An Operator Theory Problem Book:
Chapter 5

MOHAMMED HICHEM MORTAD, Ph.D.

THE UNIVERSITY OF ORAN I, AHMED BEN BELLA.
CHAPTER 5

Positive operators. Square root

5.1. Exercises with Solutions

Exercise 5.1.1. Are the matrices
\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}. \]
positive?

Exercise 5.1.2. Let \( S \) be the shift operator on \( \ell^2(\mathbb{N}) \). Is \( I - SS^* \) positive?

Exercise 5.1.3. Let \( A \in B(\ell^2) \) be the multiplication operator defined by:
\[ A(x_1, x_2, \cdots, x_n, \cdots) = (\alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, \cdots) \]
where \((\alpha_n)_n \in \ell^\infty\). Show that
\[ A \geq 0 \iff \alpha_n \geq 0, \ \forall n \in \mathbb{N}. \]

Exercise 5.1.4. Let \( A \in B(H) \) be self-adjoint. Show that \( e^A \) is positive.

Exercise 5.1.5. Let \( A, B \in B(H) \) be both positive. Does it follow that \( AB + BA \geq 0 \)?

Exercise 5.1.6. Let \( A \) and \( B \) be two bounded and positive operators on a complex Hilbert space \( H \). Show that if \( A + B = 0 \), then \( A = B = 0 \).

Exercise 5.1.7. Let \( A \) be a matrix on a finite dimensional space such that \( A \geq 0 \) and \( \text{tr} A = 0 \). Show that \( A = 0 \).

Exercise 5.1.8. Let \( A, B, T \in B(H) \) where \( A \) and \( B \) are self-adjoint.

(1) Show that:
\[ A \geq 0 \implies T^* AT \geq 0 \quad \text{and} \quad TAT^* \geq 0. \]

(2) Show that:
\[ A \geq B \implies T^* AT \geq T^* BT \quad \text{and} \quad TAT^* \geq TBT^*. \]
Exercise 5.1.9. Let $P, Q \in B(H)$ be two orthogonal projections. Show that $P - Q$ is an orthogonal projection iff $P \geq Q$.

Exercise 5.1.10. Let $A \in B(H)$ be positive.
(1) Show that
$$\left| \langle Ax, y \rangle \right|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle$$
for all $x, y \in H$.
(2) Infer that for every $x \in H$,
$$\|Ax\|^2 \leq \|A\| \langle Ax, x \rangle.$$

Exercise 5.1.11. Let $A \in B(H)$ be self-adjoint.
(1) Show that
$$-I \leq A \leq I \iff \|A\| \leq 1.$$
(2) Let $\alpha \geq 0$. Show that
$$-\alpha I \leq A \leq \alpha I \iff \|A\| \leq \alpha.$$

Exercise 5.1.12. Let $A, B \in B(H)$ be self-adjoint where $A \geq 0$. Show that
$$-A \leq B \leq A \implies \|B\| \leq \|A\|.$$

Exercise 5.1.13. Let $A, B \in B(H)$ be both positive. Show that
$$\|A - B\| \leq \max(\|A\|, \|B\|).$$

Exercise 5.1.14. Let $A, K \in B(H)$ be such that $A$ is positive and $AK$ is self-adjoint. Prove that
$$\left| \langle AKx, x \rangle \right| \leq \|K\| \|Ax, x\|$$
for all $x \in H$.

Exercise 5.1.15. (cf. Exercise 5.1.29) Let $A \in B(H)$ be positive and let $K \in B(H)$ be a contraction. Show that if $AK^* = KA$, then
$$K^2A = A(K^*)^2 = KAK^* \leq A.$$

Exercise 5.1.16. Let $A, B \in B(H)$ be commuting and positive. Using the Reid Inequality, show that $AB \geq 0$.

Exercise 5.1.17. Let $A \in B(H)$ be positive. Show that $A^n$ is also positive for each $n \in \mathbb{N}$.

Exercise 5.1.18. (cf. Exercise 5.1.19) Let $A, B \in B(H)$ be such that $A \geq B \geq 0$.
(1) Does it follow that $A^2 \geq B^2$?
(2) Show that $A^2 \geq B^2$ whenever $AB = BA$. 

Exercise 5.1.19. Let \( A, B \in B(H) \) be such that \( 0 \leq A \leq B \) and \( AB = BA \). Show that \( 0 \leq A^n \leq B^n \) for all \( n \in \mathbb{N} \).

Exercise 5.1.20. Let \( A \) be a bounded self-adjoint operator on an \( \mathbb{R} \)-Hilbert space \( H \) such that

\[
\exists c > 0, \forall x \in H : < Ax, x > \geq c\|x\|^2.
\]

(1) Show that \( A \) is invertible.
(2) Let \( p(t) = t^2 + at + b \) be a real polynomial having a strictly negative discriminant. Show that \( p(A) \) is invertible.
(3) Application: Check that \( A^2 + A + I \) is invertible whenever \( A \) is self-adjoint.
(4) Show that the hypothesis \( A \) being self-adjoint cannot be simply dropped.

Exercise 5.1.21. Let \( A \in B(H) \) be self-adjoint. Let

\[
U = (A - iI)(A + iI)^{-1}
\]

(\( U \) is called the Cayley Transform of \( A \)).

(1) Explain why \( A + iI \) is invertible (so that \( (A + iI)^{-1} \) makes sense!).
(2) Show that \( U \) is unitary.

Exercise 5.1.22. ([29]) Let \( U, V \in B(H) \) be both unitary. Show that the following assertions are equivalent:

(1) \( \|U - V\| < 2 \);
(2) \( U + V \) is invertible.

Exercise 5.1.23. Let \( A \in B(H) \). Show that

\[
\Re A \geq 0 \iff (A - \alpha I)^*(A - \alpha I) \geq \alpha^2 I, \forall \alpha < 0.
\]

Exercise 5.1.24. Find the square root (if it exists) of the following operators:

(1) \( A : \ell^2 \to \ell^2 \) defined by

\[
A(x_1, x_2, \cdots) = (0, 0, x_3, x_4, \cdots).
\]

(2) \( S \) is the shift operator on \( \ell^2 \). What about \( S^* \)?

Exercise 5.1.25. Let \( (A_n) \) be a sequence of self-adjoint operators in \( B(H) \). Prove that if \( (A_n) \) is bounded monotone increasing, then it is strongly convergent to a self-adjoint operator in \( B(H) \).

Exercise 5.1.26. Let \( A \in B(H) \) be positive.
(1) Suppose that \( \|A\| \leq 1 \). Define a sequence \((B_n)\) of operators in \( B(H) \) by

\[
\begin{align*}
B_0 &= 0, \\
B_{n+1} &= B_n + \frac{1}{2}(A - B_n^2).
\end{align*}
\]

Show that \((B_n)\) is a sequence of positive self-adjoint operators which is also bounded monotone increasing.

(2) Deduce that \((B_n)\) strongly converges to a positive \( B \in B(H) \) such that \( B^2 = A \). Infer also that any operator which commutes with \( A \) commutes with \( B \).

(3) Obtain the same conclusion by making no assumption this time on the norm \( \|A\| \).

(4) Show that if \( B \) and \( C \) are positive and such that \( B^2 = A \) and \( C^2 = A \), then \( B = C \).

Exercise 5.1.27. Give another proof of the uniqueness of the positive square root of positive operators (hint: if \( T \in B(H) \) is self-adjoint, what is \( \|T^4\| \)?)

Exercise 5.1.28. Let \( A \) and \( B \) be two positive operators on a complex Hilbert space \( H \).

(1) Show that if \( A \) and \( B \) commute, then \( AB \) (and hence \( BA \)) is positive. Infer that

\[
(AB)^{\frac{1}{2}} = A^{\frac{1}{2}}B^{\frac{1}{2}}.
\]

(2) Give an example showing the importance of the commutativity of \( A \) and \( B \) for the result to hold.

(3) Prove the converse of the result in Question 1, that is, prove that if \( A, B \) and \( AB \) are all positive operators, then \( A \) and \( B \) must commute.

Exercise 5.1.29. (cf. Exercise 5.1.15) Let \( A, K \in B(H) \) where \( A \) is positive and \( K \) is a contraction. Show that if \( AK = KA \), then \( K^*AK \leq A \).

Exercise 5.1.30. Let \( A, B \in B(H) \) be such that \( 0 \leq A \leq B \).

(1) Show that \( \sqrt{A} \leq \sqrt{B} \).

(2) If further \( A \) is taken to be invertible, then show that \( B \) too is invertible and that \( B^{-1} \leq A^{-1} \).

Exercise 5.1.31. Let \( A, B \in B(H) \) be such that \( AB = BA \) and \( A, B \geq 0 \). Show that

\[
\sqrt{A+B} \leq \sqrt{A} + \sqrt{B} \leq \sqrt{2(A+B)}.
\]
5.1. EXERCISES WITH SOLUTIONS

**Exercise 5.1.32.** Let $A$ be a self-adjoint operator on a complex Hilbert space $H$ such that $\|A\| \leq 1$. Let $I$ be the identity operator on $H$.

1. Justify the existence of $(I - A^2)^{\frac{1}{2}}$.
2. Set $U_\pm = A \pm i(I - A^2)^{\frac{1}{2}}$. Show that $U_\pm$ are unitary operators on $H$.

**Exercise 5.1.33.** Show that any $A \in B(H)$ may be written as a linear combination of four unitary operators.

**Exercise 5.1.34.** Let $H$ be a complex Hilbert space. If $A, B \in B(H)$ are self-adjoint and $BA \geq 0$, then show that
\[ \forall x \in H : \|Ax\| \leq \|Bx\| \iff \exists K \in B(H) \text{ positive contraction : } A = KB. \]

**Exercise 5.1.35.** Let $A, B \in B(H)$ be positive and commuting. Show that $0 \leq A \leq B \implies A^2 \leq B^2$.

**Exercise 5.1.36.** ([7]) Let $A, B, C \in B(H)$ be such that $A, B \geq 0$. Define an operator $T$ on $B(H \oplus H)$ by
\[ T = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}. \]
Show that
\[ T \geq 0 \iff |<Cx, y>|^2 \leq <Ax, x><By, y>, \forall x, y \in H. \]

**Exercise 5.1.37.** Let $A, B, C \in B(H)$ be such that $B$ and $C$ are positive. Show that if $BA = AC$, then
\[ \sqrt{BA} = A\sqrt{C}. \]

**Exercise 5.1.38.** ([121]). Let $A, B \in B(H)$ be such that either $A$ or $B$ is positive. We want to show that
\[ \|[A, B]\| \leq \|A\|\|B\|...(1) \]

WLOG, we choose $A \geq 0$.

1. If $B$ is a self-adjoint contraction, show that
\[ \|[A, B]\| \leq \|A\|. \]

2. Deduce that if $B$ is self-adjoint but not necessarily a contraction this time, then Inequality (1) still holds.

3. Show, via an operator matrix trick, that Inequality (1) holds for any $B \in B(H)$.
Exercise 5.1.39. ([158], cf. Exercise 5.1.40) Let \( T \in B(H) \) be such that \( T^2 = 0 \) and \( \text{Re} T \geq 0 \) (or \( \text{Im} T \geq 0 \)). Show that \( T \) is normal and so \( T = 0 \).

Exercise 5.1.40. ([77]) Let \( T = A + iB \in B(H) \) and let \( n \geq 2 \). Show that if \( T^n = 0 \) and \( A \geq 0 \) (or \( B \geq 0 \)), then \( T = 0 \).

Exercise 5.1.41. Let \( p \) and \( q \) be two relatively prime numbers, and let \( A, B \in B(H) \) be such that \( A^p = B^p \) and \( A^q = B^q \). Show that \( A = B \) whenever \( A \) is invertible.

5.2. Solutions

Solution 5.2.1. Both \( A \) and \( B \) are positive. Let \( x, y \in \mathbb{R} \). Then
\[
< \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > = < \begin{pmatrix} x + y \\ x + y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > \\
= x^2 + yx + xy + y^2 \\
= (x + y)^2 \geq 0.
\]

As for \( B \), despite the fact that
\[
< \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > = < \begin{pmatrix} x + y \\ x + y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > \\
= x^2 + yx + 2y^2 \\
= \left( x + \frac{y}{2} \right)^2 + \frac{7}{4} y^2 > 0,
\]
we cannot consider it as a positive matrix as \( B \) is not symmetric!

In fine, \( C \) is not positive because
\[
< \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > = x^2 + 4xy + 2y^2
\]
can be negative (e.g. if \( x = 1 \) and \( y = -1 \)).

Solution 5.2.2. The answer is yes. Let \( x = (x_1, x_2, \cdots) \in \ell^2 \). Then we already know that
\[
S(S^* x) = S(x_2, x_3, \cdots) = (0, x_2, x_3, \cdots).
\]
Hence
\[
(I - SS^*)(x_1, x_2, \cdots) = (x_1, x_2, \cdots) - (0, x_2, \cdots) = (x_1, 0, 0, \cdots).
\]
Thence
\[
< (I - SS^*) x, x > = < (x_1, 0, 0, \cdots), (x_1, x_2, \cdots) > = x_1 x_1 + 0 + \cdots = |x_1|^2.
\]
Therefore, \( I - SS^* \) is positive.
**Remark.** We know that $S^*S = I$. This means that we have just shown that $SS^* \leq S^*S$. In fact, any isometry $A$ verifies $AA^* \leq A^*A$. This seems to be an unnecessary observation but this shows that the shift operator belongs to an important class of operators (see Hyponormal Operators).

**Solution 5.2.3.** We know that $A$ is self-adjoint iff $\alpha_n$ is real-valued for each $n$. If $\alpha_n \geq 0$ for all $n$, then clearly for any $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2$

$$<Ax, x> = \sum_{n=1}^{\infty} \alpha_n |x_n|^2 \geq 0,$$

i.e. $A \geq 0$.

Conversely, if $A \geq 0$, then for any $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2$

$$<Ax, x> = \sum_{n=1}^{\infty} \alpha_n |x_n|^2 \geq 0.$$

In particular, for $x = e_n$ (from the usual orthonormal basis), we have that $\alpha_n \geq 0$ for all $n$, as needed.

**Solution 5.2.4.** Let $x \in H$. Since $A$ is self-adjoint, $A/2$ too is self-adjoint so that $e^{A/2}$ is self-adjoint. We may then write for all $x \in H$

$$<e^{A}x, x> = <e^{A/2}e^{A/2}x, x> = <e^{A/2}x, e^{A/2}x> = \|e^{A/2}x\|^2 \geq 0.$$

**Solution 5.2.5.** No! Consider the positive matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad BA = (AB)^* = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

But

$$AB + BA = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

is not positive (why?).

**Solution 5.2.6.** Let $x \in H$. We may write for all $x \in H$

$$0 = <(A + B)x, x> = <Ax, x> + <Bx, x>.$$

But $<Ax, x>$ and $<Bx, x>$ are two positive real numbers because $A$ and $B$ are positive operators. Therefore,

$$<Ax, x> = 0 \quad \text{and} \quad <Bx, x> = 0 \quad \text{for all} \quad x \in H,$$

i.e. $A = B = 0$. 


**Solution 5.2.7.** Since \( A \geq 0 \), \( A \) is self-adjoint. Hence it is diagonalizable (a well known fact or see e.g. [10]). Thus, for some invertible \( P \),

\[
P^{-1}AP = D,\]

where \( D \) is a diagonal matrix whose diagonal contains the eigenvalues of \( A \) which are all positive (why?). But, clearly

\[
\text{tr}D = \text{tr}(P^{-1}AP) = \text{tr}(APP^{-1}) = \text{tr}A.
\]

Since \( \text{tr}A = 0 \), \( \text{tr}D = 0 \), that is, the sum of the positive eigenvalues vanishes. This forces \( D = 0 \) or \( A = 0 \).

**Solution 5.2.8.**

(1) Let \( x \in H \). Then

\[
<T^*ATx, x> = <ATx, T**x> = <ATx, Tx> \geq 0
\]

since \( A \) is positive. A similar argument applies to prove the other inequality.

(2) Since \( A - B \geq 0 \), we may just apply the previous results to have

\[
T^*(A - B)T \geq 0 \text{ or } T^*AT \geq T^*BT
\]

(since also \( T^*AT \) and \( T^*BT \) are self-adjoint) and

\[
T(A - B)T^* \geq 0 \text{ or } TAT^* \geq TBT^*.
\]

**Solution 5.2.9.** Assume that \( P - Q \) is an orthogonal projection. Then \((P - Q)^2 = P - Q\) so that for all \( x \in H \), we have

\[
<(P - Q)x, x> = (P - Q)^2x, x> = (P - Q)x, (P - Q)x >= \| (P - Q)x \|^2 \geq 0,
\]

meaning that \( P \geq Q \).

Conversely, assume that \( P \geq Q \). Then we leave it to the reader to show that this is equivalent to saying that \( PQ = Q \), and also equivalent to \(QP = Q \). Hence

\[
(P - Q)^2 = P^2 - PQ - QP + Q^2 = P - Q - Q + Q = P - Q.
\]

Accordingly, \( P - Q \) is an orthogonal projection (because \( P - Q \) is also self-adjoint).

**Solution 5.2.10.**

(1) Let \( x, y \in H \). Define

\[
[x, y] = <Ax, y>.
\]

Then \([, ,]\) verifies all the properties of an inner product except perhaps that we may have \([x, x] = 0\) for some \( x \neq 0 \). So, to
establish the required inequality, just proceed as in the first question of Exercise 3.3.7.

**Remark.** ([132]) Another way of establishing the previous inequality is to set \(< x, y >_r = < Ax, y > + r < x, y >\) where \(r > 0\). Then show that \(< ·, · >_r\) is an inner product, apply the standard Cauchy-Schwarz Inequality to it, send \(r \to 0\) and finally get the desired generalization!

(2) Setting \(y = Ax\) in the previous result, we get
\[
\|Ax\|^4 = |< Ax, Ax >|^2 \leq < Ax, x > < A^2x, Ax > \leq < Ax, x > \|A^2x\|\|Ax\|.
\]
Whence
\[
\|Ax\|^4 \leq |< Ax, x >\|\|Ax\|\|A\| \implies \|Ax\|^4 \leq < Ax, x > \|A\|\|Ax\|^2.
\]
Thus
\[
\|Ax\|^2 \leq \|A\| < Ax, x >.
\]

**Remark.** Another way of proving the previous inequality is via the Reid Inequality (as observed in [183]). Indeed, setting \(A = K\) in the Reid Inequality gives a shorter proof of this result.

**Solution 5.2.11.**

(1) Since \(A\) is self-adjoint, \(< Ax, x >\) is real (for all \(x \in H\)). We may then write
\[
< (\pm A - I)x, x > = \pm < Ax, x > - \|x\|^2
\]
\[
= |< Ax, x >| - \|x\|^2
\]
\[
\leq \|Ax\|\|x\| - \|x\|^2 \text{ (by the Cauchy-Schwarz Inequality)}
\]
\[
= (\|Ax\| - \|x\|)\|x\|.
\]
If \(\|A\| \leq 1\), then clearly \(\|Ax\| \leq \|A\|\|x\| \leq \|x\|\) for each \(x \in H\). Hence
\[
< (\pm A - I)x, x > \leq 0\) or merely \(\pm A \leq I\), i.e. \(-I \leq A \leq I\).

To prove the other implication, notice that if \(-I \leq A \leq I\), then
\[
\forall x \in H : \pm < Ax, x > \leq \|x\|^2 \text{ or } |< Ax, x >| \leq \|x\|^2
\]
for all \(x \in H\). Passing to the supremum over \(\|x\| = 1\) yields (by taking into account the self-adjointness of \(A\))
\[
\|A\| = \sup_{\|x\| = 1} |< Ax, x >| \leq 1
\]
and this marks the end of the proof.

(2) If $\alpha = 0$, then the results is obvious. If $\alpha > 0$, then apply the previous question with $\frac{1}{\alpha} A$ instead of $A$.

**Solution 5.2.12.** By assumption, for all $x \in H$

$$ - < Ax, x > \leq < Bx, x > \leq < Ax, x > \text{ or merely } | < Bx, x > | \leq < Ax, x > . $$

Therefore,

$$ \| B \| = \sup_{\| x \| = 1} | < Bx, x > | \leq \sup_{\| x \| = 1} < Ax, x > = \| A \| , $$

as desired.

**Solution 5.2.13.** WLOG, we may assume that $\| A \| \geq \| B \|$. So we must show that

$$ \| A - B \| \leq \| A \| . $$

Since $A, B \geq 0$, they are self-adjoint, and so is then $A - B$. Again, since $A, B \geq 0$, we have

$$ -B \leq A - B \leq A. $$

Also for all $x \in H$, we have (by the Cauchy-Schwarz Inequality)

$$ < Ax, x > \leq \| Ax \| \| x \| \leq \| A \| \leq < Ix, x > = < \| A \| Ix, x > , $$

i.e. $A \leq \| A \| I$. Similarly, we find that $-B \geq -\| B \| I$. Thus,

$$ -\| B \| I \leq A - B \leq \| A \| I. $$

Taking into account the choice $\| A \| \geq \| B \|$, we find that $-\| B \| I \leq A - B \leq \| A \| I$. Thus,

$$ -\| B \| I \leq A - B \leq \| A \| I. $$

Finally, by Exercise 5.1.11, we then obtain

$$ \| A - B \| \leq \| A \| = \max(\| A \|, \| B \|). $$

**Solution 5.2.14.** The proof presented here is mostly due to Reid in [183]. WLOG, we may assume that $\| K \| \leq 1$ (why?). Therefore, we need only show

$$ | < AKx, x > | \leq < Ax, x > $$

for all $x \in H$.

Since $AK$ is self-adjoint, it follows that $AK = K^* A$. Hence

$$ AK^2 = K^* AK = (K^*)^2 A = (AK^2)^*, \ AK^3 = (K^*)^2 \ AK = (K^*)^3 A = (AK^3)^*, \ldots , $$

so by induction, for each $n$, $AK^n$ is self-adjoint.
Since $A \geq 0$, Corollary 5.2 yields for all $x \in H$:

$$| < AKx, x > | \leq \frac{1}{2} [ < Ax, x > + < AKx, Kx > ]$$

$$= \frac{1}{2} [ < Ax, x > + < K^* AKx, x > ]$$

$$= \frac{1}{2} [ < Ax, x > + < AK^2 x, x > ].$$

Thanks to the previous inequality and by doing a little induction, we get for all $n$ (and all $x$)

$$| < AKx, x > | \leq (2^{-1} + \cdots + 2^{-n}) < Ax, x > + 2^{-n} < AK^{2n} x, x > \ldots (1)$$

Since $\|K\| \leq 1$, we have by the Cauchy-Schwarz Inequality

$$| < AK^{2n} x, x > | \leq \|AK^{2n}\| \|x\| \leq \|A\| \|K\|^{2n} \|x\|^2 \leq \|A\| \|x\|^2$$

and so passing to the limit as $n \to \infty$ in (1) gives clearly

$$| < AKx, x > | \leq < Ax, x >,$$

as suggested.

**Solution 5.2.15.** First, observe that

$$AK^* = KA \implies A(K^*)^2 = KAK^* = K^2 A.$$ 

Since $A$ is positive, so is $KAK^*$ or $A(K^*)^2$. Thereupon, using Reid Inequality, we know that

$$< KAK^* x, x > = < A(K^*)^2 x, x > = | < A(K^*)^2 x, x > | \leq < Ax, x > .$$

So much for the proof.

**Solution 5.2.16.** WLOG, we may suppose that $0 \leq B \leq I$ (otherwise work with $\frac{B}{\|B\|}$). Hence $\|I - B\| \leq 1$. Since $A(I - B)$ is clearly self-adjoint and $A \geq 0$, it follows from Reid Inequality that

$$AB = A - A(I - B) \geq 0.$$ 

**Solution 5.2.17.** The proof follows by induction (using the fact that the product of two positive commuting operators remains positive). Alternatively, we can treat two cases: $n$ being even and $n$ being odd (remembering that a positive operator is self-adjoint). Details are left to the reader.

**Solution 5.2.18.**
14 5. POSITIVE OPERATORS. SQUARE ROOT

(1) The answer is no! Anticipating a little bit, we know from Question 2 that we need to choose two non-commuting $A$ and $B$. Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Observe that both $A$ and $B$ are positive. So it only remains to check that $A \geq B$ whereas $A^2 \not\geq B^2$, that is, we need to verify that $A - B \geq 0$ and that $A^2 - B^2 \not\geq 0$. We see that

$$A - B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0$$

whereas

$$A^2 - B^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \not\geq 0$$

(check it).

(2) Since $AB = BA$, we clearly have


But, $A \geq B$ means that $A - B \geq 0$. Also, it is plain that $A + B \geq 0$.

The fact that $A - B$ commutes with $A + B$ (as $AB = BA$) imply that

$$(A + B)(A - B) = A^2 - B^2 \geq 0,$$

and hence $A^2 \geq B^2$ (remember that $A^2$ and $B^2$ are self-adjoint, a simple but a crucial point!). This marks the end of the proof.

**Solution 5.2.19.** Since $AB = BA$, we have

$$0 \leq A \leq B \implies 0 \leq A^2 \leq AB$$

and

$$0 \leq A \leq B \implies 0 \leq AB \leq B^2.$$ 

Hence

$$A^2 \leq B^2$$

(which is another proof of the result of Exercise 5.1.18). Using a similar argument, and a proof by induction, we can easily prove the required inequality for $n \in \mathbb{N}$...

**Solution 5.2.20.**
(1) Let \( x \in H \). By the Cauchy-Schwarz Inequality
\[
c\|x\|^2 \leq <Ax,x> \leq \|Ax\|\|x\|.
\]
Therefore \( \|Ax\| \geq c\|x\| \). Since \( A \) is self-adjoint, the result follows.

(2) By hypothesis \( \Delta = a^2 - 4b < 0 \). Then
\[
p(A) = A^2 + aA + bI
\]
is self-adjoint. We may write
\[
A^2 + aA + bI = \left(A + \frac{a}{2}I\right)^2 + b - \frac{a^2}{4} = \left(A + \frac{a}{2}I\right)^2 - 4\Delta.
\]
Since \( A + a/2I \) is self-adjoint, \( (A + a/2I)^2 \) is positive. Hence for all \( x \in H \)
\[
< p(A)x,x> \geq -4\Delta < x,x >.
\]
Thus \( p(A) \) is invertible by the foregoing question.

(3) Straightforward!

(4) Let
\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Then \( A \) is not self-adjoint. It is also easy to see that
\[
A^2 = -I \text{ or } A^2 + I = 0.
\]
With the above notation, \( a = 0 \) and \( b = 1 \) and so \( a^2 - 4b < 0 \).
In the end, it is clear that \( A^2 + I \) is not invertible.

**Solution 5.2.21.**

(1) Let \( x \in H \). By considering
\[
\|(A + iI)x\|^2 = < (A + iI)x, (A + iI)x >,
\]
one can easily see that
\[
\forall x \in H : \|(A + iI)x\| \geq \|x\|.
\]
Hence \( A + iI \) is bounded below. Since \( A \) is self-adjoint, \( A + iI \) is normal. Therefore, \( A + iI \) is invertible.
(2) First we compute $U^*$. We have
\[
U^* = [(A - iI)(A + iI)^{-1}]^* \\
= [(A + iI)^{-1}]^*(A - iI)^* \\
= [(A + iI)^*]^{-1} (A^* + iI^*) \\
= [(A^* - iI^*)]^{-1} (A^* + iI) \\
= (A - iI)^{-1} (A + iI) \text{ (because $A$ is self-adjoint)}.
\]
Since $A$ commutes with multiples of the identity, we easily see that
\[
U^* U = [(A - iI)]^{-1} (A + iI)(A - iI)(A + iI)^{-1} \\
= [(A - iI)]^{-1} (A - iI) \left( A + iI \right) (A + iI)^{-1} \\
= I.
\]
In a similar vein, we find that $UU^* = I$, that is, $U$ is unitary.

**Solution 5.2.22.**
(1) "$(1) \Rightarrow (2)$": First, we set
\[
A = \frac{1}{2}(U + I) \text{ and } B = \frac{1}{2}(V + I).
\]
Then it is clear that
\[
\|A - B\| = \frac{1}{2}\|U - V\| < 1.
\]
Hence $\|A - B\|^2 < 1$ so that there exists some $\alpha > 0$ such that
\[
\|(A - B)^*(A - B)\| = \|A - B\|^2 \leq 1 - \alpha.
\]
Whence
\[
(A - B)^*(A - B) \leq (1 - \alpha)I
\]
or after simplification,
\[
I - A^*A - B^*B + A^*B + B^*A \geq \alpha I.
\]
It is clear that
\[
A^*A = \frac{1}{2}(A + A^*) \text{ and } B^*B = \frac{1}{2}(B + B^*).
\]
Since $U = 2A - I$ and $V = 2B - I$, we have
\[
(U + V)^*(U + V) = 4(A^* + B^* - I)(A + B - I) \\
= 4(I - A^*A - B^*B + A^*B + B^*A) \\
\geq 4\alpha I.
\]
Similarly, by considering
\[ A^* = \frac{1}{2}(U^* + I) \text{ and } B^* = \frac{1}{2}(V^* + I), \]
we may show that
\[ (U + V)(U + V)^* \geq 4\alpha I. \]
Thus \( U + V \) is invertible.

(2) The other implication may be proved by going backwards in the previous proof (do the details!).

**Solution 5.2.23.** It is clear that if \( \alpha \in \mathbb{R} \), then
\[ (A - \alpha I)^*(A - \alpha I) - \alpha^2 I = (A^* - \alpha I)(A - \alpha I) - \alpha^2 I = A^*A - \alpha(A^* + A)\cdots(1) \]
If the previous quantity is positive for all \( \alpha < 0 \), then we have
\[ \alpha(A^* + A) \leq A^*A \text{ or } A^* + A \geq \frac{1}{\alpha}A^*A. \]
Taking the limit as \( \alpha \to -\infty \) gives
\[ A + A^* \geq 0, \text{ i.e. } \text{Re}A \geq 0 \]
and this proves "⇐".

Now assume that \( \text{Re}A \geq 0 \) and let \( \alpha < 0 \). Since \( A^*A \) is positive, it is evident that
\[ A + A^* \geq 0 \geq \frac{A^*A}{\alpha}. \]
This means that the quantities on each side of the equalities involved in Equation (1) are greater than or equal to zero, so that for any \( \alpha < 0 \),
\[ (A - \alpha I)^*(A - \alpha I) \geq \alpha^2 I, \]
establishing "⇒".

**Solution 5.2.24.**

(1) It is easy to see that \( A \) is positive (do the details!). It then follows that \( A \) has one and only one positive square root. As clearly \( A^2 = A \), then \( \sqrt{A} = A \) is the (unique) positive square root of \( A \).

(2) The shift operator and its adjoint do not possess any square root whatsoever. Assume for the sake of contradiction that e.g. \( S^* \) does, i.e. \( A^2 = S^* \), where \( A \in B(H) \). Then, \( A^2S = S^*S = I \) and by the general theory \( A \) is right invertible and so it is surjective. Notice also that \( A \) cannot be injective (indeed, this would imply that \( A^2 = S^* \) is injective and this is untrue).

Now, we show that \( \ker A = \ker S^* = \mathbb{R}e_1 \), where \( e_1 = (1, 0, 0, \cdots) \). The equality \( \ker S^* = \mathbb{R}e_1 \) is known and clear.
It also implies that $\dim \ker S^* = 1$. Now, we obviously have $\ker A \subset \ker S^*$ because $A^2 = S^*$. Since $A$ is not injective, we are forced to have $\ker A = \ker S^*$ as $\ker A$ and $\ker S^*$ are vector spaces.

Since $A$ is onto, for all $y \in \ell^2$, in particular for $e_1 \in \ell^2$, there is an $x \in \ell^2$ such that $Ax = e_1$ (and so $x \not\in \ker A = \ker S^*$).

Thus (as $e_1 \in \ker A$) $A^2x = Ae_1 = 0 \neq S^*x$.

This shows that $S^*$ does not have any square root.

If $S$ had a square root, then we would have $S = B^2$, where $B \in B(\ell^2)$. Therefore, $S^* = (B^2)^* = (B^*)^2$, i.e. $S^*$ would possess a square root! This is a contradiction with what we have just seen. Accordingly, $S$ cannot have a square root either!

**Solution 5.2.25.** By assumption, we know that $A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq A$ for some self-adjoint $A \in B(H)$. WLOG we may assume that $A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq I$ (just divide each $A_i$ by $\|A\|$ and relabel $\frac{A_i}{\|A\|}$ as $A_i$). There is also no loss of generality in assume that all $A_n \geq 0$ (e.g. we could use the sequence $(A_n - A_1)_n$, say). Therefore, we may work with $0 \leq A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq I$.

The primary aim is to show that $(A_n x)$ converges for each $x \in H$. By the completeness of $H$, this means that it suffices then to show that $(A_n x)$ is Cauchy. Let $n > m$ and let $x \in H$. Then $A_n - A_m \geq 0$ and $A_n - A_m \leq I$. Hence $\|A_n - A_m\| \leq 1$. Now, we may write

$$\|A_n x - A_m x\|^2 \leq \langle (A_n - A_m) x, (A_n - A_m) x \rangle$$

$$\leq \langle (A_n - A_m) x, x \rangle \leq \langle (A_n - A_m)^2 x, (A_n - A_m) x \rangle$$

$$\leq \langle (A_n - A_m) x, x \rangle \| (A_n - A_m)^2 x \| (A_n - A_m) x \|$$

$$\leq \langle (A_n - A_m) x, x \rangle \| A_n - A_m \| (A_n - A_m) x \|^2$$

$$\leq (A_n - A_m) x, x \rangle \| A_n x - A_m x \|^2$$

where we have used Theorem ?? in the first inequality. Therefore,

$$\|A_n x - A_m x\|^2 \leq \langle (A_n - A_m) x, x \rangle \geq \langle A_n x, x \rangle - \langle A_m x, x \rangle .$$

But $(\langle A_n x, x \rangle)_n$ is an increasing real sequence which is bounded above by $(\|x\|^2)$. Whence, it converges and so it is Cauchy. Thereupon,

$$\lim_{n,m \to \infty} \|A_n x - A_m x\| = 0 .$$

This means, as already observed above, that $\lim_{n \to \infty} A_n x$ exists for each $x \in H$. 
Define now for each \( x \)
\[
Ax = \lim_{n \to \infty} A_n x
\]
(in the sense that \( \| A_n x - Ax \| \to 0 \) for all \( x \)). Then \( A \) is clearly linear. It only remains to see why \( A \) is bounded and self-adjoint. We prove these two requirements together: By the continuity of the inner product, we have for all \( x, y \in H \)
\[
< Ax, y > = \lim_{n \to \infty} < A_n x, y > = \lim_{n \to \infty} < x, A_n y > = < x, Ay >.
\]

Calling on the Hellinger-Toeplitz Theorem, we obtain that \( A \in B(H) \), and clearly \( A \) is self-adjoint.

To summarize, the bounded monotone increasing sequence \( (A_n) \) converges strongly to the self-adjoint bounded operator \( A \).

**Solution 5.2.26.**

1. Observe first that since \( A \) is positive and \( \| A \| \leq 1 \), we have \( 0 \leq A \leq I \). Another equally important observation is that the sequence \( (B_n) \) is a "polynomial" of \( A \). This implies that all of \( B_n \) are pairwise commuting.

Next, \( B_0 = 0 \) is evidently self-adjoint. So, assuming that \( B_n \) is self-adjoint (and recalling that \( A \) is self-adjoint), we can easily check that \( B_{n+1} \) too is self-adjoint. Therefore, all \( B_n \) are self-adjoint.

Now, we claim that \( B_n \leq I \) for all \( n \). This is obviously true for \( n = 0 \). Assume that \( B_n \leq I \). Observing that \( (I - B_n)^2 \geq 0 \) (why?), we then have
\[
I - B_{n+1} = I - B_n - \frac{1}{2}(A - B_n^2) = \frac{1}{2}(I - B_n)^2 + \frac{1}{2}(I - A) \geq 0.
\]

To prove that \( (B_n) \) is increasing, observe first that \( B_0 \leq \frac{1}{2} A = B_1 \). Assuming that \( B_n \geq B_{n-1} \), we may write
\[
B_{n+1} - B_n = \frac{1}{2}[(I - B_{n-1}) + (I - B_n)](B_n - B_{n-1})
\]
which, being a product of commuting positive operators, itself is positive.

Consequently, we have shown that
\[
0 = B_0 \leq B_1 \leq \cdots \leq B_n \leq \cdots \leq I,
\]
as needed.
(2) Since \((B_n)\) is bounded monotone increasing, by Theorem ?? we know that \((B_n)\) converges strongly to some self-adjoint \(B \in B(H)\). Since each \(B_n\) is positive, we have
\[
< Bx, x > = \lim_{n \to \infty} < B_n x, x > \geq 0
\]
as strong convergence implies weak one. Thus, \(B \geq 0\).

It remains to show that \(B^2 = A\). Let \(x \in H\). We have by hypothesis
\[
B_{n+1} x = B_n x + \frac{1}{2} (A x - B_n^2 x).
\]
Passing to the strong limit and using \(\|B_n^2 x - B^2 x\| \to 0\) (why?), we finally get \(B^2 = A\), as required.

Finally, assume that a \(C \in B(H)\) commutes with \(A\), i.e. \(AC = CA\). We must show that \(BC = CB\). Since \(C\) commutes with \(A\), we may easily show that \(C\) commutes with \(B_n\) too, that is, \(CB_n x = B_n C x\) (for all \(n\) and all \(x\)). On the one hand, we clearly see that \(B_n C x \to BC x\). On the other hand, invoking the (sequential) continuity of \(C\), we have that \(CB_n x \to CB x\). By uniqueness of the strong limit, we get \(BC x = CB x\), \(\forall x \in H\), as desired.

(3) If \(A = 0\), then \(B = 0\) will do. So if \(A \neq 0\), considering \(T = \frac{A}{\|A\|}\) gives \(0 \leq T \leq 1\). Then, apply what we have already done above.

(4) The proof of uniqueness here, although not being complicated, is not as direct as one is used to with other theorems.

We have already shown that \(B^2 = A\). Assume that there is another positive \(C \in B(H)\) such that \(C^2 = A\). We must show that \(Bx = Cx\) for all \(x \in H\). Observe first that \(A\) plainly commutes with \(C\). By Question (2), \(C\) commutes with \(B\) as well, i.e. \(BC = CB\). This tells us that
\[
(B + C)(B - C) = B^2 - C^2 = A - A = 0.
\]
So, if we let \(x \in H\) and set \(y = (B - C)x\), then
\[
< By, y > + < Cy, y > = < (B+C)y, y > = < (B+C)(B-C)x, y > = 0.
\]
Because both \(B\) and \(C\) are positive, we obtain (cf. Exercise 5.1.6)
\[
< By, y > = < Cy, y > = 0.
\]
By Question (2) again, \( B \geq 0 \) has a square root which we denote by \( D \), say. That is, \( D^2 = B \). Therefore,
\[
\|Dy\|^2 = \langle Dy, Dy \rangle = \langle D^2y, y \rangle = \langle By, y \rangle = 0
\]
and so \( Dy = 0 \). This implies that \( By = D^2y = D(0) = 0 \).

Using also a square root of \( C \), we may similarly show that \( Cy = 0 \). Consequently,
\[
\|Bx - Cx\|^2 = \langle (B - C)x, (B - C)x \rangle = \langle D^2y, x \rangle = 0.
\]

Accordingly, \( B = C \), i.e. we have proven that the positive \( A \) can only have one positive square root, marking the end of the proof.

**Solution 5.2.27.** Assume that \( A \in \mathcal{B}(H) \) is positive. Hence, there is a positive \( B \in \mathcal{B}(H) \) such that \( B^2 = A \). Assume that there is another positive \( C \in \mathcal{B}(H) \) such that \( A = C^2 \) and so \( B^2 = C^2 \). We ought to show that \( B = C \).

First, it is clear that
\[
CA = C^3 = AC.
\]
Hence \( C \) commutes with \( B \) as well (why?). This gives
\[
(B - C)B(B - C) + (B - C)C(B - C) = (B^2 - C^2)(B - C) = 0.
\]
As \( B, C \geq 0 \) and \( B - C \) is self-adjoint, then \( (B - C)B(B - C) \) and \( (B - C)C(B - C) \) are both positive and so
\[
(B - C)B(B - C) = (B - C)C(B - C) = 0.
\]
Thereupon,
\[
(B - C)B(B - C) - (B - C)C(B - C) = 0,
\]
that is
\[
(B - C)^3 = 0.
\]
Whence
\[
(B - C)^4 = 0.
\]
Now, if \( T \in \mathcal{B}(H) \) is self-adjoint, then \( \|T^2\| = \|T\|^2 \). Since \( T^2 \) is self-adjoint, we get \( \|T^4\| = \|T\|^4 \).

Consequently,
\[
0 = \|(B - C)^4\| = \|B - C\|^4,
\]
that is \( B = C \), as required.

**Solution 5.2.28.**
(1) Since $A$ is positive, it admits a unique positive square root, which we denote by $P$ (that is $P^2 = A$). Since $B$ commutes with $A$, it commutes with $P$ as well.

Let $x \in H$. We may write (remembering that positive operators are necessarily self-adjoint)

$$< ABx, x > = < P^2 Bx, x > = < PBx, Px > = < BPx, Px > \geq 0$$

as $B$ is positive. Therefore, $AB \geq 0$.

Since $A$ and $B$ are positive, both $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ exist and are well-defined. Since $A$ and $B$ also commute, $AB$ is positive and it makes sense then to define $(AB)^{\frac{1}{2}}$. If we come to show that

$$(A^{\frac{1}{2}} B^{\frac{1}{2}})^2 = AB,$$

then by the uniqueness of the square root, the desired result follows.

Now since $A$ and $B$ commute, so do their square roots and we have

$$(A^{\frac{1}{2}} B^{\frac{1}{2}})^2 = A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}} = A^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} = AB.$$

The proof is complete.

(2) Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$ 

Then both $A$ and $B$ are positive.

We may also check that

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 6 \end{pmatrix},$$

i.e. $AB$ is not positive because it is not even self-adjoint and

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 6 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix} = BA.$$

(3) Since $A$, $B$ and $AB$ are all positive operators, they are all self-adjoint. Accordingly,

$$BA = B^* A^* = (AB)^* = AB,$$

that is $A$ and $B$ commute.

**Solution** 5.2.29. Since $KA = AK$ and $A$ is self-adjoint, it follows that $AK^* = K^* A$. Hence $AK^* K = K^* KA$. Therefore $A^{\frac{1}{2}} K^* K = K^* KA^{\frac{1}{2}}$ as $A \geq 0$. 
Now, let \( x \in H \). By the Generalized Cauchy-Schwarz Inequality, we may write
\[
<K^*Ax, x>^2 = <AK^*Kx, x> \leq <Ax, x> <AK^*Kx, K^*Kx>.
\]
But,
\[
<AK^*Kx, K^*Kx> = <A^\frac{1}{2}K^*Kx, A^\frac{1}{2}K^*Kx> = \|A^\frac{1}{2}K^*Kx\|^2 = \|K^*KA^\frac{1}{2}x\|^2.
\]
Because \( \|K^*K\| \leq 1 \), we obtain
\[
\|K^*KA^\frac{1}{2}x\|^2 \leq \|A^\frac{1}{2}x\|^2 = <A^\frac{1}{2}x, A^\frac{1}{2}x> = <Ax, x>
\]
so that
\[
<K^*Ax, x>^2 \leq <Ax, x>^2,
\]
completing the proof.

**Solution 5.2.30.**

(1) Let \( x \in H \). Since \( 0 \leq A \leq B \), we have for all \( x \in H \)
\[
0 \leq <Ax, x> \leq <Bx, x> \iff 0 \leq \sqrt{A}x, \sqrt{A}x \leq \sqrt{B}x, \sqrt{B}x
\]
and so (for all \( x \))
\[
0 \leq \|\sqrt{A}x\|^2 \leq \|\sqrt{B}x\|^2.
\]
So, by Theorem 3.1.69, we know that \( \sqrt{A} = K\sqrt{B} \) for some contraction \( K \in B(H) \). Since \( \sqrt{A} \) is self-adjoint, it follows that \( K\sqrt{B} \) too is self-adjoint, i.e. \( K\sqrt{B} = \sqrt{B}K^* \). Since \( \sqrt{B} \geq 0 \), by the Reid Inequality we obtain:
\[
<\sqrt{A}x, x> = <\sqrt{B}K^*x, x> \leq <\sqrt{B}x, x>,
\]
that is,
\[
\sqrt{A} \leq \sqrt{B},
\]
as required.

(2) As before, we know that \( \sqrt{A} = K\sqrt{B} \) for some contraction \( K \in B(H) \). Since \( \sqrt{A} \) is invertible (as \( A \) is), it follows that \( I = (\sqrt{A})^{-1}K\sqrt{B} \), i.e. the self-adjoint \( \sqrt{B} \) is left invertible. By taking adjoints, we see that \( \sqrt{B} \) is also right invertible. Thus, \( B \) is invertible and
\[
(\sqrt{B})^{-1} = (\sqrt{A})^{-1}K = K^*(\sqrt{A})^{-1}
\]
by the self-adjointness of both \( (\sqrt{B})^{-1} \) and \( (\sqrt{A})^{-1} \).

Finally, let \( x \in H \). Then (since \( K^* \) too is a contraction)
\[
<B^{-1}x, x> = \|(\sqrt{B})^{-1}x\|^2 = \|K^*(\sqrt{A})^{-1}x\|^2 \leq \|(\sqrt{A})^{-1}x\|^2 = <A^{-1}x, x>:
\]
as needed.
5. POSITIVE OPERATORS. SQUARE ROOT

**Solution 5.2.31.** Since $AB = BA$ and $A, B \geq 0$, we have $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$. Hence $\sqrt{A}\sqrt{B} \geq 0$. Therefore,

$$A + B \leq A + 2\sqrt{A}\sqrt{B} + B = (\sqrt{A} + \sqrt{B})^2.$$

Since $\sqrt{A} + \sqrt{B} \geq 0$, we get

$$\sqrt{A} + \sqrt{B} \leq \sqrt{A} + \sqrt{B},$$

establishing half of the result.

Finally, to prove the other inequality, reason similarly using $(\sqrt{A} - \sqrt{B})^2 \geq 0$...

**Solution 5.2.32.**

(1) We need only verify that $I - A^2$ is a positive operator. Let $x \in H$. We have

$$< (I - A^2)x, x > \geq 0 \iff < x, x > - < A^2x, x > \geq 0$$

$$\iff < A^2x, x > \leq \|x\|^2$$

$$\iff < Ax, Ax > = \|Ax\|^2 \leq \|x\|^2.$$

But by hypothesis, $\|A\| \leq 1$ which leads to

$$\|Ax\|^2 \leq \|A\|^2\|x\|^2 \leq \|x\|^2.$$

Therefore, $I - A^2 \geq 0$.

(2) We only prove $U_+$ is unitary (the proof for $U_-$ is very akin). Since $A$ is self-adjoint, one has

$$U_+^* = (A + i(I - A^2)^{\frac{1}{2}})^* = A - i(I - A^2)^{\frac{1}{2}}.$$

Since $A$ and $I - A^2$ commute, so do $A$ and $(I - A^2)^{\frac{1}{2}}$ and so

$$U_+U_+^* = (A + i(I - A^2)^{\frac{1}{2}})(A - i(I - A^2)^{\frac{1}{2}})$$

$$= A^2 - iA(I - A^2)^{\frac{1}{2}} + i(I - A^2)^{\frac{1}{2}}A + I - A^2$$

$$= I.$$

Similarly, one shows that $U_+^*U_+ = I$

**Solution 5.2.33.** We already know that any $A \in B(H)$ may be written as $A = \text{Re } A + i\text{ Im } A$, that is, every $A \in B(H)$ may be expressed as a linear combination of two self-adjoint operators.

Now, suppose that $B \in B(H)$ is self-adjoint. WLOG, we may assume that $\|B\| \leq 1$ (otherwise, you know what you should do!). By Exercise 5.1.32, $B \pm i(I - B^2)^{\frac{1}{2}}$ are unitary operators and clearly

$$B = \frac{1}{2}[B + i(I - B^2)^{\frac{1}{2}}] + \frac{1}{2}[B - i(I - B^2)^{\frac{1}{2}}],$$
so that each self-adjoint operator may be expressed as a linear combination of *two unitary* operators, and this leads to the fact that any $A \in B(H)$ may be written as a linear combination of four unitary operators.

**Solution 5.2.34.**

1. "\(\Leftarrow\)" Let $x \in H$. Then

\[ 0 \leq \langle KBx, Bx \rangle = \langle Ax, Bx \rangle = \langle BAx, x \rangle, \]

that is, $BA \geq 0$.

2. "\(\Rightarrow\)" Since $BA \geq 0$, it follows that $BA$ is self-adjoint, i.e. $AB = BA$. As a consequence, ker $A$ reduces $A$ and $B$, and the restriction of $A$ to ker $A$ is the zero operator on ker $A$. Hence, we can assume that $A$ is injective. Therefore, because ker $B \subset$ ker $A = \{0\}$, we see that $B^{-1}$ is self-adjoint and *densely defined* (i.e. defined on a dense domain). Set $K_0 = AB^{-1}$. Then $K_0$ is densely defined and

\[ ||K_0(Bx)|| = ||AB^{-1}Bx|| = ||Ax|| \leq ||Bx||, \forall x \in H, \]

signifying that $K_0$ is a contraction with a unique contractive extension $K$ to the whole $H$. Since

\[ \langle K_0(Bx), Bx \rangle = \langle Ax, Bx \rangle = \langle BAx, x \rangle \geq 0 \]

for all $x \in H$, we see that $K$ is positive as well. Clearly

\[ KBx = K_0(Bx) = Ax \]

for all $x \in H$, and this completes the proof.

**Solution 5.2.35.** Since $AB \geq 0$, we know that (why?) $\sqrt{A} = K\sqrt{B}$ for some positive contraction $K \in B(H)$ and $K\sqrt{B} = \sqrt{BK}$. Hence

\[ A = K\sqrt{B}K\sqrt{B} = K^2B. \]

So for all $x \in H$:

\[ \|Ax\|^2 = \|K^2Bx\|^2 \leq \|Bx\|^2 \]

or merely

\[ \langle A^2x, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \leq \|Bx\|^2 = \langle B^2x, x \rangle, \]

as required.

**Solution 5.2.36.**
5. POSITIVE OPERATORS. SQUARE ROOT

(1) "\(\implies\)" : Assume that \(T \geq 0\). By the Generalized Cauchy-Schwarz Inequality (applied to the vectors \((x,0)\) and \((0,y)\)), we have

\[
\left| \langle T \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \rangle \right|^2 \leq \langle T \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} \rangle \langle T \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \rangle.
\]

But \(T = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}\) and so the previous inequality becomes after simplifications:

\[
| \langle Cx, y \rangle |^2 \leq \langle Ax, x \rangle \langle By, y \rangle,
\]

valid obviously for all \(x, y \in H\).

(2) "\(\Longleftarrow\)" : Now, suppose that

\[
| \langle Cx, y \rangle |^2 \leq \langle Ax, x \rangle \langle By, y \rangle, \quad \forall x, y \in H.
\]

To show that \(T\) is positive, let \(x, y \in H\) and observe that

\[
\langle T \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = \langle Ax, x \rangle + \langle C^*y, x \rangle + \langle Cx, y \rangle + \langle By, y \rangle.
\]

Since

\[
\langle C^*y, x \rangle + \langle Cx, y \rangle = \overline{\langle Cx, y \rangle} + \langle Cx, y \rangle = 2 \Re \langle Cx, y \rangle,
\]

it follows that

\[
\langle T \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = \langle Ax, x \rangle + 2 \Re \langle Cx, y \rangle + \langle By, y \rangle \geq 2 \langle Ax, x \rangle + \frac{1}{2} \langle Bx, x \rangle + 2 \Re \langle Cx, y \rangle \quad \text{(why?)}
\]

\[
\geq 2 \langle Cx, y \rangle + 2 \Re \langle Cx, y \rangle \quad \text{(by assumption)}
\]

\[
\geq 2|\langle Cx, y \rangle | - 2|\langle Cx, y \rangle | = 0,
\]

marking the end of the proof.

**Solution 5.2.37.** Set

\[
T = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}
\]

both defined on \(H \oplus H\). Since \(B, C \geq 0\), it easily follows that \(T \geq 0\) as

\[
\langle \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = \langle \begin{pmatrix} Bx \\ Cy \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle
\]

\[
= \langle Bx, x \rangle + \langle Cy, y \rangle \geq 0
\]
for all $x, y \in H$. It is also clear that the square root of $T$ is given by

$$\sqrt{T} = \begin{pmatrix} \sqrt{B} & 0 \\ 0 & \sqrt{C} \end{pmatrix}.$$ 

Since by assumption $BA = AC$, we get

$$TS = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & BA \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & AC \\ 0 & 0 \end{pmatrix} = ST.$$ 

Now, as $T \geq 0$, then we obtain $\sqrt{TS} = S\sqrt{T}$. This means that

$$\begin{pmatrix} \sqrt{B} & 0 \\ 0 & \sqrt{C} \end{pmatrix} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \sqrt{B} \\ 0 & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 0 & \sqrt{BA} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \sqrt{C} \\ 0 & 0 \end{pmatrix},$$

i.e. $\sqrt{BA} = A\sqrt{C}$, as required.

**Solution 5.2.38.** First, recall that $[A, B] = AB - BA$.

1. Let $B$ be a self-adjoint contraction. By Exercise 5.1.32, $U = B + i\sqrt{I - B^2}$ is unitary and $B = \text{Re} U = \frac{U + U^*}{2}$.

$$\|AB - BA\| = \left\| A \left( \frac{U + U^*}{2} \right) - \left( \frac{U + U^*}{2} \right) A \right\|$$

$$= \frac{1}{2} \|AU - U A + AU^* - U^* A\|$$

$$\leq \frac{1}{2} \|AU - U A\| + \frac{1}{2} \|AU^* - U^* A\|$$

$$= \frac{1}{2} \|AU - U A\| + \frac{1}{2} \|U(AU^* - U^* A)U\|$$

$$= \frac{1}{2} \|AU - U A\| + \frac{1}{2} \|(UAU^* - A)U\|$$

$$= \frac{1}{2} \|AU - U A\| + \frac{1}{2} \|UA - AU\|$$

$$= \|AU - U A\|$$

$$= \|(A - UAU^*)U\|$$

$$= \|A - UAU^*\|$$

$$\leq \max(\|A\|, \|UAU^*\|) \quad (\text{Exercises 5.1.8 & 5.1.13})$$

$$= \|A\|,$$

establishing the result.
28 5. POSITIVE OPERATORS. SQUARE ROOT

(2) Let $B$ be self-adjoint. The inequality clearly holds for $B = 0$, so assume that $\|B\| > 0$. Hence $\frac{B}{\|B\|}$ remains self-adjoint and besides, it is a contraction. Therefore, the result of the previous question applies and yields

$$\left\| \frac{A B}{\|B\|} - \frac{B}{\|B\|} A \right\| \leq \|A\|,$$

that is,

$$\|AB - BA\| \leq \|A\| \|B\|,$$

as required.

(3) Let $B \in B(H)$. Define on $H \oplus H$

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix},$$

where the 0 is the zero operator on $H$. Observe that $\tilde{B}$ is self-adjoint (even if $B$ is not one), and that $\tilde{A}$ is self-adjoint because $A$ is one! Hence, by the previous question we know that

$$\|\tilde{A}\tilde{B} - \tilde{B}\tilde{A}\| \leq \|\tilde{A}\|\|\tilde{B}\|.$$

But,

$$\tilde{A}\tilde{B} - \tilde{B}\tilde{A} = \begin{pmatrix} 0 & AB - BA \\ AB^* - B^*A & 0 \end{pmatrix}.$$ 

Also, we have

$$\left\| \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right\| = \max(||C||, ||D||).$$

Hence (why?)

$$\|\tilde{A}\| = \|A\| \text{ and } \|\tilde{B}\| = \|B\|$$

With all these observations, we infer that

$$\|\tilde{A}\tilde{B} - \tilde{B}\tilde{A}\| = \max(||AB - BA||, ||AB^* - B^*A||)$$

$$= \max(||AB - BA||, ||(AB^* - B^*A)^*||)$$

$$= \max(||AB - BA||, ||BA - AB||)$$

$$= \|AB - BA\|,$$

so that finally we get

$$\|\tilde{A}\tilde{B} - \tilde{B}\tilde{A}\| \leq \|\tilde{A}\|\|\tilde{B}\| \iff \|AB - BA\| \leq \|A\|\|B\|,$$

and this completes the proof.
5.2. SOLUTIONS

**Solution 5.2.39.** Write $T = A + iB$ where $A, B \in B(H)$ are self-adjoint with $A = \text{Re} T$ and $B = \text{Im} T$ as is known to readers. Then clearly

$$T^2 = A^2 - B^2 + i(AB + BA).$$

So, if $T^2 = 0$, then

$$A^2 - B^2 + i(AB + BA) = 0 \implies \begin{cases} A^2 = B^2, \\ AB = -BA. \end{cases}$$

Hence, if $A \geq 0$ (a similar argument works when $B \geq 0$), then

$$AB = -BA \implies A^2 B = -ABA = BA^2 \implies AB = BA.$$ 

Therefore, $T$ is normal. Accordingly

$$\|T\|^2 = \|T^2\| = 0 \implies T = 0,$$

as suggested.

**Solution 5.2.40.** The proof is carried out in two steps.

1. Let $\dim H < \infty$. The proof uses a trace argument. First, assume that $A \geq 0$. Clearly, the nilpotence of $T$ does yield $\text{tr} \, T = 0$. Hence

$$0 = \text{tr}(A + iB) = \text{tr} A + i \text{tr} B.$$ 

Since $A$ and $B$ are self-adjoint, we know that $\text{tr} \, A, \text{tr} \, B \in \mathbb{R}$. By the above equation, this forces $\text{tr} \, B = 0$ and $\text{tr} \, A = 0$. The positiveness of $A$ now intervenes to make $A = 0$. Therefore, $T = iB$ and so $T$ is normal. Thus, and as alluded above,

$$0 = \|T^n\| = \|T\|^n,$$

thereby, $T = 0$.

In the event $B \geq 0$, reason as above to obtain $T = A$ and so $T = 0$, as wished.

2. Let $\dim H = \infty$. The condition $\text{Re} T \geq 0$ is equivalent to $\text{Re} <T x, x> \geq 0$ for all $x \in H$. So if $E$ is a closed invariant subspace of $T$, then the previous condition also holds for $T|E : E \to E$.

Now, we proceed to show that $T = 0$, i.e. we must show that $T x = 0$ for all $x \in H$. So, let $x \in H$ and let $E$ be the span of $x, Tx, \cdots, T^{n-1} x$ (that is, the orbit of $x$ under the action of $T$). Hence $E$ is a finite dimensional subspace of $H$ (and so it is equally a Hilbert space). By the nilpotence assumption, we have

$$T^n x = 0,$$
from which it follows that $E$ is invariant for $T$. So, by the first part of the proof (the finite dimensional case), we know that $T = 0$ on $E$ whereby $Tx = 0$. As this holds for any $x$, it follows that $T = 0$ on $H$, as needed.

**Solution** 5.2.41. Since $A$ is invertible, it is seen that $B$ too is invertible. Indeed, by the invertibility of $A$, we get that of $A^p$ or that of $B^p$. So, $CB^p = B^pC = I$ for a certain $C \in B(H)$, and hence $(CB^{p-1})B = B(B^{p-1}C) = I$, whereby $B$ is invertible.

Since $p$ and $q$ are relatively prime numbers, Bezout’s theorem in arithmetic says that $up + vq = 1$ for some integers $u$ and $v$ (only one of them is negative). WLOG, suppose that $u$ is the negative integer. Now, $A^p = B^p$ yields $A^{up} = B^{up}$, and $A^q = B^q$ implies that $A^{vq} = B^{vq}$. Therefore, $A^{up}A^{vq} = B^{up}B^{vq}$

$$A = A^{up+vq} = B^{up+vq} = B,$$

as looked forward to.
Bibliography


[81] T. Furuta, $A \ge B \ge 0$ assures $(B^r A^P B^r)^{1/q} \ge B^{(p+2r)/q}$ for $r \ge 0, p \ge 0, q \ge 1$ with $(1+2r)q \ge p + 2r$, Proc. Amer. Math. Soc., 101/1 (1987) 85-88.


[222] https://math.berkeley.edu/sites/default/files/pages/Spring86.pdf