Abstract. This paper proves an inconsistency within ZFC by showing that a strengthened form of the strong Goldbach conjecture as well as its negation can be deduced.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1, let \( \mathbb{N}_n \) denote the natural numbers starting from \( n > 1 \) and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): Every even integer greater than 6 can be expressed as the sum of two different primes.

Theorem. Both SSGB and the negation \( \neg \text{SSGB} \) hold.

Proof. We define the set \( S_g := \{ (p_k, m_k, q_k) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \).

SSGB is equivalent to saying that every integer \( x \geq 4 \) is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a middle component \( m_k \) of \( S_g \). So, by the definitions we have

\[
\text{SSGB } \iff \forall x \in \mathbb{N}_4 \exists (p_k, m_k, q_k) \in S_g \quad x = m.
\]

\[
\neg \text{SSGB } \iff \exists x \in \mathbb{N}_4 \forall (p_k, m_k, q_k) \in S_g \quad x \neq m.
\]

The set \( S_g \) has the following two properties.

First, the whole range of \( \mathbb{N}_3 \) can be expressed by the triple components of \( S_g \) ("covering"), because every integer \( x \geq 3 \) can be written as some \( p_k \) with \( k = 1 \) when \( x \) is prime, as some \( p_k \) with \( k \neq 1 \) when \( x \) is composite and not a power of 2, or as \( (3 + 5)k / 2 \) when \( x \) is a power of 2; \( p \in \mathbb{P}_3, k \in \mathbb{N} \). So we have

\[
(C) \quad \forall x \in \mathbb{N}_3 \exists (p_k, m_k, q_k) \in S_g \quad x = pk \lor x = mk = 4k.
\]

Second, due to the definition of the set \( S_g \), all pairs \( (p, q) \) of distinct odd primes are used ("maximality"). So we have

\[
(M) \quad \forall p, q \in \mathbb{P}_3, p < q \forall k \in \mathbb{N} \quad (p_k, m_k, q_k) \in S_g, \text{ where } m = (p + q) / 2.
\]
In case of \( \neg \text{SSGB} \) there is at least one \( n \in \mathbb{N}_4 \) that is different from all the numbers \( m \) defined in \( S_g \). In case of SSGB there is no such \( n \).

The following argumentation is independent of the choice of \( n \) if there is more than one.

If (C) or (M) did not hold, an \( n \) different from all \( m \) could exist for the reason that \( n \) is different from all \( S_g \) triple components \( p_k, m_k, q_k \) or for the reason that \( n \) is the arithmetic mean of a pair of primes not used in \( S_g \). This would not lead to a contradiction. However, since both (C) and (M) hold, these two possibilities are ruled out and we can proceed. In other words, ((C) and (M)) implies the proof below, the basic idea of which is this:

Since, due to (C), every \( n \) given by \( \neg \text{SSGB} \) as well as every multiple \( nk, k \in \mathbb{N} \), equals a component of some \( S_g \) triple that exists by definition, the \( S_g \) triples are the same in the case \( n \) exists (\( \neg \text{SSGB} \)) and in the case \( n \) does not exist (SSGB). This leads to a contradiction because in the case SSGB the numbers \( m \) defined in \( S_g \) take all integer values \( x \geq 4 \) whereas in the case \( \neg \text{SSGB} \) they don’t.

We split \( S_g \) into two complementary subsets: For any \( y \in \mathbb{N}_3 \), \( S_g = S_g^+(y) \cup S_g^{-}(y) \), where
\[
S_g^+(y) := \{ (p_k, m_k, q_k) \in S_g \mid \exists k' \in \mathbb{N} \quad p_k = yk' \lor m_k = yk' \lor q_k = yk' \} \quad \text{and}
\]
\[
S_g^{-}(y) := \{ (p_k, m_k, q_k) \in S_g \mid \forall k' \in \mathbb{N} \quad p_k \neq yk' \land m_k \neq yk' \land q_k \neq yk' \}.
\]

Now, we define
\[
S_1 := \{ (p_k, m_k, q_k) \in S_g \mid \neg \text{SSGB} \text{ holds} \}
\]
\[
S_2 := \{ (p_k, m_k, q_k) \in S_g \mid \text{SSGB holds} \}.
\]

So, \( S_1 \) is defined to be the set of all \( S_g \) triples such that the numbers \( m \) don’t take all integer values \( x \geq 4 \), and \( S_2 \) is defined to be the set of all \( S_g \) triples such that the numbers \( m \) take all integer values \( x \geq 4 \).

On the other hand, the condition \( \neg \text{SSGB} \) is equivalent to the existence of an \( n \geq 4 \) that is different from all \( m \) defined in \( S_g \).

Therefore, since every triple of \( S_g \) either belongs to \( S_g^+(y) \) or to \( S_g^{-}(y) \) for any \( y \in \mathbb{N}_3 \), we obtain
\[
S_1 = S_g^+(n) \cup S_g^{-}(n) = S_g.
\]
Analogously, the condition SSGB is equivalent to the non-existence of an \( n \geq 4 \) that is different from all \( m \) defined in \( S_g \). Therefore, all triples of \( S_g \) belong to \( S_2 \), and so we trivially obtain
\[ S_2 = S_g. \]

Since \( S_g \) is non-empty, we have
\[
\begin{align*}
(1) \quad &-SSGB \iff S_g = S_1 \\
(2) \quad &SSGB \iff S_g = S_2.
\end{align*}
\]

Because of \( S_1 = S_2 = S_g \), (1) and (2) imply \((-SSGB \iff SSGB)\), which yields the contradiction \((SSGB \land -SSGB)\).

\[ \square \]

**Note.** The proof is based on the following general principle.
Suppose there is a non-empty set \( A \) and a proposition \( P \) such that
\[
\begin{align*}
A_1 &:= \{ a \in A \mid -P \text{ holds} \} = A \\
\text{and} \\
A_2 &:= \{ a \in A \mid P \text{ holds} \} = A.
\end{align*}
\]

Then, we have a contradiction for the following reason.

Since \( A \) is non-empty, we have
\[
\begin{align*}
(1) \quad &-P \iff A = A_1 \\
(2) \quad &P \iff A = A_2.
\end{align*}
\]

Because of \( A_1 = A_2 = A \), (1) and (2) imply \((-P \iff P)\), which yields \((P \land -P)\).