A Reformulation of the Riemann Hypothesis

Jose Risomar Sousa

November 28, 2020

Abstract

We present some novelties on the Riemann zeta function. Using the analytic continuation we created for the polylogarithm, $\mathrm{Li}_k(e^m)$, we extend the zeta function from $\Re(k) > 1$ to the complex half-plane, $\Re(k) > 0$, by means of the Dirichlet eta function. More strikingly, we offer a reformulation of the Riemann hypothesis through a zeta's cousin, $\varphi(k)$, a pole-free function defined on the entire complex plane whose non-trivial zeros coincide with those of the zeta function.

1 Introduction

The Riemann Hypothesis is a long-standing problem in math, which involves the zeros of the analytic continuation of its most famous Dirichlet series, the zeta function. This Dirichlet series, along with its analytic continuation, constitutes a so-called L-function, whose zeros encode information about the location of the prime numbers. Riemann provided insight into this connection through his unnecessarily convoluted prime counting functions[2].

The zeta function as a Dirichlet series is given by,

$$\zeta(k) = \sum_{j=1}^{\infty} \frac{1}{j^k},$$

and throughout here we use k for the variable instead of the usual s, to keep the same notation used in previous papers released on generalized harmonic numbers and progressions and interrelated subjects.

This series only converges for $\Re(k) > 1$, but it can be analytically continued to the whole complex plane. For the purpose of analyzing the zeros of the zeta function though, we produce its analytic continuation on the complex half-plane only, $\Re(k) > 0$, by means of the alternating zeta function, known as the Dirichlet eta function, $\eta(k)$. It's a well known fact that all the non-trivial zeros of $\zeta(k)$ lie on the critical strip $(0 < \Re(k) < 1)$. The Riemann hypothesis is then the conjecture that all such zeros have $\Re(k) = 1/2$. A somewhat convincing argument for the Riemann hypothesis was given in [4], though it's reasonable to think it will take some time for it to be acknowledged or dismissed.

We start from the formula we found for the analytic continuation of the polylogarithm function, discussed in paper [3]. The polylogarithm is a generalization of the zeta function, and it has the advantage of encompassing the Dirichlet eta function.

We then greatly simplify the convoluted expressions and remove the complex numbers out of the picture, going from four-dimensional chaos ($\mathbb{C} \to \mathbb{C}$) to a manageable two-dimensional relation.

2 The polylogarithm, $\mathrm{Li}_k(e^m)$

As seen in [3], the following expression for the polylogarithm holds for all complex k with positive real part, $\Re(k) > 0$, and all complex m (except m such that $\Re(m) >= 0$ and $|\Im(m)| > 2\pi$ – though for $|\Im(m)| = 2\pi$ one must have $\Re(k) > 1$):

$$\mathrm{Li}_k(e^m) = -\frac{m^k}{2k!} - \frac{m^{k-1}\,(1+\log(-m))}{(k-1)!} - \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}\coth\frac{mu}{2} + \frac{2\left(1-m^{-k}(m-\log(1-u))^k\right)}{k\,u^2}\,du$$

From this formula, we can derive two different formulae for $\zeta(k)$, using $m = 0$ or $m = 2\pi\boldsymbol{i}$, but both are only valid when $\Re(k) > 1$. Using $m = 2\pi\boldsymbol{i}$ is effortless, we just need to replace m with $2\pi\boldsymbol{i}$ in the above. Using $m = 0$ is not as direct, we need to take the limit of $H_k(n)$[1] as n tends to infinity:

$$\zeta(k) = \frac{1}{k!}\int_0^1 \frac{(-\log u)^k}{(1-u)^2}\,du$$

2.1 The analytic continuation of $\zeta(k)$

The analytic continuation of the zeta function to the complex half-plane can be achieved using the Dirichlet eta function, as below:

$$\zeta(k) = \frac{1}{1-2^{1-k}}\eta(k) = \frac{1}{1-2^{1-k}}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^k},$$

which is valid in the only region that matters to the zeta function's non-trivial zeros, the critical strip. The exception to this mapping are the zeros of $1-2^{1-k}$, which are also zeros of $\eta(k)$, thus yielding an undefined product.

For the purpose of studying the zeros of the zeta function, we can focus only on $\eta(k)$ and

[1] A formula derived from the partial sums of $\mathrm{Li}_k(e^m)$, as explained in [3].

ignore its multiplier. The below should hold whenever $\Re(k) > 0$:

$$\eta(k) = \frac{(\boldsymbol{i}\pi)^k}{2k!} + \frac{(\boldsymbol{i}\pi)^{k-1}\left(1+\log\left(-\boldsymbol{i}\pi\right)\right)}{(k-1)!} + \frac{(\boldsymbol{i}\pi)^k}{2(k-1)!}\int_0^1 -\boldsymbol{i}(1-u)^{k-1}\cot\frac{\pi u}{2} + \frac{2\left(1-\pi^{-k}(\pi+\boldsymbol{i}\log\left(1-u\right))^k\right)}{k\,u^2}\,du$$

3 Simplifying the problem

Before starting to solve equation $\eta(k) = 0$, let's simplify it by letting $h(k) = 2\,k!(\boldsymbol{i}\pi)^{-k}\eta(k)$.

Now, $\eta(k) = 0$ if and only if $h(k) = 0$, which implies the below equation:

$$-1+\frac{2\boldsymbol{i}\,k\left(1+\log\left(-\boldsymbol{i}\pi\right)\right)}{\pi} = \int_0^1 -\boldsymbol{i}\,k(1-u)^{k-1}\cot\frac{\pi u}{2} + \frac{2\left(1-\pi^{-k}(\pi+\boldsymbol{i}\log\left(1-u\right))^k\right)}{u^2}\,du$$

Now we need to make another transformation, with the goal of separating the real and imaginary parts. Let's set $k = r + \boldsymbol{i}\,t$, expressed in polar form, and change the integral's variable using the relation $\log\left(1-u\right) = \pi\tan v$, chosen for convenience.

$$k = r + \boldsymbol{i}\,t = \sqrt{r^2+t^2}\exp\left(\boldsymbol{i}\arctan\frac{t}{r}\right)$$

With that, taking into account the Jacobian of the transformation, our equation becomes:

$$\frac{\pi(r-1)-2\,t(1+\log\pi)}{\pi} + \boldsymbol{i}\frac{\pi\,t+2\,r(1+\log\pi)}{\pi} = \pi\int_0^{\pi/2}(\sec v)^2(-\boldsymbol{i}\sqrt{r^2+t^2}\tan\frac{\pi e^{-\pi\tan v}}{2}\exp\left(-\pi\,r\tan v+\boldsymbol{i}\left(\arctan\frac{t}{r}-\pi\,t\tan v\right)\right) + \frac{1}{2}\left(\operatorname{csch}\frac{\pi\tan v}{2}\right)^2\left(1-\exp\left(-r\log\cos v+t\,v+\boldsymbol{i}(-t\log\cos v-r\,v)\right)\right))\,dv$$

Though this expression is very complicated, it can be simplified, as we do next. Since the parameters are real, we can separate the real and imaginary parts.

3.1 The real part equation

Below we have the equation one can derive for the real part:

$$\frac{\pi(r-1)-2\,t(1+\log\pi)}{\pi} = \pi\int_0^{\pi/2}(\sec v)^2(\sqrt{r^2+t^2}\tan\frac{\pi e^{-\pi\tan v}}{2}\exp\left(-\pi\,r\tan v\right)\sin\left(\arctan\frac{t}{r}-\pi\,t\tan v\right) + \frac{1}{2}\left(\operatorname{csch}\frac{\pi\tan v}{2}\right)^2\left(1-\exp\left(-r\log\cos v+t\,v\right)\cos\left(t\log\cos v+r\,v\right)\right))\,dv$$

Any positive odd integer r satisfies this equation, when $t = 0$.

If $r + \boldsymbol{i}\, t$ is a zero of the zeta function, so is its conjugate, $r - \boldsymbol{i}\, t$. Hence, noting that the first term inside of the integral is an odd function in t, we can further simplify the above by adding the equations for t and $-t$ up as follows:

$$2(r-1) = \frac{\pi}{2}\int_0^{\pi/2}\left(\sec v \operatorname{csch}\frac{\pi \tan v}{2}\right)^2 \left(2 - (\sec v)^r \left(e^{t\, v}\cos\left(t \log\cos v + r\, v\right) + e^{-t\, v}\cos\left(t\log\cos v - r\, v\right)\right)\right) dv$$

Let's call the function on the right-hand side of the equation $f(r,t)$. After this transformation, the positive odd integers r remain zeros of $f(r,0) = 2(r-1)$.

3.2 The imaginary part equation

Likewise, for the imaginary part we have:

$$\frac{\pi\, t + 2\, r(1+\log\pi)}{\pi} =$$
$$\pi\int_0^{\pi/2}(\sec v)^2(-\sqrt{r^2+t^2}\tan\frac{\pi e^{-\pi\tan v}}{2}\exp\left(-\pi\, r\tan v\right)\cos\left(\arctan\frac{t}{r} - \pi\, t\tan v\right)$$
$$+\frac{1}{2}\left(\operatorname{csch}\frac{\pi\tan v}{2}\right)^2\exp\left(-r\log\cos v + t\, v\right)\sin\left(t\log\cos v + r\, v\right))\, dv$$

Coincidentally, any positive even integer r satisfies this equation when $t = 0$, so the two equations (real and imaginary) are never satisfied simultaneously for any positive integer.

Now the first term inside of the integral is an even function in t, so to simplify it we need to subtract the equations for t and $-t$, obtaining:

$$2\, t = \frac{\pi}{2}\int_0^{\pi/2}(\sec v)^{r+2}\left(\operatorname{csch}\frac{\pi\tan v}{2}\right)^2\left(e^{t\, v}\sin\left(t\log\cos v + r\, v\right) + e^{-t\, v}\sin\left(t\log\cos v - r\, v\right)\right) dv$$

Let's call the function on the right-hand side of the equation $g(r,t)$. After this transformation, any r satisfies $g(r,0) = 0$, whereas the roots of $f(r,0) = 2(r-1)$ are still the positive odd integers r. This means that when $t = 0$, these transformations have introduced the positive odd integers r as new zeros of the system, which weren't there before.

4 Riemann hypothesis reformulation

If we take a linear combination of the equations for the real and imaginary parts, such as $2(r-1) - 2\, t\, \boldsymbol{i} = f(r,t) - \boldsymbol{i}\, g(r,t)$, we can turn the system of equations into a simpler single equation:

$$k - 1 = \frac{\pi}{2}\int_0^{\pi/2}\left(\sec v\operatorname{csch}\frac{\pi\tan v}{2}\right)^2\left(1 - \frac{\cos k\, v}{(\cos v)^k}\right) dv$$

Going a little further, with a simple transformation ($u = \tan v$) we can deduce the following theorem.

Theorem k is a non-trivial zero of the Riemann zeta function if and only if k is a non-trivial zero of:

$$\varphi(k) = 1 - k + \frac{\pi}{2} \int_0^\infty \left(\operatorname{csch} \frac{\pi\, u}{2}\right)^2 \left(1 - (1+u^2)^{k/2} \cos\left(k \arctan u\right)\right)\, du$$

Hence the Riemann hypothesis is the statement that the zeros of $\varphi(k)$ located on the critical strip have $\Re(k) = 1/2$.

Proof All the roots of $\varphi(k)$ should also be roots of the zeta function, except for the positive odd integers and the trivial zeros of the eta function ($1+2\pi\boldsymbol{i}\, j/\log 2$, for any integer j), though this might not be true since we transformed the equations (that is, there might be other k such that $\varphi(k) = 0$ but $\zeta(k) \neq 0$).

However, a little empirical research reveals the following relationship between $\zeta(k)$ and $\varphi(k)$:

$$-\frac{2\,\Gamma(k+1)\left(2^{1-k}-1\right)}{\pi^k} \cos\frac{\pi k}{2}\, \zeta(k) = \varphi(k), \text{ for all complex } k \text{ except where undefined.} \quad (1)$$

This relationship was derived from the observation that $\varphi(k) = \Re(h(k))$ for all real k. $\square$

Note this functional equation breaks down at the negative integers ($k! = \pm\infty$ but $\zeta(k) = 0$ or $\cos \pi k/2 = 0$, whereas $\varphi(k) \neq 0$) and at 1. There is a zeros trade-off between these two functions (the negative even integers for the positive odd integers). Note also that while the convergence domain of $h(k)$ is $\Re(k) > 0$, the domain of $\varphi(k)$ is the whole complex plane.

If we combine equation (1) with Riemann's functional equation, we can obtain the following simpler equivalence, valid for all complex k except the zeta pole:

$$2(k-1)(1-2^{-k})\zeta(k) = \varphi(1-k),$$

which in turn implies Riemann's functional equation when combined with (1).

4.1 Particular values of $\varphi(k)$ when k is integer

From the functional equations, we can easily find out the values of $\varphi(\pm k)$ when k is a non-negative integer:

$$\begin{cases} \varphi(2k) = \left(2 - 2^{2k}\right) B_{2k} \\ \varphi(2k+1) = 0 \\ \varphi(-k) = 2k\left(1 - 2^{-k-1}\right)\zeta(k+1) \end{cases}$$

And from these formulae we conclude that for large positive real k, $\varphi(-k) \sim 2k$.

One can also create a generating function for $\varphi(k)$, based on the following identities:

$$\cos \arctan u = \frac{1}{\sqrt{1+u^2}}, \text{ and } \cos\left(k \arctan u\right) = T_k(\cos \arctan u),$$

where $T_k(x)$ is the Chebyshev polynomial of the first kind.

Therefore, using the generating function of $T_k(x)$ available in the literature, for the positive integers we have:

$$\sum_{k=0}^{\infty} \left(x\sqrt{1+u^2}\right)^k T_k(\cos \arctan u) = \frac{1-x}{(1-x)^2+(x\,u)^2},$$

from which it's possible to produce the generating function of $\varphi(k)$ (let it be $q(x)$):

$$\sum_{k=0}^{\infty} x^k \varphi(k) = \frac{2}{1-x} - \frac{1}{(1-x)^2} + \frac{\pi\, x^2}{2(1-x)} \int_0^{\infty} \left(\operatorname{csch} \frac{\pi\, u}{2}\right)^2 \frac{u^2}{(1-x)^2+(x\,u)^2}\, du$$

The k-th derivative of $q(x)$ yields the value of $\varphi(k)$, and obtaining it is not very hard (we just need to decompose the functions in x into a sum of fractions whose denominators have degree 1, if the roots are simple – so we can easily generalize their k-th derivative). After we perform all the calculations and simplifications we find that:

$$\varphi(k) = \frac{q^{(k)}(x)}{k!} = 1 - k - \frac{\pi\, k!}{2} \int_0^{\infty} \left(\operatorname{csch} \frac{\pi\, u}{2}\right)^2 \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^j\, u^{2j}}{(2j)!(k-2j)!}\, du,$$

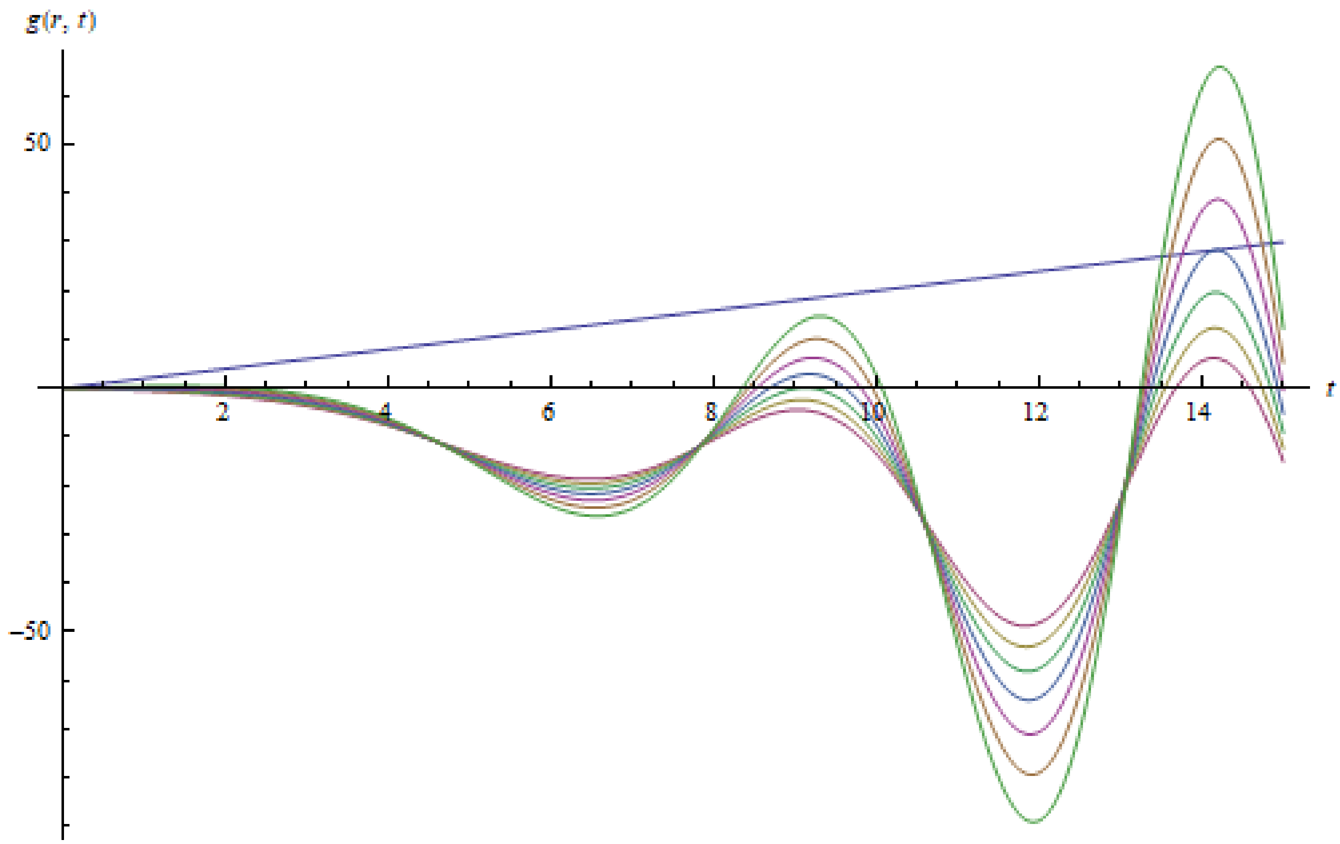
where $\lfloor k/2 \rfloor$ means the integer division.

In here we went from an expression that holds for all k, to an expression that only holds for k a positive integer, the opposite of analytic continuation.

5 Graphics plotting

First we plot the curves obtained with the imaginary part equation, $g(r, t)$, with values of r starting at 1/8 with 1/8 increments, up to 7/8, for a total of 7 curves plus the $2t$ line. The points where the line crosses the curves are candidates for zeros of the Riemann zeta function (they also need to satisfy the real part equation, $f(r, t) = 2(r - 1)$).

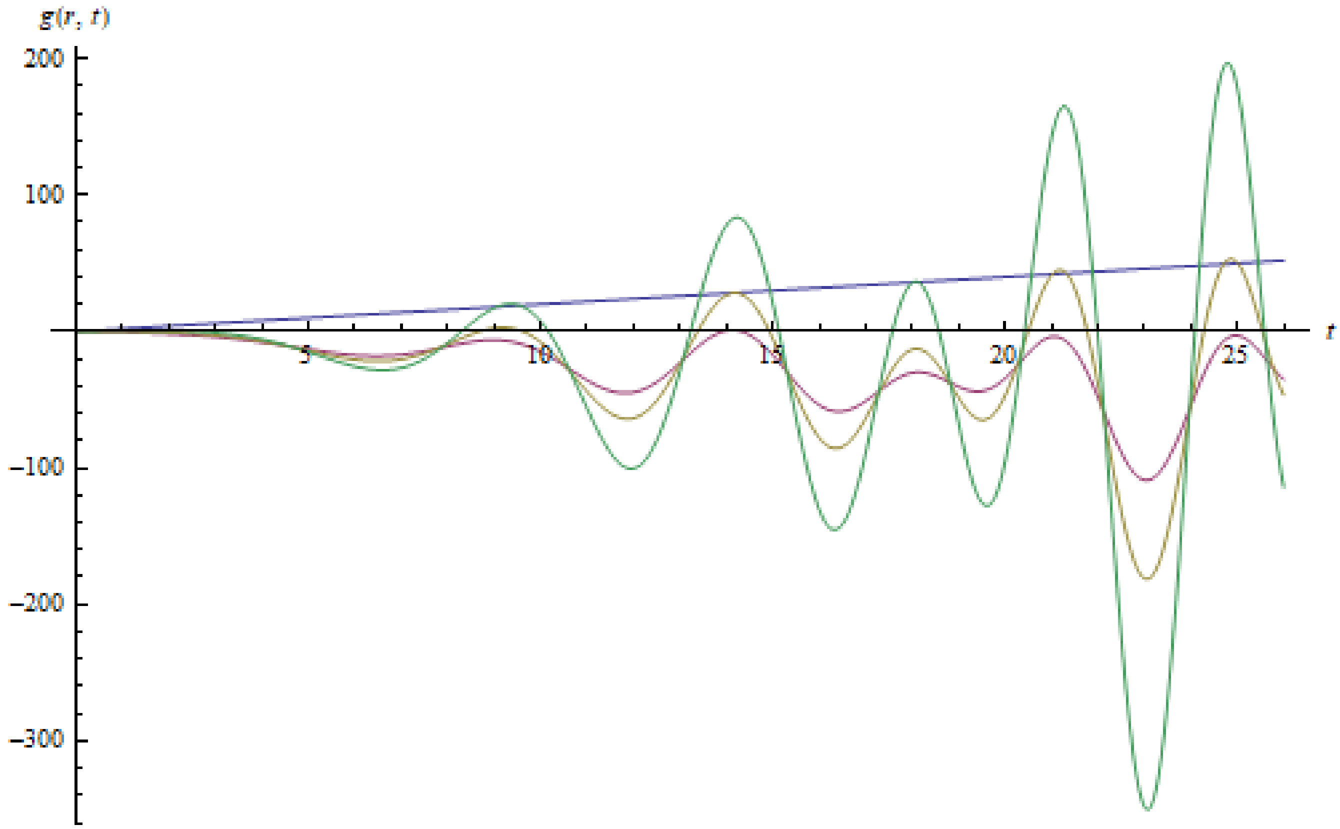
Let's see what we obtain when we plot these curves with t varying from 0 to 15. In the graph below, higher curves have greater r, though not always, below the x-axis it's vice-versa – but generally the more outward the curve, the greater the r:



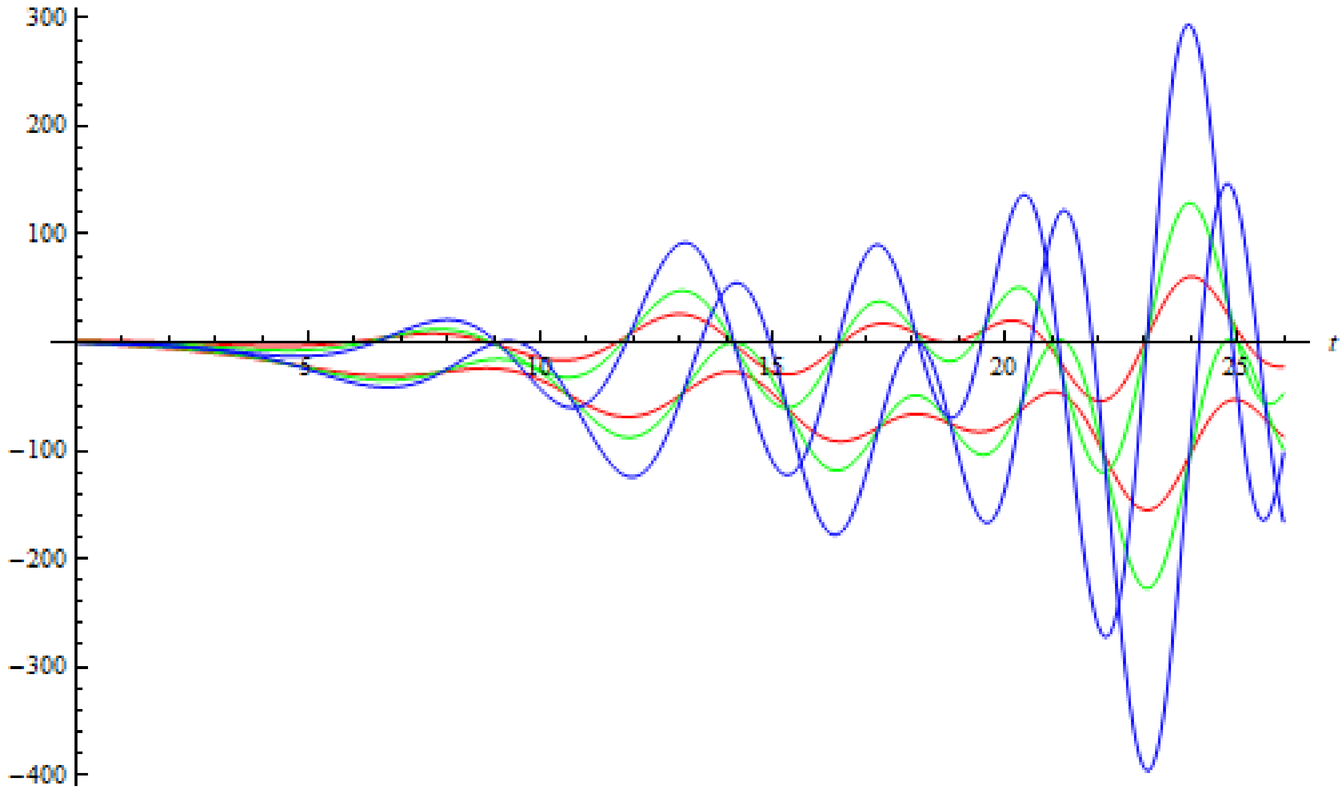
As we can see, it seems the line crosses the curve for $r = 1/2$ at its local maximum, which must be the first non-trivial zero (that is, its imaginary part). The line also crosses 3 other curves (all of which have $r > 1/2$), but these are probably not zeros due to the real part equation. Also, it seems there must be a line that unites the local maximum points of all the curves, though that is just a wild guess.

One first conclusion is that one equation seems to be enough for $r = 1/2$, the line seems to only cross this curve at the zeta zeros. Another conclusion is that apparently curves with $r < 1/2$ don't even meet the first requirement, and also apparently $r = 1/2$ is just right. A third conclusion is that all curves seem to have the same inflection points.

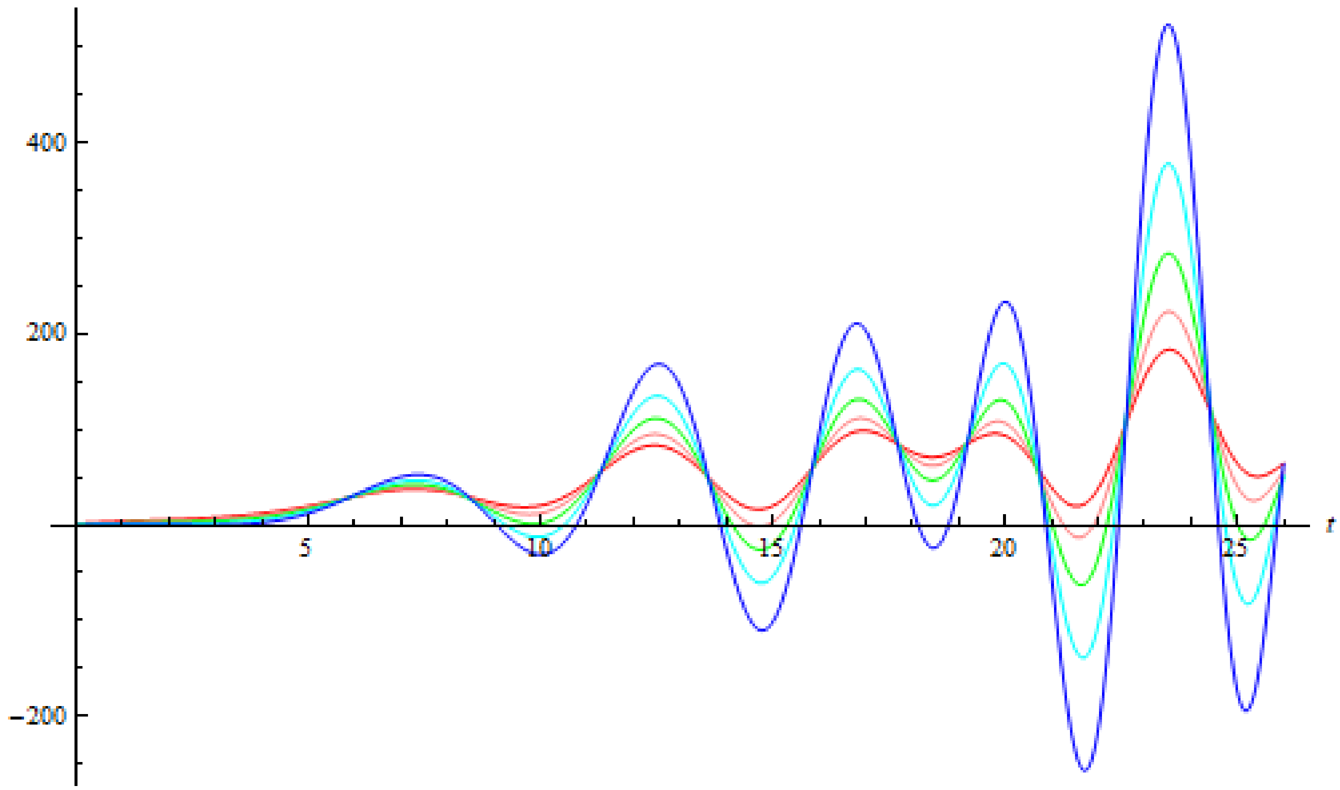
In the below graph we plotted $g(r,t)$ for the minimum, middle and maximum points of the critical strip (0, 1/2 and 1), with t varying from 0 to 26, for further comparison (0 is pink, 1 is green):



Now, the graph below shows plots for curves $-2(r-1)+f(r,t)$ and $-2t+g(r,t)$ together. The plots were created for $r=0$ (red), $r=1/2$ (green) and $r=1$ (blue) (curves with the same color have the same r). A point is a zero of the zeta function when both graphs cross the x-axis at the same point (three zeta zeros are shown).



And finally, graphs for the difference of the two functions, $-2(r-1)+f(r,t)+2t-g(r,t)$, were created for the same r's as before and with the same colors as before (but now we also have pink ($r=1/4$) and cyan ($r=3/4$)).



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