alpha i],\[\alpha pointH]}];*)

AppendTo[coarseGrid, \{idB, \[\alpha i], \[\delta j], \[\alpha pointH], \[\delta pointH], er[\[\alpha pointH], \[\delta pointH]\}];

idB = idB + 1
]

In[73]:= lpCoarse1 = ListPointPlot3D[Table[coarseGrid[[i, 6]], {i, 1, Length[coarseGrid], 10}],
PlotRange \rightarrow \{\{-1.2, 1.2\}, \{-1.2, 1.2\}, \{-1.2, 1.2\}},
AxesLabel \rightarrow \{"x", "y", "z"\}, BoxRatios \rightarrow \{1, 1, 1\}];

Length[coarseGrid];

(*Show[{lpFine,lpCoarse}]*)

Now we need to rotate the combined fine/coarse grid ‘gridN’ so that it is centered on the sample, the sourceCenter .

In[75]:= rotzToSample = RotationMatrix[{{\[\theta], \[\theta], 1}, sourceCenter}];

%.{\[\theta], \[\theta], 1};
sourceCenter;

In[77]:= gridN = Join[fineGrid, coarseGrid];
grid = Table[gridN[[i, 1]], gridN[[i, 2]], gridN[[i, 3]], gridN[[i, 4]],
gridN[[i, 5]], rotzToSample.gridN[[i, 6]]], {i, Length[gridN]}];
lpgrid = ListPointPlot3D[Table[grid[[i, 6]], {i, 1, Length[grid], 10}],
PlotRange \rightarrow \{\{-1.2, 1.2\}, \{-1.2, 1.2\}, \{-1.2, 1.2\}},
AxesLabel \rightarrow \{"x", "y", "z"\}, BoxRatios \rightarrow \{1, 1, 1\}];
In[81]:= lpgrid
Print[
  "Figure 10: The grid. The grid is centered on the source sample, with a finely spaced cap. The grid covers one hemisphere, centered on the sample. The fine and coarse grids are off-set, so they do not share any grid points. There are ",
  nGrid, " grid points on the hemisphere."
]

Out[81]=

Figure 10: The grid. The grid is centered on the source sample, with a finely spaced cap. The grid covers one hemisphere, centered on the sample. The fine and coarse grids are off-set, so they do not share any grid points. There are nGrid grid points on the hemisphere.

In[83]:= αGrid = Table[αFROMr[grid[[j, 6]]], {j, Length[grid]}];
δGrid = Table[δFROMr[grid[[j, 6]]], {j, Length[grid]}];
rGrid = Table[grid[[j, 6]], {j, Length[grid]}];
nGrid = Length[grid];

4b. Section Summary

In[87]:= Print["The fine grid on the 'cap' has ", Length[fineGrid], " grid points.""]
Print["The grid points on the cap are separated by gridSpacing = ",
  gridSpacing, "° in latitude and longitude.""]
Print["On the entire hemisphere, there is a second set of grid points that are separated by gridSpacing = ",
  coarseGridSpacing, "° in latitude and longitude. The two sets do not share any grid points.""
Print["The second set has ", Length[coarseGrid], " grid points.""]
Print["The total grid, 'grid', has ", Length[fineGrid],
  "+ ", Length[coarseGrid], " = ", Length[grid], " grid points.""]
The fine grid on the 'cap' has 7459 grid points.

The grid points on the cap are separated by gridSpacing = 0.6° in latitude and longitude.

On the entire hemisphere, there is a second set of grid points that are separated by gridSpacing = 2.° in latitude and longitude. The two sets do not share any grid points.

The second set has 5026 grid points.

The total grid, 'grid', has 7459 + 5026 = 12485 grid points.

5. The alignment function $\eta(H)$ for the sample of sources

"Best" means we use the $\psi$Src that were listed in the catalog. We calculate the alignment function $\eta(H)$ at the grid points $H$.

Given the alignment function $\eta(H)$, one can find the smallest alignment angle $\eta_{\min}$ and the largest avoidance angle $\eta_{\max}$ and determine the significances for the alignment and avoidance of the polarization directions.

5a. Determine the alignment angle $\eta(H)$

First find $\eta(H_j)$ on the grid and find the smallest and largest values of the alignment function on the grid. Then use the function “$\etaBarAtHwithAny$" derived in Secs. 2 and 3 to go between grid points and locate the smallest and largest angles, $\eta_{\min}$ and $\eta_{\max}$, and their locations, the hubs $H_{\min}$ and $H_{\max}$. These are the extremes for convergence and divergence of the polarization directions.

Definitions:

$v\psi$Src unit vectors along the polarization directions $\psi$ in the tangent planes of the sources
eN local unit vectors along local North
eE local unit vectors along local East
grid$\eta$BarHj \{ j, $\eta(H_j)$ \}, where j is the index for grid point $H_j$ and $\eta(H)$ is the average alignment angle at $H_j$. See Eq. (1).
sortgrid$\eta$BarH {j, $\eta(H_j)$ }, with smallest angles $\eta(H)$ first.
grid$\eta$BarMin {j,$\eta(H_j)$}, the j and $\eta$ for the smallest value of $\eta(H)$, best alignment
grid$\eta$BarMin index j for the grid point $H$ with the smallest value of $\eta(H)$
grid$\eta$BarMin smallest $\eta(H)$ on grid
grid$\eta$BarMax {j,$\eta(H_j)$}, the j and $\eta$ for the largest value of $\eta(H)$, best alignment
grid$\eta$BarMax index j for the grid point $H$ with the largest value of $\eta(H)$
grid$\eta$BarMax largest $\eta(H)$ on grid

$\eta_{\min}$Obs smallest $\eta(H)$ and H, local min near grid$\eta$BarMin (use “$\etaBarAtHwithAny$" off-grid)

$\eta_{\max}$Obs largest $\eta(H)$ and H, local max near grid$\eta$BarMax

funcDataObs off-grid data for extreme alignment angles $\eta$ and their hubs H

$\eta$BarMinObs $\eta_{\min}$
$\eta$BarMaxObs $\eta_{\max}$
HminObs $H_{\min}$ location RA $\alpha$ in radians
HminObs $H_{\min}$ location dec $\delta$ in radians
HminObs $H_{\min}$ location (RA,dec) = ($\alpha$, $\delta$) in radians
Hmax\_\text{funDataObs} \quad H_{\text{max}} \text{ location } \alpha \text{ in radians}

Hmax\_\text{deltafunDataObs} H_{\text{max}} \text{ location } \delta \text{ in radians}

Hmax\_\text{alphaFunDataObs} \quad H_{\text{max}} \text{ location } (\text{RA,}\delta) = (\alpha, \delta) \text{ in radians}

\text{In[92]} = \begin{align*}
(* \upsilon_{\phi}, \epsilon_{n}, \epsilon_{e} \text{ unit vectors in the tangent plane of each source } S_{i}, \\
\text{pointing along the polarization direction, local North,} \\
\text{and local East, respectively. See Fig. 2.} *)
\upsilon_{\phi}\_\text{Src} &= \text{Table}[\text{Cos}[\psi\_\text{Src}[\text{[i]}]], \text{eN}\_\text{[aSrc[\text{[i]}]]}, \text{dSrc[\text{[i]}]]}] + \\
\text{Sin}[\psi\_\text{Src}[\text{[i]}]], \text{eE[ aSrc[\text{[i]}]], dSrc[\text{[i]}]]}, \text{i, nSrc}];
\end{align*}

\text{In[93]} = \begin{align*}
(* \text{Analysis using Eq (5) in Ref. 14 to get } \mathbf{\hat{\nu}}(H_{j}). \text{ First } \eta_{\text{JH}}, \cos(\eta_{\text{JH}}) = |\hat{\nu}_{H}\_\text{Hphi}|, \\
\text{where } \nu_{H} \text{ was called } "\nu_{H}\_\text{perp}\_\text{S}" \text{ in a previous discussion. Thus,} \\
\text{we can get } \mathbf{\hat{\nu}}(H_{j}), \text{ by Eq. (2):} *)
\text{gridj}\_\text{BarHj} &= \begin{align*}
\text{Table}[\text{ArcCos}[\text{Abs}[\text{rGrid[\text{[j]}]}.\upsilon\_\text{Src[\text{[i]}]]}], \text{Abs}[\text{rGrid[\text{[j]}]}.\upsilon\_\text{Src[\text{[i]}]]}] / \left((\text{rGrid[\text{[j]}]}.\upsilon\_\text{Src[\text{[i]}]} - \text{rGrid[\text{[j]}]}.\upsilon\_\text{Src[\text{[i]}]}).\upsilon\_\text{Src[\text{[i]}]}\right) \right]^{1/2} - 0.000001, \text{j, nSrc}], \text{i, nSrc}]}, \text{j, nGrid}];
\text{sortgridj}\_\eta\_\text{BarMin} &= \text{Sort[gridj}\_\eta\_\text{BarHj][\text{[1]}]}, \text{\{j, gridj}\_\eta(H_{j})\}} \text{ for smallest } \mathbf{\hat{\nu}}(H_{j}), \text{*)}
\text{gridj}\_\eta\_\text{BarMin} &= \text{gridj}\_\eta\_\text{BarMin}[\text{[2]}];
\text{gridj}\_\eta\_\text{BarMax} &= \text{sortgridj}\_\eta\_\text{BarHj}[-\text{[1]}]; \text{\{j, gridj}\_\eta(H_{j})\} \text{ for largest } \mathbf{\hat{\nu}}(H_{j}), \text{*)}
\text{gridj}\_\eta\_\text{BarMax} &= \text{gridj}\_\eta\_\text{BarMax}[\text{[2]}];
\text{sortgridj}\_\eta\_\text{BarHj} = \text{Sort[gridj}\_\text{BarHj][\text{[2]}]};
\text{gridj}\_\eta\_\text{BarMin} = \text{gridj}\_\eta\_\text{BarMin}[\text{[2]}];
\text{gridj}\_\eta\_\text{BarMax} = \text{gridj}\_\eta\_\text{BarMax}[\text{[2]}];
\end{align*}

\text{The results just found on the grid should be close to the results. Use FindMinimum and FindMaximum to go off-grid and get closer.}

\text{In[96]} = \begin{align*}
\eta_{\text{min}}\_\text{deltaHObS} &= \text{FindMinimum}[\eta\_\text{BarAtHwithAnyPsi}[\alphaH, \deltaH, \psi\_\text{Src}], \\
\{\alphaH, \text{aGrid[ gridj}\_\text{BarMin[\text{[1]}]]}}, \{\deltaH, \text{dGrid[ gridj}\_\text{BarMin[\text{[1]}]]}}]\};
\eta_{\text{max}}\_\text{deltaHObS} &= \begin{align*}
\text{FindMaximum}[\eta\_\text{BarAtHwithAnyPsi}[\alphaH, \deltaH, \psi\_\text{Src}], \\
\{\alphaH, \text{aGrid[ gridj}\_\text{BarMax[\text{[1]}]]}}, \{\deltaH, \text{dGrid[ gridj}\_\text{BarMax[\text{[1]}]]}}]\};
\text{funcDataObS} &= \{1, \{\eta_{\text{min}}\_\text{deltaHObS}[\text{[1]}]}, \{\alphaH, \deltaH\} / \eta_{\text{min}}\_\text{deltaHObS[\text{[2]}]}, \\
\{\eta_{\text{max}}\_\text{deltaHObS}[\text{[1]}]}, \{\alphaH, \deltaH\} / \eta_{\text{max}}\_\text{deltaHObS[\text{[2]}]}\}\}
\end{align*}

\text{\textbf{FindMinimum:} The line search decreased the step size to within the tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the function. You may need more than MachinePrecision digits of working precision to meet these tolerances.}

\text{\textbf{FindMaximum:} The line search decreased the step size to within the tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient increase in the function. You may need more than MachinePrecision digits of working precision to meet these tolerances.}

\text{Out[101]} = \{1, \{0.189626, \{3.13394, 0.854791\}}, \{1.09371, \{2.93195, 0.485158\}\}\}
In[102]:=
\[\eta\]BarMin\[funDataObs = funcDataObs[[2, 1]];\]
\[\eta\]BarMax\[funDataObs = funcDataObs[[3, 1]];\]
Hmin\[funDataObs = funcDataObs[[2, 2, 1]];\]
Hmin\[\delta\]funcDataObs = funcDataObs[[2, 2, 2]];\]
Hmin\[\alpha\delta\]funcDataObs = funcDataObs[[2, 2, 1]];\]
Hmax\[\alpha\]funcDataObs = funcDataObs[[3, 2, 1]];\]
Hmax\[\delta\]funcDataObs = funcDataObs[[3, 2, 2]];\]
Hmax\[\alpha\delta\]funcDataObs = {funcDataObs[[3, 2, 1]], funcDataObs[[3, 2, 2]]};

In[110]:= Print["When moving off-grid, check that the hubs Hmin and Hmax did not move more than a grid spacing:"]
Print["When we found a local minimum, the hub H min moved off-grid by ",
ArcCos[er[\[\alpha\]Grid[[ gridj\[\eta\]BarMin[[1]] ]], \[\delta\]Grid[[ gridj\[\eta\]BarMin[[1]] ]]]] \( \frac{360.}{2.\pi} \), "°."]
Print["When we found a local maximum, the hub H max moved off-grid by ",
ArcCos[er[\[\alpha\]Grid[[ gridj\[\eta\]BarMax[[1]] ]], \[\delta\]Grid[[ gridj\[\eta\]BarMax[[1]] ]]]] \( \frac{360.}{2.\pi} \), "°."]
Print["The alignment hub H min is ",
ArcCos[er[\[\alpha\]funDataObs, Hmin\[\delta\]funcDataObs].\sourceCenter \( \frac{360.}{2.\pi} \), "° from the source center."]
Print["The alignment hub H min is ",
ArcCos[er[\[\alpha\]funDataObs, Hmax\[\delta\]funcDataObs].\sourceCenter \( \frac{360.}{2.\pi} \), "° from the source center."]
Print["Now compare that with the grid: The fine grid spacing close to the sources is ",
gridSpacing, "°. If the hub is more than ", fineCapRadius \( \frac{360.}{2.\pi} \), "° from the sample center, then the grid spacing is ", coarseGridSpacing, "°."

When moving off-grid, check that the hubs Hmin and Hmax did not move more than a grid spacing:
When we found a local minimum, the hub Hmin moved off-grid by 0.133122°.
When we found a local maximum, the hub Hmax moved off-grid by 0.378856°.
The alignment hub Hmin is 5.34907° from the source center.
The alignment hub Hmin is 21.0509° from the source center.
Now compare that with the grid: The fine grid spacing close to the sources is 0.6°. If the hub is more than 28.6479° from the sample center, then the grid spacing is 2°.

5b. Plot the Alignment Angle Function \( \eta(H) \)

Definitions

\( \alpha_{HminDegrees} \quad H_{min} \) location RA \( \alpha \) in degrees
\( \alpha_{HminHours} \quad H_{min} \) location RA \( \alpha \) in hours
δHminDegrees  $H_{\text{min}}$ location Dec $\delta$ in degrees
αHmaxDegrees  $H_{\text{max}}$ location RA $\alpha$ in degrees
αHmaxHours  $H_{\text{max}}$ location RA $\alpha$ in hours
δHmaxDegrees  $H_{\text{max}}$ location Dec $\delta$ in degrees
rHmin, rHmax radial unit vectors to the alignment and avoidance hubs $H_{\text{min}}$ and $H_{\text{max}}$
nrPerpHmin (max) a unit vector in the plane of the great circle combining rCenterSrc and rHmin (max)
rGreatMinCircle($\theta$) (Max) radial unit vector to a point on the great circle
αGreatMin (Max) longitude at the point for $\theta$
δGreatMin (Max) latitude at the point for $\theta$
xyAitoffGreatMin (Max) Aitoff plot coordinates for the great circles
crossMin (Max) unit vector perpendicular, normal to the plane of the great circle
$\theta_{\text{min MAXgreatcircles}}$ angle between the vectors normal to the planes of the two great circles

In[116]:= (* Equatorial coordinates ($\alpha$, $\delta$) for the hubs $H_{\text{min}}$ and $H_{\text{max}}$ in other units.*)
αHminDegrees = HminαfunDataObs (360 / (2 $\pi$));
αHminHours = HminαfunDataObs (24 / (2 $\pi$)); (*$H_{\text{min}}$*)
δHminDegrees = HminδfunDataObs (360 / (2 $\pi$));

αHmaxDegrees = HmaxαfunDataObs (360 / (2 $\pi$)); (*$H_{\text{max}}$*)
αHmaxHours = HmaxαfunDataObs (24 / (2 $\pi$));
δHmaxDegrees = HmaxδfunDataObs (360 / (2 $\pi$));

In[122]:= rHmin = er $\left[ \alphaHminDegrees \left( \frac{2 \cdot $\pi$}{360}. \right) + \pi, -\deltaHminDegrees \left( \frac{2 \cdot $\pi$}{360}. \right) \right]$;
rPerpHmin0 = rHmin - (rHmin.sourceCenter) sourceCenter;
rPerpHmin = rPerpHmin0 / (rPerpHmin0.rPerpHmin0) $^{1/2}$;
rGreatMinCircle[{$\theta$}] := Cos[{$\theta$}] sourceCenter + Sin[{$\theta$}] rPerpHmin
αGreatMin[{$\theta$}] := αFROMr[rGreatMinCircle[{$\theta$}]]
δGreatMin[{$\theta$}] := δFROMr[rGreatMinCircle[{$\theta$}]]
xyAitoffGreatMin = Table[[xH180[ αGreatMin[{$\theta$}] (360 / (2 $\pi$)), δGreatMin[{$\theta$}] (360 / (2 $\pi$)) ]],
yH180[ αGreatMin[{$\theta$}] (360 / (2 $\pi$)), δGreatMin[{$\theta$}] (360 / (2 $\pi$)) ]], {{$\theta$, 1, 360}};
\(r_{\text{Hmax}} = e^r \left[ \alpha_{\text{HmaxDegrees}} \left( \frac{2 \pi}{360} \right) + \pi, -\delta_{\text{HmaxDegrees}} \left( \frac{2 \pi}{360} \right) \right];\)

\(r_{\text{PerpHmax0}} = r_{\text{Hmax}} - (r_{\text{Hmax.sourceCenter}}) \text{ sourceCenter};\)

\(r_{\text{PerpHmax}} = \left( r_{\text{PerpHmax0}} \cdot r_{\text{PerpHmax0}} \right)^{1/2};\)

\(r_{\text{GreatMaxCircle}[\theta]} := \cos[\theta] \text{ sourceCenter} + \sin[\theta] \cdot r_{\text{PerpHmax}}\)

\(\alpha_{\text{GreatMax[\theta]}} := \alpha_{\text{FROMr}[r_{\text{GreatMaxCircle}[\theta]}]}\)

\(\delta_{\text{GreatMax[\theta]}} := \delta_{\text{FROMr}[r_{\text{GreatMaxCircle}[\theta]}]}\)

\(xy_{\text{AitoffGreatMax}} = \{x_{180} \alpha_{\text{GreatMax[\theta]}}, \left(360 / (2 \pi)\right), \delta_{\text{GreatMax[\theta]}}, \left(360 / (2 \pi)\right)\}, \{\theta, 1, 360\};\)

\(\text{crossMin} = \text{Cross}[r_{\text{Hmin}}, \text{sourceCenter}];\)

\(\text{crossMin} = \frac{\text{crossMin0}}{\left(\text{crossMin0} \cdot \text{crossMin0}\right)^{1/2}};\)

\(\text{crossMax} = \text{Cross}[r_{\text{Hmax}}, \text{sourceCenter}];\)

\(\text{crossMax} = \frac{\text{crossMax0}}{\left(\text{crossMax0} \cdot \text{crossMax0}\right)^{1/2}};\)

\(\Theta_{\text{minMAXgreatestcircles}} = \text{ArcCos}[\text{crossMax} \cdot \text{crossMin}] \left(\frac{360.}{2. \pi}\right);\)

(* The following table \(\alpha_{\text{ojn}}\text{BarHjTable} is created to generate a map of the alignment angle \(\eta_{\text{BarRgnkj}}\) over the sphere.*)

(* Table \(\alpha_{\text{ojn}}\text{BarHjTable}
entries: 1. \(\alpha, 2. \delta, 3.\) alignment angle \(\eta_{\text{BarRgnkj}}\) at grid point (all in degrees)*)

\(\alpha_{\text{ojn}}\text{BarHjTable} = \{\alpha_{\text{ojn}}\text{BarHjTable}0 = \{\};\)

\(\text{For} [j = 1, j \leq \text{Length}[\text{gridj}\text{BarHj}], j++,\)

\(\text{AppendTo}[\alpha_{\text{ojn}}\text{BarHjTable0}, \{\alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)), \delta_{\text{grid}}[[j]] \cdot (360. / (2. \pi)), \}

\text{gridj}\text{BarHj}[[j, 2]] \cdot (360. / (2. \pi))] ; \text{If} [360. \geq \alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) > 180.0, \text{AppendTo}[\alpha_{\text{ojn}}\text{BarHjTable0}, \{\alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) - 180., \}

\text{gridj}\text{BarHj}[[j, 2]] \cdot (360. / (2. \pi))] ; \text{If} [180. > \alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) > 0, \text{AppendTo}[\alpha_{\text{ojn}}\text{BarHjTable0}, \{\alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) + 180., \}

\text{gridj}\text{BarHj}[[j, 2]] \cdot (360. / (2. \pi))] ; \text{If} [360. \geq \alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) > 354, \text{AppendTo}[\alpha_{\text{ojn}}\text{BarHjTable0}, \{\alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) - 360., \}

\text{gridj}\text{BarHj}[[j, 2]] \cdot (360. / (2. \pi))] ; \text{If} [+6. > \alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) > 0, \text{AppendTo}[\alpha_{\text{ojn}}\text{BarHjTable0}, \{\alpha_{\text{grid}}[[j]] \cdot (360. / (2. \pi)) + 360, \text{gridj}\text{BarHj}[[j, 2]] \cdot (360. / (2. \pi)))] ; \alpha_{\text{ojn}}\text{BarHjTable0};\)
(*The grid does not cover the sphere. Check that the \( \alpha_j \beta_j \gamma \) table covers the entire Celestial Sphere.*)

```
ListPlot[Table[{\( \alpha_j \beta_j \gamma \)BarHjTable[[i, 1]], \( \alpha_j \beta_j \gamma \)BarHjTable[[i, 2]]}, {i, Length[\( \alpha_j \beta_j \gamma \)BarHjTable]}]]
```

Print["Figure 11: Check. Since the grid does not cover the sphere, only half, we should check that the \( \alpha_j \beta_j \gamma \)BarHjTable table covers the entire Celestial Sphere. "]

![Figure 11: Check. Since the grid does not cover the sphere, only half, we should check that the \( \alpha_j \beta_j \gamma \)BarHjTable table covers the entire Celestial Sphere.](image)

(*Transcribe the alignment function \( \eta(H) \), the location of the sources, and the Celestial Equator onto an Aitoff plot.*)

```
xy\( \eta \)BarAitoffTable = Table[{xH180[\( \alpha \)Src[[n]] (360/(2\( \pi \))), \( \delta \)Src[[n]] (360/(2\( \pi \)))], yH180[\( \alpha \)Src[[n]] (360/(2\( \pi \))), \( \delta \)Src[[n]] (360/(2\( \pi \)))}, {n, nSrc}];
```

(*The Aitoff coordinates for the sources' locations.*)

(*Contour plot of the alignment angle function \( \eta(H) \) on the grid.*)

```
d\( \eta \)ContourPlot = 6;
(\(*\), in degrees. *)listCP = ListContourPlot[Union[xy\( \eta \)BarAitoffTable, {{xH180[\( \alpha \)MinDegrees, \( \delta \)MinDegrees], yH180[\( \alpha \)MinDegrees, \( \delta \)MinDegrees], \( \eta \)Min*(360./(2\( \pi \))-1.0)}}, {{xH180[\( \alpha \)MaxDegrees, \( \delta \)MaxDegrees], yH180[\( \alpha \)MaxDegrees, \( \delta \)MaxDegrees], \( \eta \)Max*(360./(2\( \pi \)))+1.0)}}, AspectRatio -> 1/2, Contours -> Table[\( \eta \), {\( \eta \), Floor[grid\( \eta \)BarMin[[2]]*(360./(2\( \pi \)))] + 1, Ceiling[grid\( \eta \)BarMax[[2]]*(360./(2\( \pi \)))] - 1, d\( \eta \)ContourPlot}], ColorFunction -> "TemperatureMap", PlotRange -> {(-4.0, 3.5), 7.5 11.0}, Axes -> False, Frame -> False, PlotLegends -> Placed[BarLegend[Automatic, LegendMargins -> {0, 0}, {10, 5}], LegendLabel -> "\( \eta(H) \), °", LabelStyle -> {Plain, FontFamily -> "Times"}, Right] ];
```
mapOf\eta\text{Bar} =
Show[
{listCP, Table[ParametricPlot[{xH180[\alpha], yH180[\alpha, \delta]},
\{\delta, -90, 90\}, PlotStyle \to \{Black, Thickness[0.002]\}, (*Mesh \to \{(11, 5, 0), (23, 11, 0)\}*) PlotPoints \to 60], \{\alpha, 0, 360, 30\}],
PlotStyle \to \{Black, Thickness[0.002]\}, (*Mesh \to \{(11, 5, 0), (23, 11, 0)\}) PlotPoints \to 60, \{\delta, -90, 90\}, Graphics[
PointSize[0.004], Text[StyleForm["N", FontSize \to 1.85, FontWeight \to "Plain"], \{0, 1.85\}], Text[StyleForm["Equatorial Coordinate System", FontSize \to 1.4, FontWeight \to "Plain"], \{0, -1.85\}], (*Sources S:*) Green[xyAitoffSources], Gray, Point[xyAitoffGreatMin], Point[xyAitoffGreatMax], Black, Text[StyleForm["H_{\max}"], FontSize \to 12, FontWeight \to "Bold"], \{-3.3, 1.0\}],
{Arrow[BezierCurve[\{-3.3, 1.2\}, \{-1.3, 3.0\}],
{\alpha_{\max}Degrees, \delta_{\max}Degrees}], yH180[\alpha_{\max}Degrees, \delta_{\max}Degrees]]]}],
Text[StyleForm["H_{\min}"], FontSize \to 12, FontWeight \to "Bold"], \{3.3, 1.0\}],
{Arrow[BezierCurve[\{3.3, 1.2\}, \{0.3, 3.0\}],
{\alpha_{\min}Degrees, \delta_{\min}Degrees}], yH180[\alpha_{\min}Degrees, \delta_{\min}Degrees]]]}],
Text[StyleForm["H_{\min}"], FontSize \to 12, FontWeight \to "Bold"], \{-3.3, -1.0\}],
{Arrow[BezierCurve[\{-3.3, -1.2\}, \{-2.3, -2.5\}],
\{\alpha_{\min}Degrees - 180, \delta_{\min}Degrees],
\{\alpha_{\max}Degrees - 180, \delta_{\max}Degrees]]}\}, (**)
Text[StyleForm["H_{\max}"], FontSize \to 12, FontWeight \to "Bold"], \{3.3, -1.0\}],
{Arrow[BezierCurve[\{3.3, -1.2\}, \{2.3, -2.0\}],
\{\alpha_{\max}Degrees + 180, \delta_{\max}Degrees],
\{\alpha_{\max}Degrees + 180, \delta_{\max}Degrees]]}\}]], ImageSize \to 0.9 \times 432];

SetDirectory[homeDirectory]
Export["20210517Q5OnearbyHmin.pdf", mapOf\eta\text{Bar}]
*)

5c. Section Summary
In[149]:= mapOf\[\eta\]Bar

Print[
"Figure 12: The alignment function \(\eta(H)\), Eq. (1). The map is centered on \((\alpha,\delta) = (180^\circ,0^\circ)\),"
]
Print["with \(\alpha = 0^\circ\) on the left and \(\alpha = 360^\circ\) on the right, Equatorial Coordinates."
]
Print["The sources are located at the dots, shaded \(\text{"},\text{Green},\text{"").}"
]
Print["The smallest alignment angle is \(\eta_{\text{min}} = \), \(\etaBarMinfunDataObs(360./\(2.\pi\))\), " located at the"
]
Print["alignment hubs \(H_{\text{min}}\) and \(-H_{\text{min}}\) in the areas shaded \(\text{"},\text{Blue},\text{"").}"
]
Print["The hubs \(H_{\text{min}}\) and \(-H_{\text{min}}\) are located at \((\alpha,\delta) = \), Round[\{\alphaHminDegrees, \deltaHminDegrees\}], " and ", Round[\{\alphaHminDegrees - 180, \deltaHminDegrees\}], " , in degrees.""
]
Print["The angle between the sample's center and the closest alignment hub \(H_{\text{min}}\) is ",
\[\text{ArcCos}\left(-\frac{rHmin.sourceCenter}{360.2.\pi}\right)\], "
]
Print["The largest avoidance angle is \(\eta_{\text{max}} = \), \(\etaBarMaxfunDataObs(360./\(2.\pi\))\), " located at the"
]
Print["avoidance hubs \(H_{\text{max}}\) and \(-H_{\text{max}}\) in the areas shaded \(\text{"},\text{Red},\text{"").}"
]
Print["The hubs \(H_{\text{max}}\) and \(-H_{\text{max}}\) are located at \((\alpha,\delta) = \), Round[\{\alphaHmaxDegrees + 180, \deltaHmaxDegrees\}], " and at ", Round[\{\alphaHmaxDegrees, \deltaHmaxDegrees\}], " , in degrees.""
]
Print["The angle between the sample's center and the closest avoidance hub \(H_{\text{max}}\) is ",
\[\text{ArcCos}\left(-\frac{-rHmax.sourceCenter}{360.2.\pi}\right)\], "
]
Print["To guide the eye, two Great Circles are plotted, one through the sources' center and the avoidance hubs \(H_{\text{max}}\) and \(-H_{\text{max}}\). The other connects the center of the sources' locations with the alignment hubs \(H_{\text{min}}\) and \(-H_{\text{min}}\). The Great Circles are shaded Gray, \(\text{"},\text{Gray},\text{"").}"
]
Print["The angle between the normals to the planes of the two great circles is ",
\(\theta_{\text{minMAXgreatcircles}}\), ""
]
Print["Notes: Although somewhat obscured by the distortion needed to plot a sphere on a flat surface, the function \(\eta(H)\) is symmetric across diameters: Diametrically opposite points \(-H\) and \(H\) have the same alignment angle \(\eta(H)\)."
]
Figure 12: The alignment function $\eta(H)$, Eq. (1). The map is centered on $(\alpha, \delta) = (180^\circ, 0^\circ)$, with $\alpha = 0^\circ$ on the left and $\alpha = 360^\circ$ on the right, Equatorial Coordinates.

The sources are located at the dots, shaded ■. The smallest alignment angle is $\eta_{\text{min}} = 10.8648^\circ$, located at the alignment hubs $H_{\text{min}}$ and $-H_{\text{min}}$ in the areas shaded ■.

The hubs $H_{\text{min}}$ and $-H_{\text{min}}$ are located at $(\alpha, \delta) = (180, 49)$ and $(0, -49)$, in degrees.

The angle between the sample's center and the closest alignment hub $H_{\text{min}}$ is 5.34907°.

The largest avoidance angle is $\eta_{\text{max}} = 62.6651^\circ$, located at the avoidance hubs $H_{\text{max}}$ and $-H_{\text{max}}$ in the areas shaded ■.

The hubs $H_{\text{max}}$ and $-H_{\text{max}}$ are located at $(\alpha, \delta) = (348, -28)$ and at $(168, 28)$, in degrees.

To guide the eye, two Great Circles are plotted, one through the sources' center and the avoidance hubs $H_{\text{max}}$ and $-H_{\text{max}}$. The other connects the center of the sources' locations with the alignment hubs $H_{\text{min}}$ and $-H_{\text{min}}$. The Great Circles are shaded Gray, ■.

The angle between the normals to the planes of the two great circles is 104.78°.

Notes: Although somewhat obscured by the distortion needed to plot a sphere on a flat surface, the function $\eta(H)$ is symmetric across diameters: Diametrically opposite points $-H$ and $H$ have the same alignment angle $\eta(H)$.
(*Plot polarization directions*)

$\eta[i_-, d_-] := (rSrc[[i]] + d \psi Src[[i]]) / ((rSrc[[i]] + d \psi Src[[i]]) \cdot (rSrc[[i]] + d \psi Src[[i]]))^{1/2}$

polarLines[d_] := Table[Line[{{xH180[αFROM[ rPlus[ i, d ]] \[ 360. \pi/2 ]] \[ 360. \pi/2 ]}, δFROM[ rPlus[ i, d ]] \[ 360. \pi/2 ]}], {i, nSrc}]

(*Construct the map of $\pi(H)$.*)

mapOfηBarLocal = Show[{listCPlocal, Table[ParametricPlot[{{xH180[α, δ], yH180[α, δ]}, {δ, 2θ, 90}], PlotStyle -> {Black, Thickness[0.002]}, PlotPoints -> 60}, {α, 12θ, 24θ, 3θ}], Graphics[PointSize[0.009], Black, {Thick, polarLines[0.03], (*Sources S:* Green, PointSize[0.012], Point[ xyAitoffSources ], Gray, PointSize[0.005], Point[ xyAitoffGreatMin], Point[ xyAitoffGreatMax ], Black, Text[StyleForm["X", FontSize -> 12, FontWeight -> "Bold"]], {xH180[αminDegrees, δminDegrees], yH180[αminDegrees, δminDegrees]}], Text[StyleForm["X", FontSize -> 12], FontWeight -> "Bold"], {xH180[αmaxDegrees, δmaxDegrees], yH180[αmaxDegrees, δmaxDegrees]}], Arrow[BezierCurve[{-3.3, 1.2}, {-1.3, -3.0}, {xH180[αmaxDegrees, δmaxDegrees] - 0.01, yH180[αmaxDegrees, δmaxDegrees] + 0.03}]}, Arrow[BezierCurve[{-3.3, 1.2}, {0.3, 3.0}, {xH180[αminDegrees, δminDegrees] - 0.005, yH180[αminDegrees, δminDegrees] + 0.02}]]}, ImageSize -> 0.9\times432];
Figure 13: Map of the alignment angle function \( \eta(H) \) in the neighborhood of the sources. The polarization directions display parallax, generally pointing toward the alignment hub \( H_{\text{min}} \).

Note how close three of the sources are to the hub \( H_{\text{min}} \).

6. Uncertainty Runs

6a. Creating and Storing Uncertainty Runs

For each “uncertainty run”, the polarization direction \( \psi \) for each source is allowed to differ from the best value \( \psi_{\text{Src}} \) by an amount \( \delta \psi \) chosen according to a Gaussian distribution with a mean equal to the best value \( \psi_{\text{Src}} \) and half-width \( \sigma_{\psi_{\text{Src}}} \), \( \psi = \psi_{\text{Src}} + \delta \psi \). Both values \( \psi_{\text{Src}} \) and \( \sigma_{\psi_{\text{Src}}} \) are taken from the JVAS1450 catalog.

The notebook .nb version generates new uncertainty runs. The pdf version uses old uncertainty runs that are uploaded from previously saved files that are not publically available. Thus both versions have some cells commented out: (* comments are not processed by Mathematica*).

Definitions:

- \( r_{\text{Src}}x_{\text{Grid}} \) unit vector \( S_i \times H_j \), the cross product of the radial unit vector to source \( S_i \) with the radial unit vector to grid point \( H_j \)
- \( nR \) number of uncertainty runs
- \( nRun \) sequential index labeling the runs
- \( \psi_{\text{Data}} \) table \{nRun, \( \psi \}\} of polarization directions \( \psi = \psi_{\text{Src}} + \delta \psi \) for each run
- \( \text{runData} \) collection of data to save from the uncertainty runs, see below for content list
- \( nRunPrint \) dummy index controlling when current TimeUsed and MemoryInUse are printed
- \( \psi_{\text{SrcU}} \) the polarization direction \( \psi \) for the run.
- \( r_{\text{Src}}x_{\psi_{\text{Src}}} \) unit vector, \( S_i \times \psi_{\text{Src}} \), cross product of the radial vector \( S_i \) to the source with the vector \( \hat{v}_\psi \) in the direction of the polariza-
jηBarToGridU \{j, \bar{\eta}(H_j)\}, where \(j\) is the index for the grid point \(H_j\) and \(\bar{\eta}(H_j)\) is the alignment angle function, (1), at \(H_j\).

sortjηBarToGridU sort \{j, \eta(H_j)\}, with the smaller angle \(\eta(H)\) first.

jηBarMinU \{j, \eta(H)\} for the smallest value of \(\eta(H)\), best alignment

jηBarMaxU \{j, \eta(H)\}, for the largest value of \(\eta(H)\), most avoided

ηminHU off-grid local min data \{\eta_{\text{min}}, \{\alpha, \delta\} at \(H_{\text{min}}\}\}

ηmaxHU off-grid local max data \{\eta_{\text{max}}, \{\alpha, \delta\} at \(H_{\text{max}}\}\}

funcDataU off-grid, superior values of \{nRun, \eta_{\text{min}}HU, \eta_{\text{max}}HU\} collected results

HminfunDataU values of \(\alpha = \alpha\) for hub \(H_{\text{min}}\) from uncertainty runs, alignment

HminfunDataU values of \(\delta = \delta\) for hub \(H_{\text{min}}\) from uncertainty runs, alignment

HmaxfunDataU values of \(\alpha = \alpha\) for hub \(H_{\text{max}}\) from uncertainty runs, avoidance

HmaxfunDataU values of \(\delta = \delta\) for hub \(H_{\text{max}}\) from uncertainty runs, avoidance


tables:

ψData entries: 1. Run # 2. ψSrcU, list of polarization position angles ψ

gridDataUn on-grid, entries: 1. Run # 2. \{\eta_{\text{min}}, \{\alpha, \delta\} at \(H_{\text{min}}\}\} 3. \{\eta_{\text{max}}, \{\alpha, \delta\} at \(H_{\text{max}}\}\}

funcDataU off-grid, (better) entries: 1. Run # 2. \{\eta_{\text{min}}, \{\alpha, \delta\} at \(H_{\text{min}}\}\} 3. \{\eta_{\text{max}}, \{\alpha, \delta\} at \(H_{\text{max}}\}\}

To generate your own Uncertainty Runs:
First calculate “rSrcxGrid” and then evaluate the “For” statement in the following two cells.
One can save the results with the “Put[]” statements.
Once saved, there is no need to repeat the runs. Comment out the “rSrcxGrid” and “For” statements by enclosing them in (*comment brackets*).
The data can be retrieved with the “Get” statements.

\[\text{ln}[172]= hasteRemove comment marks, "(*) and ")", below to generate your own tables. \(*\)\]
(* Evaluate this cell for the notebook .nb version *)
(*
nR=500;
t1=TimeUsed[];
rSrcxGrid=Table[ Cross[ rSrc[[i] ],rGrid[[j] ] ] , (i,nSrc),(j,nGrid)];
(* first step: aw cross product, not unit vectors*)
rSrcxGrid=Table[ rSrcxGrid[[i,j]]/
{rSrcxGrid[[i,j]], rSrcxGrid1[[i,j]]+ 0.000001}^{1/2} , (i,nSrc),(j,nGrid)];
Clear[rSrcxGrid1];
gridDataUn=();ψData=();funcDataU=();nRunPrint=0;
For[nRun=1,nRun<nR,nRun++,
If[nRun>nRunPrint,Print["At the start of run ",nRun," , the time is ",
TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."]; nRunPrint=nRunPrint+100];
ψSrcU=Table[RandomVariate[NormalDistribution[ψSrc[[i]],ψSrc[[i]]]],[i,nSrc]];
(*table of PPA angles ψ for the sources in region j0, in radians*)
rSrcxψSrc = Table[ Sin[(ψSrc[[i]]),eNSrc[[i]]]-
Cos[(ψSrc[[i]]),eSrc[[i]]]} , (i,nSrc)];
(*table of the cross product of rSrc and vector in direction of ψSrcU, a
unit vector*) jηBarToGridU = Table[ [j,1/nSrc]Sum[ ArcCos[
Abs[ rSrcxψSrc[[i]].rSrcxGrid[[i,j]] ] - 0.000001 ],(i,nSrc)],(j,nGrid)];
(*
{grid point η, value of the alignment angle ηHj[j] averaged over all sources,
in radians]*) sortjηBarToGridU=Sort[ jηBarToGridU,#1[[2]]<#2[[2]] ];
(* jηBarToGridU, {j,ηj}, but sorted with the smallest alignment angles first *)
 jηBarMinU=sortjηBarToGridU[[1]]; (* {j,ηj}, at the grid point Hj with minimum η*)
 jηBarMaxU=sortjηBarToGridU[[-1]]; (* {j,ηj}, at the grid point Hj with maximum η*)
AppendTo[ψData,{nRun,ψSrcU}];
AppendTo[gridDataUn,{nRun, jηBarMinU[[2]]},
{αGrid [[ jηBarMinU[[1]] ]],δGrid [[ jηBarMinU[[1]] ]]}],
{ jηBarMaxU[[2]],{αGrid [[ jηBarMaxU[[1]] ]],δGrid [[ jηBarMaxU[[1]] ]]}},];
(*collect discrete (on-grid) data*)
ηminαδHU=FindMinimum[ jηBarAtWithAnyψ[αH,δH,ψData[[nRun,2]]],
{{αH,gridDataUn[[nRun,2,2,1]]},{δH,gridDataUn[[nRun,2,2,2]]}}];
ηmaxαδHU= FindMaximum[ jηBarAtWithAnyψ[αH,δH,ψData[[nRun,2]]],
{{αH,gridDataUn[[nRun,3,2,1]]},{δH,gridDataUn[[nRun,3,2,2]]}}];
AppendTo[funcDataU,[nRun, ηminαδHU[[1]],(αH,δH)/.ηminαδHU],[ηmaxαδHU[[1]],
(αH,δH)/.ηmaxαδHU[[2]]]] (*collect continuous (function-based) data*)
]
t2=TimeUsed[];
Print["Time used to compute ψData, gridDataUn, and funcDataU: t2 - t1 = ",t2-t1]
*)

Hint: You can save memory if you do not get the “ψData”. The table ψData is needed to reconstruct the exact values of the gridDataUn table, but it is not needed in any following calculation.
In[174]:= SetDirectory[homeDirectory];
(*Save a new data file*)
(*
Put[ψData,"20211005PsiDataUqsoClump2U4000.dat" ]
*)
(*
Put[gridDataUn,"20211005gridDataUnqsoClump2U4000.dat" ]
*)
(*
Put[funcDataU,"20211005funcDataQSON13Un10000b.dat" ]
*)

Hint: Saving “gridDataUn” to a file avoids the time it takes to complete the “For” statement. Make the above “For” statement into a remark so that it doesn’t evaluate.

In[175]:= SetDirectory[homeDirectory];
(*Retrieve an old data file*)
(*
ψData4000=Get["20211004PsiDataUqsoClump2U4000.dat"];
ψData6000=Get["20210928PsiDataUqsoClump2U6000.dat"];
*)
(*
gridDataUn4000=Get["20211004gridDataUnqsoClump2U4000.dat"];
gridDataUn6000=Get["20210928runDataUqsoClump2U6000.dat"];
*)
(*Get the funcDataU file for the pdf version:*)

funcDataU = Get["20211005funcDataQSON13Un10000a.dat"];

In[177]:= (*If needed, edit the following to collect data files together.*)
(*
ψData=Join[ψData4000,ψData6000];
Length[ψData]
ψData[[1]]
griddDataUn=Join[gridDataUn4000,gridDataUn6000];
Length[gridDataUn]
griddDataUn[[1]]
*)

In[178]:= (*nR may not be previously defined, depending on what cells have been processed.*)
(*Define nR for the pdf version.*)

nR = Length[funcDataU]

Out[178]= 10000
In[179]:= (* Define quantities based on the function continuous results. The
continuous results should be better than the on-grid quantities. *)

In[179]:=

ηBarMinfunDataU = Table[funcDataU[[i1, 2, 1]], {i1, Length[funcDataU]}];
ηBarMaxfunDataU = Table[funcDataU[[i1, 3, 1]], {i1, Length[funcDataU]}];
HminαfunDataU = Table[funcDataU[[i1, 2, 2]], {i1, Length[funcDataU]}];
HminδfunDataU = Table[funcDataU[[i1, 2, 2]], {i1, Length[funcDataU]}];
HmaxαδfunDataU = Table[{funcDataU[[i1, 2, 2]], funcDataU[[i1, 2, 2]]}, {i1, Length[funcDataU]}];
HmaxδαfunDataU = Table[funcDataU[[i1, 3, 2]], {i1, Length[funcDataU]}];
HmaxαfunDataU = Table[funcDataU[[i1, 3, 2]], {i1, Length[funcDataU]}];

In[430]:= ListPlot[{HminαδfunDataU, HmaxαδfunDataU}, PlotRange -> All,
PlotStyle -> {{Blue, PointSize[0.01]}, {Red, PointSize[0.01]}},
PlotLabel -> "The hubs from the uncertainty runs", AxesLabel -> {"α (rad)", "δ (rad)"}]

Out[430]=

The hubs from the uncertainty runs

Figure 14: Uncertainty run hubs. The alignment hubs $H_{\text{min}}$ are in blue,
The avoidance hubs $H_{\text{max}}$ are in red,
Symmetry across a diameter means there are hubs
diametrically opposed to these. Including any diametrically
opposed hubs would ruin the statistical calculations for hubs.

6b. The Effects of Uncertainty on the Smallest Alignment Angle $\eta_{\text{min}}$

This section fits a Gaussian distribution to the $\eta_{\text{min}}$ from the uncertainty runs.

Definitions

- sortηBarMin: sort the list of $\eta_{\text{min}}$ from the uncertainty runs
- η0minU: estimated mean of the Gaussian fit
- σminU: estimated half-width of the Gaussian fit
- hlminU0, hlminU: histogram \{η, bin height\} tables needed to set up the NonlinearModelFit
nlmmin\(\) non-linear model fit of a Gaussian to the \(\eta_{\text{min}}\) histogram

showNLMB plot of Gaussian and histogram

pTableNLmminU table of parameter attributes, including standard error

\(\sigma\eta_{\text{BarminUFit}}, \eta_{\text{BarminUFit}}\) - half-width, and mean of the Gaussian fit

In[189]:=
Print["The number of uncertainty runs is ", Length[funcDataU], "."]
The number of uncertainty runs is 10000.

In[190]:= sort\(\eta\)BarMinU = Sort[\(\eta\)BarMinfunDataU];
\(\eta\)0minU = mean[\(\eta\)BarMinfunDataU]; (*Guess the mean for the Gaussian.*)
\(\sigma\)minU = standDev[\(\eta\)BarMinfunDataU]; (*Guess the half-width.*)

hlmin\(\)0 = HistogramList[sort\(\eta\)BarMinU, \{\(\eta\)0minU - 5 \(\sigma\)minU, \(\eta\)0minU + 5 \(\sigma\)minU, 0.4 \(\sigma\)minU\}];
hlminU = Table[[(1/2) (hlminU0[[1, i1]] + hlminU0[[1, i1 + 1]]), hlminU0[[2, i1]]],
{i1, Length[hlminU0[[2]]]}];

nlmminU = NonlinearModelFit[hlminU, a Exp[-(1/2.) (\(x\) - \(x\)0) / b)\(^2\)],
\{\{a, Length[sort\(\eta\)BarMinU/6]\}, \{b, \(\sigma\)minU\}, \{\(x\)0, \(\eta\)0minU\}\}, \(x\); (*\(x\) is \(\eta\)BarMin*)

In[195]:= pTableNLmminU = nlmminU["ParameterTable"]
\{\(\sigma\)BarminUFit, \(\eta\)BarminUFit\} = (b, \(x\)0) /. nlmminU["BestFitParameters"]; (*radians*)

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1580.79</td>
<td>8.78314</td>
<td>179.98</td>
</tr>
<tr>
<td>b</td>
<td>0.0187055</td>
<td>0.000120009</td>
<td>155.867</td>
</tr>
<tr>
<td>(x)0</td>
<td>0.198773</td>
<td>0.000120009</td>
<td>1656.32</td>
</tr>
</tbody>
</table>

In[197]:= showNLMB = Show[
Histogram[sort\(\eta\)BarMinU, \{\(\eta\)0minU - 5 \(\sigma\)minU, \(\eta\)0minU + 5 \(\sigma\)minU, 0.4 \(\sigma\)minU\}],
PlotLabel -> "Uncertainty run \(\eta_{\text{min}}\) ", AxesLabel -> [{\(\eta_{\text{min}}\), radians", "\(\Delta R\)"}],
Plot[Normal[nlmminU], \{\(x\), \(\eta\)0minU - 5 \(\sigma\)minU, \(\eta\)0minU + 5 \(\sigma\)minU\}], PlotLabel -> "\(\eta_{\text{min}}\) ",
ListPlot[hlminU, PlotLabel -> "\(\eta_{\text{min}}\) "];
In[198] = showNLMB
Print["Figure 15: The Gaussian fit to the alignment angle \( \eta_{\text{min}} \) histogram. The height is the number of runs \( \Delta R \) in each bin. Note how nicely symmetric this is."
Print["The total number of runs is \( R = \Sigma (\Delta R) = \), Length[funcDataU], ".

Out[198] =

\begin{align*}
0.15 & \quad 0.20 & \quad 0.25 & \quad \eta_{\text{min}}, \text{ radians} \\
0 & \quad 500 & \quad 1000 & \quad 1500 & \quad \Delta R & \quad \text{Uncertainty run} \ \\
6c. \text{ The Effects of Uncertainty on the Largest Avoidance Angle } \eta_{\text{max}} \text{ This section fits a Gaussian distribution to the } \eta_{\text{max}} \text{ returned by the uncertainty runs.}

Definitions: Similar to the definitions in Sec. 6b.

In[201] = sort\etaBarMaxU = Sort[\etaBarMaxfunDataU];
\eta0maxU = mean[\etaBarMaxfunDataU]; (*Guess the mean for the Gaussian. *)
\sigma\text{maxU} = stanDev[\etaBarMaxfunDataU]; (*Guess the half-width.*)
histogramrangemaxU = [\eta0maxU - 5 \sigma\text{maxU}, \eta0maxU + 5 \sigma\text{maxU}, 0.4 \sigma\text{maxU}];
hlmaxU = HistogramList[sort\etaBarMaxU, histogramrangemaxU];
hl0maxU = Table[[(1/2) (hlmaxU[[1, i1]] + hlmaxU[[1, i1 + 1]]), hlmaxU[[2, i1]]],
{i1, Length[ hlmaxU[[2]] ]}];
nlmmaxU = NonlinearModelFit[hlmaxU, a Exp[- (1/2) ((x - x0)/b)^2],
{{a, 300.}, {b, \sigma\text{maxU}}, {x0, \eta0maxU}}, x]; (*x is \eta_{\text{BarMaxU} \ast}*

nlmBmaxU = NonlinearModelFit[hlmaxU, \left(a \left(1 + e^{- \frac{4 (x-x0)}{x}} \right)^{-1} \exp\left[- \frac{1}{2} \left( \frac{x-x0}{b} \right)^2 \right]\right) \ast \ast (x > 0)\ast, x];

pTableNLMmaxU = nlmBmaxU["ParameterTable"]
\{\etaBarMaxFitU, \etaBarMaxFitU\} =
ParametersNLMmaxU = {b, x0} /. nlmBmaxU["BestFitParameters"]; (*radians*)

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a 1559.75</td>
<td>17.1284</td>
<td>91.0624</td>
<td>7.49644 \times 10^{-30}</td>
</tr>
<tr>
<td>b 0.0168288</td>
<td>0.000206563</td>
<td>81.4703</td>
<td>8.61619 \times 10^{-29}</td>
</tr>
<tr>
<td>x0 1.10294</td>
<td>0.000172835</td>
<td>6381.43</td>
<td>1.92418 \times 10^{-70}</td>
</tr>
</tbody>
</table>
6d. The Effects of Uncertainty on the Locations \((\alpha, \delta)\) of the Alignment Hubs \(H_{\text{min}}\)

Each uncertainty run returns an alignment hub \(H_{\text{min}}\). In this section, we investigate the distribution of the locations the alignment hubs \(H_{\text{min}}\).

There are two hubs, \(H_{\text{min}}\) and \(-H_{\text{min}}\) for each uncertainty run, by the symmetry across a diameter. So we collect the data together by moving the \(-H_{\text{min}}\) hubs across a diameter to join the \(H_{\text{min}}\) hubs. See Fig. 14.

```mathematica
ln[213] = sortHminαδfunDataU = Sort[Union[HminαδfunDataU]];
lPHminU = ListPlot[Union[HminαδfunDataU], PlotRange -> All, PlotStyle -> {Blue, PointSize[0.01]}, PlotLabel -> "The alignment hubs from the uncertainty runs", AxesLabel -> {"\(\alpha\) (rad)"", "\(\delta\) (rad)"}];
```
\[\text{In[215]}:=\text{sortHmin} = \text{Sort[Hmin} \alpha \text{funDataU]}; (*\text{Guess the mean for the Gaussian. *}) \]
\[dx0Hmin = \text{stanDev[Hmin} \alpha \text{funDataU]}; (*\text{Guess the half-width. *}) \]
\[\text{histogramrangeRAHminU} = \{x0Hmin - 5 \ dx0Hmin, x0Hmin + 5 \ dx0Hmin, 0.4 \ dx0Hmin\}; \]
\[h1xHmin = \text{HistogramList[sortHmin, histogramrangeRAHminU]}; \]
\[\{i1, \text{Length[h1xHmin[[2]]]}\}]; \]
\[\text{nlmxHmin} = \text{NonlinearModelFit[h1xHmin, a \ \text{Exp[-}(1/2).((x-x0)/b)^2)],} \]
\[\{\{a, \text{Length[sortHmin/6]}\}, \{b, dx0Hmin\}, \{x0, x0Hmin\}\}, x]; (*x is Hmin \alpha *) \]

\[\text{In[221]}:=\text{pTablenlmxHmin} = \text{nlmxHmin["ParameterTable"]} \]
\[\{\alpha Hmin\text{Fit}, \alpha Hmin\text{Fit}\} = \text{Parameters[nlmxHmin]} = \{b, \ x0\} /. \text{nlmxHmin["BestFitParameters"]}; \]
\[\alpha Hmin\text{Fit} = \text{Normal[nlmxHmin]} \]
\[\text{expOfnlmxHmin[x_]} := -（1/2.）(（x-x0）/b）^2 /. \text{nlmxHmin["BestFitParameters"]} \]

\[\text{Out[221]}= \text{Estimate Standard Error t-Statistic P-Value} \]
\[a \quad 4684.95 \quad 588.845 \quad 7.95618 \quad 6.46607 \times 10^{-8} \]
\[b \quad 0.000479938 \quad 0.0000791483 \quad 6.06379 \quad 4.20082 \times 10^{-6} \]
\[x0 \quad 3.13413 \quad 0.000130467 \quad 24022.4 \quad 4.15789 \times 10^{-83} \]

\[\text{Out[223]} = 4684.95 e^{-2.1707 \times 10^6 (-3.13413 x)^2} \]
\[\text{Out[225]} = -2.1707 \times 10^6 (-3.13413 + x)^2 \]

\[\text{In[226]}=\text{shownlmxHmin} = \text{Show[\{\text{Histogram[sortHmin, histogramrangeRAHminU,}} \]
\[\text{PlotLabel \rightarrow "\alpha Hmin ", AxesLabel \rightarrow \{"\alpha Hmin, radians", "\Delta R"\}, PlotRange \rightarrow \text{All\},} \]
\[\text{Plot}\[\text{Normal[nlmxHmin], \{x, 3.12, 3.145\}, PlotRange \rightarrow \text{All\}, PlotLabel \rightarrow "\alpha Hmin"\},} \]
\[\text{ListPlot[h1xHmin, PlotLabel \rightarrow "\alpha Hmin"]} \}; \]

\[\text{In[227]}=\text{sortHmin} = \text{Sort[Hmin} \delta \text{funDataU]}; (*\text{Guess the mean for the Gaussian. *}) \]
\[dy0Hmin = \text{stanDev[Hmin} \delta \text{funDataU]}; (*\text{Guess the half-width. *}) \]
\[\text{histogramrangeDecHminU} = \{y0Hmin - 5 \ dy0Hmin, y0Hmin + 5 \ dy0Hmin, 0.4 \ dy0Hmin\}; \]
\[h1yHmin = \text{HistogramList[sortHmin, histogramrangeDecHminU]}; \]
\[\{i1, \text{Length[h1yHmin[[2]]]}\}]; \]
\[\text{nlmyHmin} = \text{NonlinearModelFit[h1yHmin, a \ \text{Exp[-}(1/2.)(y-y0)/b)^2)],} \]
\[\{\{a, \text{Length[sortHmin/6]}\}, \{b, dy0Hmin\}, \{y0, y0Hmin\}\}, y]; (*y is Hmin \delta *) \]
\[\text{In}[,233] = \text{pTablenlmyHmin} = \text{nlmyHmin}[\text{"ParameterTable"}];
\]
\[\{\text{nlmHmin}, \text{nlmyHmin}\} = \text{nlmyHmin}[\text{"BestFitParameters"}];\]

\[\text{*radians*}
\]
\[\text{Normal[nlmyHmin]}
\]
\[\text{expOfnlmyHmin[y_]} := -\left(\frac{1}{2}\right) \left((y - y0) / b\right)^2 / \text{nlmyHmin}[\text{"BestFitParameters"}]
\]

\[\text{expOfnlmyHmin[y]}
\]

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>7835.08</td>
<td>122.439</td>
<td>63.9917</td>
<td>1.71076 \times 10^{-26}</td>
</tr>
<tr>
<td>0.000909269</td>
<td>0.0000386478</td>
<td>23.5271</td>
<td>4.36022 \times 10^{-17}</td>
</tr>
<tr>
<td>0.855277</td>
<td>0.0000977768</td>
<td>8747.24</td>
<td>1.86759 \times 10^{-73}</td>
</tr>
</tbody>
</table>

\[\text{Out}[233] = \text{7835.08 e^{-604.763. (-0.855277 + y)^2}}
\]

\[\text{Out}[237] = -604.763. \left(-0.855277 + y\right)^2
\]

\[\text{In}[238] = \text{shownlmyHmin} = \text{Show[}\{\text{Histogram}\left[\text{sortHmin}, \text{histogramrangeDecHminU},\right.\text{PlotLabel} \to \"\text{δHmin}\", \text{AxesLabel} \to \{\"\text{δHmin, radians}\", \"\Delta R\"\}], \text{Plot}\left[\text{Normal[nlmyHmin]}, \{y, 0.82, 0.88\}\right\}, \text{PlotRange} \to \text{All}, \text{PlotLabel} \to \"\text{δHmin}\"]\};
\]

\[\text{... General: Exp[-752.569] is too small to represent as a normalized machine number; precision may be lost.}
\]

\[\text{In}[239] = \text{GraphicsRow}\left[\{\text{shownlmxHmin, shownlmyHmin}\}\right]
\]

\[\text{Print[}\text{"Figure 17: The Gaussian fits to the Hmin RA and DEC histograms, where the height is the number of runs ΔR in each bin. "}]
\]

\[\text{Print[}\text{"In both graphs, the total number of runs is R = Σ(ΔR) = \", Length[funcDataU], "."]}
\]

\[\text{Out}[239] = \text{Figure 17: The Gaussian fits to the Hmin RA and DEC histograms, where the height is the number of runs ΔR in each bin.}
\]

\[\text{In both graphs, the total number of runs is R = Σ(ΔR) = 10\,000.}
\]

\[\text{In}[242] = \text{expoHminU[x_, y_] := -}\left(\text{expOfnlmxHmin[x]} + \text{expOfnlmyHmin[y]}\right)
\]

\[\text{Print[}\text{"The exponent of the probability distribution for Hmin, i.e. the negative log of the distribution: ", expoHminU[\alpha, \delta]\}]
\]

\[\text{The exponent of the probability distribution for Hmin, i.e. the negative log of the distribution: 2.1707\times10^6 (-3.13413 + \alpha)^2 + 604.763. (-0.855277 + \delta)^2}
\]
In[244]:= Plot3D[{expoHminU[x, y], 0.5}, {x, x0 - 0.0010, x0 + 0.0010} /. nlmxHmin["BestFitParameters"], {y, y0 - 0.0015, y0 + 0.0015} /. nlmyHmin["BestFitParameters"], PlotLabel -> "Negative log of the probability of (α, δ) for H_{min}", AxesLabel -> {"α (rad)", "δ (rad)"}] Print["Figure 18: The negative log of the likelihood of (RA, dec) for H_{min}, as a function of RA and dec. Where the likelihood is down by a factor e^{-1/2}, the negative log is 0.5 and that defines the half-width σ of the distribution."]

Out[244]= Figure 18: The negative log of the likelihood of (RA, dec) for H_{min}, as a function of RA and dec. Where the likelihood is down by a factor e^{-1/2}, the negative log is 0.5 and that defines the half-width σ of the distribution.

In[246]= (*Find the curve for the intersection in Fig. 18*)
frHmin[r_, θ_] := Simplify[{expoHminU[x, y] - 0.5 /. {x -> HminαFit + r Cos[θ], y -> HminδFit + r Sin[θ]}] frHmin[r, θ];
solverHmin[θ_] := Solve[frHmin[r, θ] == 0, r];
solverHmin[θ];
rHmin[θ_] := Abs[r /. solverHmin[θ][[2]]] rHmin[θ];
rHmin[0.8];
Plot[rHmin[θ], {θ, 0, 2.π}];

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.
Show[
{lpHminU, ParametricPlot[
{Hmin\[Alpha]Fit + rHmin\[Theta]Cos[\[Theta]],
Hmin\[Delta]Fit + rHmin\[Theta]Sin[\[Theta]],
{\[Theta], 0, 2.\[Pi]}, PlotStyle \[Rule] Orange, PlotRange \[Rule] All(*{(3.12,3.14),\{0.84,0.90\})*)}]
}
]
Print["Figure 19: All of the alignment hubs \(H_{\text{min}}\) from uncertainty runs. The ellipse encloses the most likely locations of the hubs. Symmetry across diameters means there is another set diametrically opposite those displayed here."
]

Figure 19: All of the alignment hubs \(H_{\text{min}}\) from uncertainty runs. The ellipse encloses the most likely locations of the hubs. Symmetry across diameters means there is another set diametrically opposite those displayed here.

6e. The Effects of Uncertainty on the Locations \((\alpha, \delta)\) of the Avoidance Hubs \(H_{\text{max}}\)

Each uncertainty run returns an avoidance hub \(H_{\text{max}}\). In this section, we investigate the distribution of the locations the avoidance hubs \(H_{\text{max}}\).

There are two hubs, \(H_{\text{max}}\) and \(-H_{\text{max}}\) for each uncertainty run, by the symmetry across a diameter. So we collect all the hubs together by moving the \(-H_{\text{max}}\) hubs across a diameter to join the \(H_{\text{max}}\) hubs. See Fig. 14.

(*Check that \(0^\circ \leq \alpha < 180^\circ\) and \(-90^\circ \leq \delta < 90^\circ\)*)

\[
\text{sortHmax} = \text{Sort}[\text{Union}[\text{Hmax}\[\alpha]\[\delta]\text{funDataU}]]
\]

\[
lpHmaxU = \text{ListPlot}[\text{Union}[\text{Hmax}\[\alpha]\[\delta]\text{funDataU}], \text{PlotRange} \rightarrow \text{All}, \text{PlotStyle} \rightarrow \{\text{Red}, \text{PointSize}[0.01]\}]
\]

\[
\text{PlotLabel} \rightarrow \"\text{The avoidance hubs from the uncertainty runs}, \"\text{AxesLabel} \rightarrow \{\"\alpha\ (\text{rad})\", \"\delta\ (\text{rad})\"\}]
\]

\[
\text{histogramrange} = \{\text{x0Hmax} - 5 \text{dx0Hmax}, \text{x0Hmax} + 5 \text{dx0Hmax}, \text{dx0Hmax}\}
\]

\[
\text{hl0xHmax} = \text{HistogramList}[\text{sortHmax}, \text{histogramrange}];
\]

\[
\text{hlxHmax} = \text{Table}[\{1/2\} \text{hl0xHmax[[1, i1]]} + \text{hl0xHmax[[1, i1+1]]}, \text{hl0xHmax[[2, i1]]}],
\{i1, \text{Length}[\text{hl0xHmax[[2]]]}\}];
\]

\[
\text{nlmxHmax} = \text{NonlinearModelFit}[\text{hlxHmax}, \alpha \text{Exp}[-(1/2.) \((x - x0)\^2\)],
\{\{a, \text{Length}[\text{sortHmax}\[6]\}], \{b, \text{dx0Hmax}\}, \{x0, \text{x0Hmax}\}\};(*x\ is\ Hmax\*)
\]
\begin{verbatim}
In[264]=
{\sigma Hmax\alpha, Hmax\alpha Fit} = ParametersnlmxHmax = \{b, x0\} /. nlmxHmax["BestFitParameters"]; (*radians*)
Normal[nlmxHmax]
exp0fnlmxHmax[x_] := -(1/2.) ((x - x0)/b)^2 /. nlmxHmax["BestFitParameters"]

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>3279.85</td>
<td>1349.65</td>
<td>2.43016</td>
</tr>
<tr>
<td>b</td>
<td>0.0508147</td>
<td>0.0241455</td>
<td>2.10452</td>
</tr>
<tr>
<td>x0</td>
<td>2.90428</td>
<td>0.0241444</td>
<td>120.288</td>
</tr>
</tbody>
</table>

Out[266]= 3279.85 e^{-193.638 (-2.90428 + x)^2}

Out[268]= -193.638 (-2.90428 + x)^2

In[269]= shownlmxHmax = Show[
Histogram[sortHmax\alpha, histogramrange, PlotLabel \rightarrow "\alpha Hmax", AxesLabel \rightarrow \{"\alpha Hmax, radians", "\Delta R"\}, PlotRange \rightarrow All],

Plot[Normal[nlmxHmax], \{x, 2.7, 3.1\}, PlotRange \rightarrow All, PlotLabel \rightarrow "\alpha Hmax"],

ListPlot[hlxHmax, PlotLabel \rightarrow "\alpha Hmax"] ]

In[270]= sortHmax\delta = Sort[Hmax\delta funDataU];
y0Hmax = mean[Hmax\delta funDataU]; (*Guess the mean for the Gaussian. *)
dy0Hmax = stdDev[Hmax\delta funDataU]; (*Guess the half-width.*)
histogramrange = \{y0Hmax - 5 dy0Hmax, y0Hmax + 5 dy0Hmax, 0.4 dy0Hmax\};

hl0yHmax = HistogramList[sortHmax\delta, histogramrange];

hlyHmax = Table[\{(1/2) (hl0yHmax[[1, i1]] + hl0yHmax[[1, i1 + 1]]), hl0yHmax[[2, i1]]\},
{i1, Length[ hl0yHmax[[2]] ]}];
nlmyHmax = NonlinearModelFit[hlyHmax, a Exp[-(1/2.) ((y - y0)/b)^2],
\{a, Length[ sortHmax\delta / 6\}, \{b, dy0Hmax\}, \{y0, y0Hmax\}\}, y]; (*x is Hmax\delta*)

In[276]= pTablenlmyHmax = nlmyHmax["ParameterTable"]
{\alpha Hmax\delta\alpha, Hmax\delta\alpha Fit} = ParametersnlmyHmax = \{b, y0\} /. nlmyHmax["BestFitParameters"]; (*radians*)
Normal[nlmyHmax]
exp0fnlmyHmax[y_] := -(1/2.) ((y - y0)/b)^2 /. nlmyHmax["BestFitParameters"]

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1252.9</td>
<td>520.149</td>
<td>2.40874</td>
</tr>
<tr>
<td>b</td>
<td>0.235369</td>
<td>0.112836</td>
<td>2.08594</td>
</tr>
<tr>
<td>y0</td>
<td>0.589584</td>
<td>0.11283</td>
<td>5.22543</td>
</tr>
</tbody>
</table>

Out[278]= 1252.9 e^{-9.02549 (-0.589584 - y)^2}

Out[280]= -9.02549 (-0.589584 + y)^2
\end{verbatim}
shownlmxHmax = Show[
  {Histogram[sortHmaxx, histogramrange,
    PlotLabel -> "δHmax ", AxesLabel -> {"δHmax, radians", "ΔR"},
    PlotRange -> All],
   Plot[Normal[nlmyHmax], {y, 0., 1.5}, PlotRange -> All,
    PlotLabel -> "δHmax"]},
  GraphicsRow[{shownlmxHmax, shownlmxHmax}]];

Print["Figure 20: The Gaussian fits to the Hmax RA and DEC
histograms, where the height is the number of runs ΔR in each bin. "]
Print["In both graphs, the total number of runs is R = Σ(ΔR) = ",
  Length[funcDataU], "."]

Figure 20: The Gaussian fits to the Hmax RA and DEC
histograms, where the height is the number of runs ΔR in each bin.
In both graphs, the total number of runs is R = Σ(ΔR) = 10000.

expoHmaxU[x_, y_] := -expOfnlmxHmax[x] + expOfnlmyHmax[y]
Print["The exponent of the probability distribution for
Hmax, i.e. the negative log of the distribution: ",
  expoHmaxU[α, δ]]

The exponent of the probability distribution for Hmax, i.e. the negative log of the distribution:
193.638 (-2.90428 + α)^2 + 9.02549 (-0.589584 + δ)^2
In[287]:= Plot3D[{expoHmaxU[x, y], 0.5}, {x, x0 - 0.08, x0 + 0.08} /. nlmxHmax["BestFitParameters"], {y, y0 - 0.5, y0 + 0.5} /. nlmyHmax["BestFitParameters"], PlotLabel -> "Negative log of the probability of (\(\alpha, \delta\)) for H_{max}", AxesLabel -> {"\(\alpha\) (rad)"", "\(\delta\) (rad)"}]
Print["Figure 21: The negative log of the likelihood of (RA,dec) for H_{max}, as a function of RA and dec. Where the likelihood is down by a factor e^{-1/2}, the negative log is 0.5 and that defines the half-width \(\sigma\) of the distribution.""]

![Figure 21](image.png)

Figure 21: The negative log of the likelihood of (RA,dec) for H_{max}, as a function of RA and dec. Where the likelihood is down by a factor e^{-1/2}, the negative log is 0.5 and that defines the half-width \(\sigma\) of the distribution.

In[289]:= (*Find the curve for the intersection in Fig. 21*)

frHmax[r_, \[Theta]_] := 
  Simplify[(expoHmaxU[x, y]) - 0.5 /. {x -> Hmax\[Alpha]Fit + r Cos[\[Theta]], y -> Hmax\[Delta]Fit + r Sin[\[Theta]]}]
frHmax[r, \[Theta]];
solverHmax[\[Theta]_] := Solve[frHmax[r, \[Theta]] == 0, r];
solverHmax[\[Theta]]; rHmax[\[Theta]] := Abs[r /. solverHmax[\[Theta]][[2]]]
rHmax[\[Theta]]; rHmax[0.8];
Plot[rHmax[\[Theta]], {\[Theta], 0, 2.\[Pi]}];

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.
In[297]:=
Show[
{lphmaxU, ParametricPlot[
{HmaxFit + rHmaxθ[θ] Cos[θ], HmaxFit + rHmaxθ[θ] Sin[θ]},
{θ, θ, 2. π}, PlotStyle → Orange, PlotRange → All]]}
Print["Figure 22: Avoidance hubs \(H_{\text{max}}\) from uncertainty runs. The ellipse
encloses the most likely locations of the hubs. Symmetry across diameters
means there is another set diametrically opposite those displayed here."]

The avoidance hubs from the uncertainty runs

Out[297]=

Figure 22: Avoidance hubs \(H_{\text{max}}\) from uncertainty runs. The ellipse
encloses the most likely locations of the hubs. Symmetry across diameters
means there is another set diametrically opposite those displayed here.

6f. The Effects of Uncertainty on the angle \(\theta\) between the planes of the Sample to \(H_{\text{min}}\) Great Circle and the Sample to \(H_{\text{max}}\) Great Circle.

These are the Gray lines in Figs. 3, 4, 12, 13. However, in Sec. 7 below, we see that the avoidance angle \(\eta_{\text{max}}\) is not significant, random \(\phi\) would be likely to yield a \(\eta_{\text{max}}\) that is as large or larger. Also, we see a lot of scatter in Fig. 22 for the avoidance hubs \(H_{\text{max}}\). Conversely, the alignment angle \(\eta_{\text{min}}\) is very significant and the alignment hubs collect in a tight formation. Compare the axes scales in Figs. 19 and 22. The Great Circle from the Sample to \(H_{\text{max}}\) is not well-defined. So the angle \(\theta\) varies over a wide range.

Definitions:

“uRuns” prefix results from the uncertainty runs
uRunsCrossMin unit vector normal to the Great Circle connecting the center of the source region with the alignment hub \(H_{\text{min}}\)
uRunsCrossMax unit vector normal to the Great Circle connecting the center of the source region with the alignment hub \(H_{\text{max}}\)
uRunsθminmaxUgreatcircles angle between the two normals in degrees
sortθminmaxU sort “uRunsθminmaxUgreatcircles”, smallest \(\theta\) first

See Definitions above in Secs. 6a,6b for other quantities below. There you should find similarly named quantities.
\[\text{Out}[310]=\] 
\[\text{In}[312]=\] 
\[\text{In}[310]=\] 
\[\text{In}[299]=\] 
\[\text{In}[294]=\] 
\[\text{In}[290]=\] 
\[\text{In}[285]=\] 
\[\text{In}[281]=\] 
\[\text{In}[276]=\] 
\[\text{θ} = 170.689, 3.26776, 52.2344, 5.45207, 10\times 10^{-10}\] 
\[\text{θ} = 7.6131, 2.48322, 3.06581, 6.36138, 10^{-10}\] 
\[\text{θ} = 1015.42, 125.289, 8.10463, 1.38104, 10^{10}\] 
\[\text{θ} = 10.2389, 3.59334, 2.01426, 0.196727, 10^{-10}\] 
\[\text{θ} = 10\times 10^{-10}\] 
\[\text{θ} = \frac{360}{2\pi}\] 
\[\text{θ} = 2.101246, 0.196727, -10.2389, 3.59334, 10^{-9}\] 
\[\text{θ} = 668.742, 5.30798, 10^{-43}\] 
\[\text{θ} = 8.10463, 1.38104, 10^{-7}\] 
\[\text{θ} = 3.06581, 0.00636138\] 
\[\text{θ} = 52.2344, 5.45207, 10^{-22}\] 
\[\text{θ} = \text{Exp[-1360.92]}\] is too small to represent as a normalized machine number; precision may be lost.
In[313]:= showNLM
Print["Figure 23: The Gaussian fit to the angle \( \theta \) histogram. We fit two angles \( \theta \),
corresponding to the two likely locations of the avoidance hubs \( H_{\text{max}} \)."]

\[ \text{Angle } \theta \text{ between the Two Gray Great Circles in Figs. 3, 4, 12, 13.} \]

Outlet[313]=

Figure 23: The Gaussian fit to the angle \( \theta \) histogram. We fit two angles \( \theta \),
corresponding to the two likely locations of the avoidance hubs \( H_{\text{max}} \).

6g. Map of the Hubs for the Uncertainty Runs

In this subsection, we map the locations of the many alignment hubs \( H_{\text{min}} \) and the avoidance hubs \( H_{\text{max}} \) that are found in the
uncertainty runs.

Definitions:
\( \psi \)SrcBig, Small \quad \text{unit vectors, } v(\psi \pm \sigma_{\psi}) \text{, large & small, the one-sigma range of polarization directions } \psi

In[315]:= (*The Aitoff coordinates for the hubs \( H_{\text{min}} \) locations.*)
xyAitoffHminU = Table[{x180[ Hmin\[n\] (360/2\( \pi \))], y180[ Hmin\[n\] (360/2\( \pi \))]}, {n, Length[Hmin\[n\]]}];

In[316]:= (*The Aitoff coordinates for the hubs \( H_{\text{max}} \) locations.*)
xyAitoffHmaxU = Table[{x180[ Hmax\[n\] (360/2\( \pi \))], y180[ Hmax\[n\] (360/2\( \pi \))]}, {n, Length[Hmax\[n\]]}];

In[317]:= (*The Aitoff coordinates for the hubs \(-H_{\text{min}} \) locations.*)
xyAitoffOppositeHminU = Table[{x180[ If[0 <= Hmin\[n\] (360/2\( \pi \)) < 180, Hmin\[n\] (360/2\( \pi \)) + 180, If[360 > Hmin\[n\] (360/2\( \pi \)) > 180, Hmin\[n\] (360/2\( \pi \)) - 180], -Hmin\[n\] (360/2\( \pi \))]}, {n, Length[Hmin\[n\]]}];
(*The Aitoff coordinates for the hubs - $H_{\text{max}}$ locations.*)

\[\text{xyAitoffOppositeHmaxU = Table}\left[\left[\right.\begin{array}{c}
\text{xFH}\[180\]_\{\text{If}[0 \leq H_{\text{max}}\text{funDataU}[\text{[}n\text{]]}] (360/\left(2\pi\right)) < 180, 180, \text{If}[360 > H_{\text{max}}\text{funDataU}[\text{[}n\text{]]}] (360/\left(2\pi\right)) > 180, -H_{\text{max}}\text{funDataU}[\text{[}n\text{]]}] (360/\left(2\pi\right)) \right], \text{yFH}\[180\]_\{\text{If}[0 \leq H_{\text{max}}\text{funDataU}[\text{[}n\text{]]}] (360/\left(2\pi\right)) < 180, H_{\text{max}}\text{funDataU}[\text{[}n\text{]]}] (360/\left(2\pi\right)) + 180, \text{If}[360 > H_{\text{max}}\text{funDataU}[\text{[}n\text{]]}] (360/\left(2\pi\right)) > 180, -H_{\text{max}}\text{funDataU}[\text{[}n\text{]]}] (360/\left(2\pi\right)) \right]\right], \{n, \text{Length}[H_{\text{max}}\text{funDataU}]\}];

(* $\psi$ unit vectors pointing along the polarization direction, have an experimental uncertainty. These are their plus/minus values.*)

\[\text{vSrcBig = Table}[\cos[\left(\psi_{\text{Src}}[\text{[}i\text{]]}] + \sigma_{\psi_{\text{Src}}}[\text{[}i\text{]]}\right)], eE[\alpha_{\text{Src}}[\text{[}i\text{]]}], \delta_{\text{Src}}[\text{[}i\text{]]}], \{i, \text{nSrc}\}];\]

\[\text{vSrcSmall = Table}[\cos[\left(\psi_{\text{Src}}[\text{[}i\text{]]}] - \sigma_{\psi_{\text{Src}}}[\text{[}i\text{]]}\right)], eE[\alpha_{\text{Src}}[\text{[}i\text{]]}], \delta_{\text{Src}}[\text{[}i\text{]]}], \{i, \text{nSrc}\}];\]

(*Plot polarization direction Uncertainty in Sec. 6*)

\[\text{rPluspsiBig[i_, d_] := \left(\text{rSrc}[\text{[}i\text{]]} + d \text{vSrcBig}[\text{[}i\text{]]}\right)}/\sqrt{\left(\text{rSrc}[\text{[}i\text{]]} + d \text{vSrcBig}[\text{[}i\text{]]}\right)}^2;\]

\[\text{polarLinesBig[d_] := Table}[\text{Line}[\left[\right.\left[\text{xFH}\text{[}\alpha \text{FROM}[\text{rPluspsiBig}[\text{[}i\text{], d]], \text{yFH}\text{[}\alpha \text{FROM}[\text{rPluspsiBig}[\text{[}i\text{], d]]\right] \right]];\]

(*Plot polarization direction Uncertainty in Sec. 6*)

\[\text{rPluspsiSmall[i_, d_] := \left(\text{rSrc}[\text{[}i\text{]]} + d \text{vSrcSmall}[\text{[}i\text{]]}\right)}/\sqrt{\left(\text{rSrc}[\text{[}i\text{]]} + d \text{vSrcSmall}[\text{[}i\text{]]}\right)}^2;\]

\[\text{polarLinesSmall[d_] := Table}[\text{Line}[\left[\right.\left[\text{xFH}\text{[}\alpha \text{FROM}[\text{rPluspsiSmall}[\text{[}i\text{], d]], \text{yFH}\text{[}\alpha \text{FROM}[\text{rPluspsiSmall}[\text{[}i\text{], d]]\right] \right]];\]

(* Local contour plot of the alignment angle function $\bar{\bar{\eta}}(H)$ on the grid. *)

(*d|ContourPlot = 6;*) (*, in degrees.*)

\[\text{ContourPlot}[\left(\text{yFH}[\text{[}150\text{, 22.5\text{, 30 \degree}}], \text{yFH}[\text{[}150\text{, 48.5\text{, 60 \degree}}], \text{None}], \{\text{xFH}[\text{[}150\text{, 15\text{, 10 \degree}}], \text{xFH}[\text{[}180\text{, 15\text{, 12 \degree}}], \text{xFH}[\text{[}210\text{, 15\text{, 14 \degree}}], \text{(None)}]\right));\]
In[326]:= listCPlocalU = Show[
    Table[
        ParametricPlot[
            {xH180[α, δ], yH180[α, δ]}, {δ, 10, 90}, PlotStyle -> {Black, Thickness[0.002]},
            PlotPoints -> 60, PlotRange -> {{xH180[(*)135+150, 30], xH180[(*)225+190, 30]},
            {yH180[180, (*15+30), yH180[180, 62]}}, Axes -> False, Frame -> True,
            FrameLabel -> {"α", "δ", "Close-Up View"}, FrameTicks -> frameticks, {α, 120, 240, 30}],
        PlotStyle -> Black, Thickness[0.002],
        Graphics[{PointSize[0.01], Red, (*Hmax:*Point[ xyAitoffHmaxU ],
            PointSize[0.009], Gray, {Thick, polarLines[0.03]}, {Thick, polarLinesBig[0.03]},
            {Thick, polarLinesSmall[0.03]}, (*Sources S:*Green, PointSize[0.012],
            Point[ xyAitoffSources ],
            PointSize[0.01], Blue, (*Hmin:*Point[ xyAitoffHminU ],
            Gray, PointSize[0.005])}],
        ParametricPlot[
            {xH180[(HminFit + rHminθ[θ] Cos[θ]) (360. / 2. π)],
            (HminFit + rHminθ[θ] Sin[θ]) (360. / 2. π)},
            yH180[(HminFit + rHminθ[θ] Cos[θ]) (360. / 2. π)],
            (HminFit + rHminθ[θ] Sin[θ]) (360. / 2. π)],
            {θ, 0, 2. π}, PlotStyle -> {Orange, Thickness[0.01]}],
        ParametricPlot[
            {xH180[(HmaxFit + rHmaxθ[θ] Cos[θ]) (360. / 2. π)],
            (HmaxFit + rHmaxθ[θ] Sin[θ]) (360. / 2. π)},
            yH180[(HmaxFit + rHmaxθ[θ] Cos[θ]) (360. / 2. π)],
            (HmaxFit + rHmaxθ[θ] Sin[θ]) (360. / 2. π)],
            {θ, 0, 2. π}, PlotStyle -> {Orange, Thickness[0.005]}],
        ImageSize -> 0.9 × 432 ];
Figure 24: Uncertainty plot. The sources are shaded green, **Green**,. Polarization directions for the reported value 
\( \psi \), and the one-sigma values \( \psi \pm \sigma \psi \) are plotted as gray, **Gray**, , line segments through the sources. All of the alignment hubs \( H_{\text{min}} \) from the uncertainty runs are plotted as overlapping blue, **Blue**, , dots, with the orange, **Orange**, , spot denoting the tiny ellipse of highest hub density. Many of the avoidance red dots, **Red**, , for the \( H_{\text{max}} \) are off-graph. The big orange ellipse encloses the likely locations for avoidance hubs.
To estimate the effects of experimental uncertainty, there were 10,000 uncertainty runs.

Uncertainty runs have polarization directions \( \psi = \psi_{\text{Src}} + \delta \psi \), where \( \delta \psi \) is chosen with a normal distribution of half-width \( \sigma \psi \) about the best value \( \psi_{\text{Src}} \).

The uncertainty runs determine the smallest alignment angle to be \( \eta_{\text{min}} = 11.389^\circ \pm 1.072^\circ \).

The uncertainty runs determine the largest avoidance angle to be \( \eta_{\text{max}} = 63.193^\circ \pm 0.964^\circ \).

Note, from Fig. 24, the avoidance hubs \( H_{\text{max}} \) from uncertainty runs separate into two distinct blobs. Thus, the uncertainty runs determine the angle \( \theta \) between the two grey Great Circles in Figs. 3, 4, 12, 13, to be centered around two different values.

For the more likely \( H_{\text{max}} \)s, we have \( \theta = 105.092^\circ \pm 2.014^\circ \).

The less likely group of \( H_{\text{max}} \) hubs give the angle \( \theta \) between the two grey Great Circles \( \theta = 170.689^\circ \pm 7.613^\circ \) from the sample to the avoidance hubs \( H_{\text{max}} \) is drawn in the figures.

7. Probability and Significance

The problem of “significance” is to determine the likelihood that random polarizations directions would produce better alignment or avoidance than the observed polarization directions.

To determine the probability distributions and related formulas, we made many runs with random data and fit the results. In this effort, as has occurred previously elsewhere, one finds that the probability distributions for the smallest alignment angle \( \eta_{\text{min}} \) and the largest avoidance angle \( \eta_{\text{max}} \) are not well-described by Gaussian functions. Better fits have the Gaussian multiplied by a step-function. The fitting functions are based on the following distribution,

\[
f(y) = \frac{1}{(2\pi)^{1/2}} \left( 1 + e^{4(y-1)} \right)^{-1} e^{-\frac{y^2}{2}}
\]
More discussion appears below when the function (4) is needed.

For example, random polarization directions are well-fit by a probability distribution for the smallest alignment angle $\eta_{\text{min}}$ that takes the form

$$P_{\text{min}}(\eta) = \left( \frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left( 1 + e^{\frac{\left[ \eta - (\eta_0 - \sigma) \right]}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left( \frac{\eta - (\eta_0 - \sigma)}{\sigma} \right)^2},$$

(5)

where norm makes the integral of distribution equal to unity, $\eta_0$ and $\sigma$ are parameters that are adjusted to fit the random run results.

7a. Probability and Significance Formulas

Definitions:

- **norm**: a constant used to normalize the distribution so the integral of probability is 1.
- **probMIN0, probMAX0**: probability distributions for alignment (MIN) and avoidance (MAX), functions of $\eta, \eta_0, \sigma$
- **signiMIN0, signiMAX0**: significance as a function of $(\eta, \eta_0, \sigma)$

In[336]:= (* $y = \left( \eta - \eta_0 \right) / \sigma$; $dy = d\eta / \sigma$ *)

(* The normalization factor "norm" is needed for the probability density *)

\[
\text{norm} = \left( \frac{1}{(2\pi)^{1/2}} \right) \frac{\text{norm}}{\sigma} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ 1 + e^{\frac{\left[ y - (\eta_0 - \sigma) \right]}{\sigma}} \right]^{-1} e^{-\frac{1}{2} \left( \frac{y - (\eta_0 - \sigma)}{\sigma} \right)^2} dy ;
\]

norm;(*Constant needed to make the integral of the probability distribution equal to unity.*)

In[338]:= probMIN0[$\eta_-$, $\eta_0$, $\sigma$] := \( \left( \frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left( 1 + e^{\frac{\left[ \eta - (\eta_0 - \sigma) \right]}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left( \frac{\eta - (\eta_0 - \sigma)}{\sigma} \right)^2} \)

signiMIN0[$\eta_-$, $\eta_0$, $\sigma$] := \( \text{NIntegrate}[\text{probMIN0}[\eta, \eta_0, \sigma], \{\eta, -\infty, \eta\}] \)

probMAX0[$\eta_-$, $\eta_0$, $\sigma$] := \( \left( \frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left( 1 + e^{\frac{\left[ \eta - (\eta_0 - \sigma) \right]}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left( \frac{\eta - (\eta_0 - \sigma)}{\sigma} \right)^2} \)

signiMAX0[$\eta_-$, $\eta_0$, $\sigma$] := \( \text{NIntegrate}[\text{probMAX0}[\eta, \eta_0, \sigma], \{\eta, \eta, \infty\}] \)

The significance signiMIN0[$\eta, \eta_0, \sigma$] is the Integral of probMIN0, i.e. signiMIN0 = $\int_{-\infty}^{\eta} P_{\text{MIN}}(\eta) \, d\eta$.

The significance signiMAX0[$\eta, \eta_0, \sigma$] is the Integral of probMAX0, i.e. signiMAX0 = $\int_{\eta}^{\infty} P_{\text{MAX}}(\eta) \, d\eta$.

7b. Generating random $\psi$ runs

The notebook .nb version generates new random runs. The pdf version uses old random runs that are uploaded from previously saved files that are not publically available. Thus both versions have some cells commented out: (* comments are not processed by Mathematica*).
Definitions:

- **nRunMax** number of random runs to be generated
- **ρRgnRadius** distance to furthest source from sourceCenter, radians
- **minGridCenterToHmin**, **max GridCenterToHmax** - minimum number of grid spaces between Hmin, Hmax and sources’ center
- **gridjηBarMinRand**
- **iSminmas** parameters for center to hub distance
- **nRunPrint** dummy index to control printing frequency
- **rSrcxGrid** unit vector perpendicular to the plane of rSrc for each source
- **ψSrcRand** random polarization directions for the sources
- **rSrcxψSrc** cross product of rSrc and the vector in direction of ψSrc, both are unit vectors
- **ηBarToGrid** \{grid point #, value of the alignment angle Eq. (1) averaged over all sources Si, in radians\}
- **sortηBarToGrid** - sort ηBarToGrid, smallest alignment angles ηj first
- **gridjηBarMinRand** - \{j, ηj\} for the grid point Hj with the smallest alignment angle ηj, not counting Hj that are too close to the sample
- **gridjηBarMaxRand** - \{j, ηj\} for the grid point Hj with the largest avoidance angle ηj, not counting Hj that are too close to the sample

niSnrData 1. run # 2. iSmin 3. iSmax 4. nSrc 6. ρRgnRadius
ψDataRand 1. run # 2. ψSrcRand table
runData 1. run # 2. sourceCenter 3. \{j, ηj\} at point Hj where smallest ηj 4. \{j, ηj\} at point Hj where largest ηj 5. nSrc 6. ρRgnRadius

(*Remove comment marks, "(*) and "*") below to generate your own table "runData". *)
(* Evaluate this cell for the notebook .nb version *)
(*
 nRunMax=500;
niSnrData={};
ψDataRand={};
rnData={};
times={};
(*Set up the For statement.*)
nRunPrint=0;
minGridCenterToHmin = 2;
(*minimum number of grid spaces between Hmin and sources' center*)
minGridCenterToHmax = 2;
(*minimum number of grid spaces between Hmax and sources' center*)
*)
\textbf{ln(343)=}

(* Evaluate this cell for the notebook .nb version *)
(* You may have found rSrcxrGrid already with uncertainty. Here it is again: *)
(*
  rSrcxrGrid1 =Table[ Cross[ rSrc[[i]],rGrid[[j]] ] , {i,nSrc},{j,nGrid}]
  (*first step: raw cross product, not unit vectors*)
  rSrcxrGrid =Table[ rSrcxrGrid1[[i,j]]/
     (rSrcxrGrid1[[i,j]].rSrcxrGrid1[[i,j]]+0.000001)^{1/2} , {i,nSrc},{j,nGrid}];
(*
  (*rSrcxrGrid: table of the unit vectors perpendicular to the plane
   of the great circle containing the source \( S_i \) and the grid point \( H_j \)*)

\textbf{56 | 20211009Clump2PaperFirst.nb}
(* Evaluate this cell for the notebook .nb version *)
(*
 t[1]=TimeUsed[];
 For[nRun=1,nRun\leq nRunMax,nRun++,
 If[nRun>nRunPrint,Print["At the start of run ",nRun," , the time is ",
 TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."];nRunPrint=nRunPrint+100];

\psi SrcRand=Table[RandomReal[{-0.001,\pi-0.001}],{i,nSrc}];
(*table of PPA angles \psi for the sources, in radians*)
\psi Src\times Src =
 Table[ Sin[\psi SrcRand[[i]]] eNSrc[[i]]-Cos[\psi SrcRand[[i]]] eESrc[[i]], {i,nSrc}];
(*table of the cross product of rSrc and vector in direction of \psi SrcRand, a unit vector*)
j\eta BarToGrid = Table[[j,(i/nSrc)\Sigma BarToGrid[\eta]
 Abs[ rSrc\times Src[[i]].rSrc\times Grid[[i,j]] ] - 0.000001 ],{i,nSrc}]];{j,nGrid}];
(*
 (grid point i, value of the alignment angle \eta H[j] averaged over all sources, in radians)*)
\eta j BarToGrid=Sort[j\eta BarToGrid,\eta[[2]]<\eta[[2]]&];
(*\eta BarToGrid, (j,\eta j), but sorted with the smallest alignment angles first*)
\eta i Smin=
 Catch[Do[If[ArcCos[sourceCenter.rGrid[[sort\eta BarToGrid[[i,1]] ]] ] -0.000001 ]/\theta\leq
 minGridCenterToHmin,Throw[[i]],{i,100}]];
grid\eta BarMinRand=sort\eta BarToGrid[[\eta i Smin]]; (* {i,\eta j}, at the grid point H_j with minimum \eta, not counting the center j0*)
\eta i Smax=
 Catch[Do[If[ArcCos[sourceCenter.rGrid[[sort\eta BarToGrid[[[-1,1]] ]] ] -0.000001 ]/\theta\leq
 minGridCenterToHmax,Throw[[i]],{i,100}]];
grid\eta BarMaxRand=sort\eta BarToGrid[[-\eta i Smax]]; (* {j,\eta j}, at the grid point H_j with maximum \eta, not counting the center j0*)
AppendTo[niSnrData,{nRun,\eta i Smin,\eta i Smax,nSrc,\rho RgnRadius}];
AppendTo[\psi DataRand,{nRun,\psi SrcRand}];
AppendTo[runData,
 {nRun,sourceCenter,[grid[[  grid\eta BarMinRand[[1]] ]], grid\eta BarMinRand[[2]]],
 {grid[[ grid\eta BarMaxRand[[1]] ]], grid\eta BarMaxRand[[2]]},nSrc,\rho RgnRadius } ]
(*collect data for saving in a file.*)
)

(* Evaluate this cell for the notebook .nb version *)
(*
 t[2]=TimeUsed[];
 Print["Computer time needed to generate random runs: ",t[2]-t[1]," seconds."]*)

(*Save a new table*)
SetDirectory[homeDirectory];
(*Put[niSnrData,"20211012niSnrDataQSON13Random2000a.dat" ]*)
(*Put[\psi DataRand,"20211012\psi DataRandQSON13Random2000a.dat" ]*)
(*Put[runData,"20211012runDataQSON13Random2000a.dat"]*)
(* Get and old ψ DataRand table*)
SetDirectory[homeDirectory];
(*niSnrData = Get["20210917niSnrDataQSON13Random2000a.dat"] *)
(*ψDataRand = Get["20210917niSnrDataQSON13Random2000a.dat"] *)
(* Get the runData files for the pdf version:*)

runData2000a = Get["20210917runDataQSON13Random2000a.dat"];
runData8000a = Get["20210917runDataQSON13Random8000a.dat"];

(* Edit the following statements to Join separate data files, if needed*)
(* Join the runData files for the pdf version:*)

runData = Join[runData2000a, runData8000a];
nRunMax = Length[runData];

7c. Analyzing random ψ runs

Definitions:

ηBarminData  η_{min} in order of random runs
sortηBarmin  sorted
η0Bmin, σBmin  mean and standard deviation of ηBarminData
hlmin, hlmin0  histogram data
nlmBmin  fit to η_{min} histogram
{a,b,x0}  best fit parameters
showNlmBmin  figure displaying the fit to the η_{min} from random runs
nlmBminPtable  Parameter table for the fit

ηBarmaxData  η_{max}
sortηBarmax  sorted
η0Bmax, σBmax  mean and standard deviation of ηBarmaxData
hlmax, hlmax0  histogram data
nlmBmax  fit to η_{max} histogram
{a,b,x0}  best fit parameters
showNlmBmax  figure displaying the fit to the η_{max} from random runs
nlmBmaxPtable  Parameter table for the fit

rHminR  rGrid at H_{min}
anglerHminToCenter  θ from H_{min} to sourceCenter
θrHminToCenter, σθrHminToCenter  - mean and standard deviation of θ
Print["There are ", Length[runData], ", ")
There are 10,000 random runs to analyze.

runData
1. nRun 2. r at Region Center 3a. grid data for Hmin 3b. \(\eta_{min}\) 4a. grid data for Hmax 4b. \(\eta_{max}\) 5. nSrc 6. radius \(\rho\)RgnRadius

\(\eta\)BarminData = Table[runData[[i1, 3, 2]], {i1, Length[runData]}];
\(\eta\)BarmaxData = Table[runData[[i1, 4, 2]], {i1, Length[runData]}];
runRmax = Table[runData[[i1, 3, 1, 6]], {i1, Length[runData]}];
runRmin = Table[[(1/2) \[ \text{hlmin0}[1, i1] + hlmin0[[1, i1 + 1]]], hlmin0[[2, i1]]],
{i1, Length[ hlmin0[[2]] ]}];

nlmBmin = NonlinearModelFit[hlmin, \(\{a, \eta_{min}\} / \) \(\{b, \sigma_{min}\}, \{x0, \eta_{min}\}\}, x];

\{a, Length[runData] \}/12,
{b, \sigma_{min}}, \{x0, \eta_{min}\}, x];

{amin, bmin, x0min} = \{a, b, x0\} / . nlmBmin["BestFitParameters"]; {amin, bmin, x0min} = nlmin["ParameterErrors"]; (*x is \(\eta\)Barmin*)

sort\(\eta\)Barmax = Sort[\(\eta\)BarmaxData];
\(\eta\)Bmax = mean[\(\eta\)BarmaxData]; (*Guess the mean for the Gaussian.*)
\(\sigma\)Bmax = stanDev[\(\eta\)BarmaxData]; (*Guess the half-width.*)
hlmax0 = HistogramList[sort\(\eta\)Barmin, \{\(\eta\)Bmin - 5 \(\sigma\)Bmax, \(\eta\)Bmin + 5 \(\sigma\)Bmax, 0.4 \(\sigma\)Bmin\}];
hlmax = Table[[(1/2) \[ \text{hlmax0}[1, i1] + hlmax0[[1, i1 + 1]]], hlmax0[[2, i1]]],
{i1, Length[ hlmax0[[2]] ]}];

nlmBmax = NonlinearModelFit[hlmax, \(\{a, \eta_{max}\} / \) \(\{b, \sigma_{max}\}, \{x0, \eta_{max}\}\}, x];

{amax, bmax, x0max} = \{a, b, x0\} / . nlmBmax["BestFitParameters"]; {amax, bmax, x0max} = nlmBmax["ParameterErrors"]; (*x is \(\eta\)Barmax*)
anglerHminToCenter = Table[ArcCos[Abs[rHminR[[i]].sourceCenter - 0.00001]], {i, Length[rHminR]}];

θrHminToCenter = mean[anglerHminToCenter];
σθrHminToCenter = stanDev[anglerHminToCenter];

anglerHmaxToCenter = Table[ArcCos[Abs[rHmaxR[[i]].sourceCenter - 0.00001]], {i, Length[rHmaxR]}];

θrHmaxToCenter = mean[anglerHmaxToCenter];
σθrHmaxToCenter = stanDev[anglerHmaxToCenter];

fitData = {{nSrc, ρRgnRadius, ρRMS}, {x0min, dx0min}, {bmin, dbmin}, {amin, damin},
{x0max, dx0max}, {bmax, dbmax}, {amax, damax}, {σθrHminToCenter, θrHminToCenter}, {σθrHmaxToCenter, θrHmaxToCenter}} (*collect data for saving in a file.*);

ListPlot[{sortηBarmin, sortηBarmax}];
ListPlot[hlmin];
Normal[nlmBmin];
Print["The parameter table for the fit to η\_min: "]

nlmBminPtable = nlmBmin["ParameterTable"]
Normal[nlmBmax];
Print["The parameter table for the fit to η\_max: "]

nlmBmaxPtable = nlmBmax["ParameterTable"]

The parameter table for the fit to η\_min:

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1591.22</td>
<td>9.66492</td>
<td>164.639</td>
<td>1.67881 \times 10^{-35}</td>
</tr>
<tr>
<td>b</td>
<td>0.0823933</td>
<td>0.000559361</td>
<td>147.299</td>
<td>1.93825 \times 10^{-34}</td>
</tr>
<tr>
<td>x0</td>
<td>0.531878</td>
<td>0.000468054</td>
<td>1136.36</td>
<td>5.90205 \times 10^{-54}</td>
</tr>
</tbody>
</table>

The parameter table for the fit to η\_max:

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1592.54</td>
<td>14.7099</td>
<td>108.263</td>
<td>1.67991 \times 10^{-31}</td>
</tr>
<tr>
<td>b</td>
<td>0.0807127</td>
<td>0.000833291</td>
<td>96.8601</td>
<td>1.93433 \times 10^{-30}</td>
</tr>
<tr>
<td>x0</td>
<td>1.0418</td>
<td>0.000697268</td>
<td>1494.12</td>
<td>1.432 \times 10^{-56}</td>
</tr>
</tbody>
</table>

"fitData" table

1a. nSrc, number of sources  1b. rgnRadius, nominal radius of region  1c. RMS radius
2a. x0min: x0 = η\_0 align (min)  2b. dx0min error: dx0 - σ for x0 = η\_0 align (min)
3a. bmin: b = σ align (min)  3b. dbmin: err: db - σ for b = σ align (min)
4a. amin: a = Amplitude align (min)  4b. damin: err: da - σ for a = Amplitude align (min)
5a. x0max: x0 = η\_0 avoid (max)  5b. dx0max0max: err: dx0 - σ for x0 = η\_0 avoid (max)
6a. bmax: b = σ avoid (max)  6b. dbmax: err: db - σ for b = σ avoid (max)
7a. amax: a = Amplitude avoid (max)  7b. damax: err: da - σ for a = Amplitude avoid (max)
8a. σθHminToCenter: stanDev[anglerHminToCenter] - σ for θ to H  8b. θHminToCenter: mean[anglerHminToCenter] - θ to H
9a. σθHmaxToCenter: stanDev[anglerHmaxToCenter] - σ for θ to H  9b. θHmaxToCenter: mean[anglerHmaxToCenter] - θ to H
7d. Significance of the alignment and avoidance Hub Test metrics for the sample studied in this work

Definitions

fitting function parameters from random runs:

$\eta_{\text{min}}$ mean of probability distribution for smallest alignment angle $\eta_{\text{min}}$
d$\eta_{\text{min}}$ standard error in the mean as reported by Mathematica
$\sigma_{\text{min}}$ half-width of probability distribution for smallest alignment angle $\eta_{\text{min}}$
d$\sigma_{\text{min}}$ standard error in the half-width as reported by Mathematica
\( \eta_{0\text{max}} \) mean of probability distribution for largest avoidance angle \( \bar{\eta}_{\text{max}} \)

\( d\eta_{0\text{max}} \) standard error in the mean as reported by Mathematica

\( \sigma_{\text{max}} \) half-width of probability distribution for largest avoidance angle \( \bar{\eta}_{\text{max}} \)

\( d\sigma_{\text{max}} \) standard error in the half-width as reported by Mathematica

\( \text{probmin} \) probability distribution for smallest alignment angle \( \eta_{\text{min}} \). This depends on the random runs.

\( \text{signimin} \) significance, integral of probmin over smaller values of \( \eta_{\text{min}} \)

\( \text{probmax} \) probability distribution for largest avoidance angle \( \eta_{\text{max}} \)

\( \text{signimax} \) significance, integral of probmax over larger values of \( \eta_{\text{max}} \)

\( \text{sigBarMinfunDataObs} \) Significance of the smallest alignment angle \( \eta_{\text{min}} \)

\( \text{sigRangeBarMinfunDataObs} \) standard errors in \( \eta_{\text{0min}} \) and \( \sigma_{\text{min}} \), i.e. \( d\eta_{\text{0min}} \) and \( d\sigma_{\text{min}} \), give the significances plus/minus values

\( \text{sigSmallBarMinfunDataObs, Big} \) extremes of significance assuming one standard error

\( \text{sigBarMaxfunDataObs} \) Significance of the largest avoidance angle \( \eta_{\text{max}} \)

\( \text{sigRangeBarMaxfunDataObs} \) standard errors in \( \eta_{\text{0max}} \) and \( \sigma_{\text{max}} \), i.e. \( d\eta_{\text{0max}} \) and \( d\sigma_{\text{max}} \), give the significances plus/minus values

\( \text{sigSmallBarMaxfunDataObs, Big} \) extremes of significance assuming one standard error

\begin{verbatim}
In[391]= (*Parameters \( \eta_0 \) and \( \sigma \) from random runs, together with their standard errors.*)
 \[ \eta_{0\text{min}} = x_{0\text{min}}; d\eta_{0\text{min}} = dx_{0\text{min}}; \]
 \[ \eta_{0\text{max}} = x_{0\text{max}}; d\eta_{0\text{max}} = dx_{0\text{max}}; \]
 \[ \sigma_{\text{min}} = b_{\text{min}}; d\sigma_{\text{min}} = db_{\text{min}}; \]
 \[ \sigma_{\text{max}} = b_{\text{max}}; d\sigma_{\text{max}} = db_{\text{max}}; \]

In[395]= probmin[\_\_] := probMIN0[\_\_, \eta_{0\text{min}}, \sigma_{\text{min}}]
 signimin[\_\_] := signiMIN0[\_\_, \eta_{0\text{min}}, \sigma_{\text{min}}]
 probmax[\_\_] := probMAX0[\_\_, \eta_{0\text{max}}, \sigma_{\text{max}}]
 signimax[\_\_] := signiMAX0[\_\_, \eta_{0\text{max}}, \sigma_{\text{max}}]
\end{verbatim}
Print["For this sample, but with random polarization directions $\psi$, the random runs give the mean value $\eta_{0\text{min}}$ and the half-width $\sigma_{\text{min}}$ of the fitting function of random runs for the smallest alignment angle $\eta_{\text{min}}$:"]

Print[" $\eta_{0\text{min}} = \eta_0 \text{min} \left(\frac{360.}{2.\pi}\right), \sigma_{\text{min}} \text{ and } \sigma_{\text{min}} = \eta_{0\text{min}} (\text{Random } \psi \text{ distribution})"]

Print[""]

Print["For this sample, but with random polarization directions $\psi$, the random runs give the mean $\eta_{0\text{max}}$ and the half-width $\sigma_{\text{max}}$ for the distributions for avoidance :"]

Print[" $\eta_{0\text{max}} = \eta_0 \text{max} \left(\frac{360.}{2.\pi}\right), \sigma_{\text{max}} \text{ and } \sigma_{\text{max}} = \eta_{0\text{max}} (\text{Random } \psi \text{ distribution})"]

For this sample, but with random polarization directions $\psi$, the random runs give the mean $\eta_{0\text{min}}$ and the half-width $\sigma_{\text{min}}$ of the fitting function of random runs for the smallest alignment angle $\eta_{\text{min}}$:

$\eta_{0\text{min}} = 30.4744^\circ \pm 0.0268175^\circ$ and $\sigma_{\text{min}} = 4.72079^\circ \pm 0.032049^\circ$. (Random $\psi$ distribution)

For this sample, but with random polarization directions $\psi$, the random runs give the mean $\eta_{0\text{max}}$ and the half-width $\sigma_{\text{max}}$ for the distributions for avoidance :

$\eta_{0\text{max}} = 59.6907^\circ \pm 0.0399505^\circ$ and $\sigma_{\text{max}} = 4.6245^\circ \pm 0.0477441^\circ$. (Random $\psi$ distribution)

(*Significance of the smallest alignment angle $\eta_{\text{min}}$.*)

sigBarMinfunDataObs = signimin[\etaBarMinfunDataObs];
sigrangeBarMinfunDataObs =
Sort[Partition[Flatten[Table[{signiMIN0[\etaBarMinfunDataObs, \eta0\text{min} + \gamma1 \text{ d}\eta\text{min}, \sigma\text{min} + \gamma2 \text{ d}\sigma\text{min}}, \gamma1, \gamma2}, \{\gamma1, -1, 1\}, \{\gamma2, -1, 1\} ]}, 3]]];
sigSmall\etaBarMinfunDataObs = sigrangeBarMinfunDataObs[[-1]]];
sigBig\etaBarMinfunDataObs = sigrangeBarMinfunDataObs[[1, 1]]];

(*Significance of the largest avoidance angle $\eta_{\text{max}}$.*)

sigBarMaxfunDataObs = signimax[\etaBarMaxfunDataObs];
sigrangeBarMaxfunDataObs =
Sort[Partition[Flatten[Table[{signiMAX0[\etaBarMaxfunDataObs, \eta0\text{max} + \gamma1 \text{ d}\eta\text{max}, \sigma\text{max} + \gamma2 \text{ d}\sigma\text{max}}, \gamma1, \gamma2}, \{\gamma1, -1, 1\}, \{\gamma2, -1, 1\} ]}, 3]]];
sigSmall\etaBarMaxfunDataObs[1];
sigBig\etaBarMaxfunDataObs = sigrangeBarMaxfunDataObs[[1]];
In[414]:= (*The names "gridj
BarMinRan", "j
BarMax" are, or perhaps were, similar to quantities below, so save the current values labeled by "Best".*)
(* j
Bar entries: 1. grid point %, 2. alignment angle *)
{j
BarMinBest, j
BarMaxBest} = {η
BarMinfunDataObs, η
BarMaxfunDataObs};

In[415]:= Print["The smallest alignment angle is η
min = ", η
BarMinfunDataObs ∗ (360. ∕ (2. π)), "°, which has a significance of sig. = ", sig
BarMinfunDataObs, ", plus/minus = + ", sigBig
η
BarMinfunDataObs - sig
η
BarMinfunDataObs, ", and - ", sig
η
BarMinfunDataObs - sigSmal
η
BarMinfunDataObs, ", giving a range from sig. = ", sigSmall
η
BarMinfunDataObs, ", to ", sigBig
η
BarMinfunDataObs, "."]
Print["The largest avoidance angle is η
max = ", η
BarMaxfunDataObs ∗ (360. ∕ (2. π)), "°, which has a significance of sig. = ", sig
η
BarMaxfunDataObs, ", plus/minus = + ", sigBig
η
BarMaxfunDataObs - sig
η
BarMaxfunDataObs, ", and - ", sig
η
BarMaxfunDataObs - sigSmal
η
BarMaxfunDataObs, ", giving a range from sig. = ", sigSmall
η
BarMaxfunDataObs, ", to ", sigBig
η
BarMaxfunDataObs, "."]
Print["These uncertainties are due to the standard errors for the parameters in the fit to the random runs."]

The smallest alignment angle is η
min = 10.8648 °, which has a significance of sig. = 0.0000199444, plus/minus = ± 3.14916 ∙ 10⁻⁶ and - 2.77319 ∙ 10⁻⁶, giving a range from sig. = 0.0000171712 to 0.0000230936.
The largest avoidance angle is η
max = 62.6651 °, which has a significance of sig. = 0.317233 , plus/minus = ± 0.00660697 and - 0.00670283 , giving a range from sig. = 0.31116 to 0.32324 .

These uncertainties are due to the standard errors for the parameters in the fit to the random runs.

Print["More Statistics of the Alignment Function \( \bar{\eta}(H) \) : "]
Print[" "]
Print["The min alignment angle, η
min = ", η
BarMinfunDataObs ∗ (360. ∕ (2. π)), "°, is \( \Delta \eta \) = ", (η
0
min - η
BarMinfunDataObs) ∗ (360. ∕ (2. π)), "° below the most likely value, ", η
0
min ∗ (360. ∕ (2. π)), "°, for random runs."]
Print["Since the half-width \( \sigma \) is ", \( \sigma \) ∗ (360. ∕ (2. π)), "°, the difference, \( \Delta \eta \) = ", (η
0
min - η
BarMinfunDataObs) ∗ (360. ∕ (2. π)), "° makes \( \eta \)\/min separated from the most likely random run value by ", (η
0
min - η
BarMinfunDataObs) ∕ \( \sigma \) ∗ "°."
Print["Thus, the smallest alignment angle \( \bar{\eta}_{\text{min}} \) is ", (η
0
min - η
BarMinfunDataObs) ∕ \( \sigma \) ∗ "° below the most likely random run value. (Very Significant)"
Print[" "]
Print["The max avoidance angle, η
max = ", η
BarMaxfunDataObs ∗ (360. ∕ (2. π)), "°, is \( \Delta \eta \) = ", -(η
0
max - η
BarMaxfunDataObs) ∗ (360. ∕ (2. π)), "° above the most likely value, " , η
0
max ∗ (360. ∕ (2. π)), "°, for random runs."]
Print["Since the half-width \( \sigma \) is ", \( \sigma \) ∗ (360. ∕ (2. π)), "°, the difference \( \Delta \eta \) = ", -(η
0
max - η
BarMaxfunDataObs) ∗ (360. ∕ (2. π)), "° makes \( \eta \)\/max separated from the most likely random run value by ", -(η
0
max - η
BarMaxfunDataObs) ∕ \( \sigma \) ∗ "°."
Print["Thus, the smallest avoidance angle \( \bar{\eta}_{\text{max}} \) is ", -(η
0
max - η
BarMaxfunDataObs) ∕ \( \sigma \) ∗ "° above the most likely random run value. (Not significant) "]
More Statistics of the Alignment Function $\eta(H)$:

The min alignment angle, $\eta_{\text{min}} = 10.8648^\circ$, is $\Delta \eta = 19.6096^\circ$ below the most likely value, $30.4744^\circ$, for random runs. Since the half-width $\sigma$ is $4.72079^\circ$, the difference $\Delta \eta = 19.6096^\circ$ makes $\eta_{\text{min}}$ separated from the most likely random run value by $4.15388\sigma$s.

Thus, the smallest alignment angle $\eta_{\text{min}}$ is $4.15388\sigma$s below the most likely random run value. (Very Significant)

The max avoidance angle, $\eta_{\text{max}} = 62.6651^\circ$, is $\Delta \eta = 2.97431^\circ$ above the most likely value, $59.6907^\circ$, for random runs. Since the half-width $\sigma$ is $4.6245^\circ$, the difference $\Delta \eta = 2.97431^\circ$ makes $\eta_{\text{max}}$ separated from the most likely random run value by $0.643165\sigma$s.

Thus, the smallest avoidance angle $\eta_{\text{max}}$ is $0.643165\sigma$s above the most likely random run value. (Not significant)

7c. Conclusion

The avoidance of the polarization directions for points on the Celestial Sphere is not significant, with $S = 0.32$. That means about one in three random runs would avoid some place on the Celestial Sphere better than the sample avoids $H_{\text{max}}$. That is not significant.

The polarization directions are very significantly aligned, with $S = 2.10^{-5}$. That means about one random run in $50,000$ would produce better alignment, a “$4\sigma$” result.

The polarization directions converge on the hub $H_{\text{min}}$ with a smallest alignment angle $\eta_{\text{min}}$ that is very significant. They are therefore correlated.

```
In[427]:= Print["The computer time on my computer is about one minute because I have uploaded the uncertainty runs and random runs from saved data files.""]
Print["The same computer takes about 10 minutes to complete the .nb version with the bulk of the time needed to generate 500 uncertainty runs and 500 random runs."]
```

The computer time on my computer is about one minute because I have uploaded the uncertainty runs and random runs from saved data files.
The same computer takes about 10 minutes to complete the .nb version with the bulk of the time needed to generate 500 uncertainty runs and 500 random runs.

```
In[429]:= Print["The computer time expended so far is ", TimeUsed[], " seconds."]
```

The computer time expended so far is 74.067 seconds.