Odd or Even

Prime Factor Game

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Elementary proof of “When choosing any natural number greater than 
or equal to 2 that does not have a power of a prime number as a 
divisor, the probability that the number of prime factors is odd or even 
is equal to 1/2, the probability of getting heads or tails tossing a coin.”

When choosing any number from natural numbers greater than or equal to 2
The probability of being a multiple of ...

2 is \(\frac{1}{2}\)

2 or 3 is \(\frac{1}{2} \times \frac{1}{3} - \frac{1}{2 \times 3}\)

2 or 3 or 5 is \(\frac{1}{2} + \frac{1}{3} \times \frac{1}{2 \times 3} + \frac{1}{5} \times \frac{1}{2 \times 5} - \frac{1}{2 \times 3 \times 5}\)

any prime multiple is

\(\frac{1}{2} + \frac{1}{3} - \frac{1}{2 \times 3} + \frac{1}{5} - \frac{1}{2 \times 5} + \frac{1}{3 \times 5} + \frac{1}{7} - \frac{1}{2 \times 7} - \frac{1}{3 \times 7} + \frac{1}{5 \times 7} + \frac{1}{2 \times 3 \times 7} + \frac{1}{3 \times 5 \times 7} - \frac{1}{2 \times 3 \times 5 \times 7} + \ldots = 1\)

Using the above formula, we can find the probability that the number of prime 
factors is odd.

\(\frac{1}{2} + \frac{1}{3} - \frac{1}{2 \times 3} + \frac{1}{5} - \frac{1}{2 \times 5} + \frac{1}{3 \times 5} + \frac{4}{7} - \frac{1}{2 \times 7} - \frac{2}{3 \times 7} + \frac{4}{5 \times 7} + \frac{2}{2 \times 3 \times 7} + \frac{4}{3 \times 5 \times 7} - \frac{8}{2 \times 3 \times 5 \times 7} + \ldots\)

The expression is rearranged, all but the first term, \(\frac{1}{2}\), are eliminated as follows.

\(\frac{1}{2} + \left(\frac{1}{3} - \frac{1}{2 \times 3}\right) + \left(\frac{1}{5} - \frac{1}{2 \times 5}\right) + \left(\frac{2}{3 \times 5} + \frac{4}{2 \times 3 \times 5}\right) + \left(\frac{1}{7} - \frac{2}{2 \times 7}\right) + \left(-\frac{2}{3 \times 7} + \frac{4}{2 \times 3 \times 7}\right) + \left(-\frac{2}{5 \times 7} + \frac{4}{2 \times 5 \times 7}\right) + \left(-\frac{8}{3 \times 5 \times 7} - \frac{2}{2 \times 3 \times 5 \times 7}\right) + \ldots\)

\(\therefore\) When choosing any natural number greater than or equal to 2, the 
probability that the number of prime factors is odd or even is \(\frac{1}{2}\)
Let \( P_{(n)} \) be the probability that the number of prime factors is odd when choosing any natural number that is a multiple of the \((n)\)th prime, then

\[
P_{(1)} = 1 - \frac{1}{3} + \frac{2}{5} - \frac{1}{7} + \frac{2}{3 \times 7} - \frac{2}{5 \times 7} - \frac{2}{3 \times 5 \times 7} - \cdots = \frac{1}{2} \\
P_{(2)} = 1 - \frac{1}{2} + \frac{2}{5} - \frac{1}{7} + \frac{2}{2 \times 7} - \frac{2}{5 \times 7} - \frac{4}{2 \times 5 \times 7} - \cdots = \frac{1}{2} \\
P_{(3)} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{2}{2 \times 3} - \frac{1}{7} + \frac{2}{2 \times 7} + \frac{2}{3 \times 7} - \frac{4}{2 \times 3 \times 7} - \cdots = \frac{1}{2} \\
\vdots
\\nP_{(n)} = \frac{1}{2}
\]

Probability that the number of prime factors is odd when choosing any natural number that is a multiple of the \((n)\)th prime

\[= \text{Probability that the number of prime factors is odd when choosing any natural number that is a multiple of the power of the } (n)\text{th prime} \]

\[\therefore \quad \text{When choosing any natural number that has a power of a prime number as a factor, the probability that the number of prime factors is odd or even is } \frac{1}{2}\]

When choosing any number from natural numbers greater than or equal to 2, let \(C\) be the probability that it is a multiple of the power of a prime number. The probability of the number of prime factors is odd or even that are not multiple of the power of a prime number is \(\frac{1-C}{2}\)

\[\therefore \quad \text{When choosing any number from natural numbers greater than or equal to 2, except for numbers that a power of a prime number as divisors, the probability that the number of prime factors is odd or even is } \frac{1}{2}, \text{ equal to the probability of getting heads or tails tossing a coin.}\]

Goodbye Riemann hypothesis