On consecutive special primes

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Abstract

We establish a number-theoretic conjecture about primes with a special property and give a hint for the proof.

Preliminaries

Let $w: \mathbb{N}^* \to \mathbb{N}^*$, where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, that $\forall n := \sum_{i=0}^{j} \alpha_i \beta^i$ be $w(n) = \sum_{i=0}^{j} \alpha_i$.[1] Then there exist $S \in \mathbb{P}^s \subset \mathbb{P}^* := \mathbb{P} \setminus \{2,3\}$ that $w(S_{i+1}) = w(S_i) \in \mathbb{P}^*$ for $i \in \mathbb{N}^*$. For such special primes let $\Omega := S_{i+1} - S_i$ be a special prime gap. Here $\Omega = \sum_{i=1}^{j} g_i$ for $j \neq 1$ is composed of consecutive prime gaps.[2] Obviously, there exists the congruence relation $\Omega \equiv 0 \pmod{3\# \cdot 3}$ whereby we consider the case $\Omega = 3\# \cdot 3$ for special twin prime pairs. We also can introduce the counting function $\pi_\Omega(x, p) := \#\{S \leq x: w(S) = p \text{ with } p \in \mathbb{P}^*\}$.

Conjecture. There exist infinitely many Ω such that $(\Omega - 1,\ \Omega + 1) \in \mathbb{P}^*$.

Note on the process of proof

One may consider as highly relevant to prove the conjecture the statement of the theorem of Drmota, Mauduit and Rivat [1] [2] that there are infinitely many $p \in \mathbb{P}$ with $w(p) \in \mathbb{P}$, and, since Ω does indeed satisfy the introduced congruence relation, that clearly every twin prime pair except the very first one of the form $(3\# \cdot n - 1,\ 3\# \cdot n + 1)$ for some $n \in \mathbb{N}^*$ [3].

References

[1] Drmota, M.; Mauduit, C.; Rivat, J.: Primes with an average sum of digits. *Compositio Mathematica*. 145, pp. 271–292, 2009.

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[1] Here we consider the expansion for the base $\beta = 10$.

[2] That is $\pi(S_{i+1}) - \pi(S_i) > 1$, where per usual one writes $\pi(x)$ for the prime counting function.

[2] Harman, G.: Counting Primes whose Sum of Digits is Prime. *Journal of Integer Sequences*, Vol. 15, 2012.
[3] Fine, B.; Rosenberger, G.: Number Theory: An Introduction via the Density of Primes. Birkhäuser, 2016.