Lerch's Φ and the Polylogarithm at the Positive Integers

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Abstract

We review the closed-forms of the partial Fourier sums associated with $HP_k(n)$ and create an asymptotic expression for $HP(n)$ as a way to obtain formulae for the full Fourier series (if b is such that $|b| < 1$, we get a surprising pattern, $HP(n) \sim H(n) - \sum_{k\geq 2}(-1)^k\zeta(k)b^{k-1}$). Finally, we use the found Fourier series formulae to obtain the values of the Lerch transcendent function, $\Phi(e^m, k, b)$, and by extension the polylogarithm, $\text{Li}_k(e^m)$, at the positive integers k.

Summary

1 Introduction

Since the Basel problem in 1650, scholars have been eager to find closed-forms for similar infinite series, especially Dirichlet series. In this article, we create formulae for the Lerch transcendent function, $\Phi(e^m, k, b)$, and the polylogarithm, $\text{Li}_k(e^m)$, that hold at the positive integers k. Conversely, a formula for the Hurwitz zeta function at the negative integers, $\zeta(-k, b)$, is also created, to complement a formula at the positive integers produced in [5].

The advantage of formulae that only hold at the positive integers is the fact we expect them to be simpler and easier to work with. It's an obvious statement if, for example, we think about the closed-forms of the zeta function at the positive integers greater than one, $\zeta(k)$, and its general integral, valid for $\Re(k) > 1$.

The formulae derived here are based on new expressions for the generalized harmonic progressions:

$$HP_k(n) = \sum_{j=1}^{n} \frac{1}{(a\,n + b)^k},$$

which have been extensively studied in two previous papers, and vary depending on whether the parameters, a and b, are integer[3] or complex[4]. When $a = 1$ and $b = 0$, we have a notable particular case, the generalized harmonic numbers, $H_k(n)$.

In [3] we derived expressions for the partial Fourier sums, $C_k^m(a, b, n)$ and $S_k^m(a, b, n)$, associated with $HP_k(n)$, which we reproduce again in the next section, with a short description.

Our objective in this paper is to obtain the limit of those expressions as n gets large, and then combine them to obtain the Lerch transcendent function, Φ, at the positive integers.

In the process, we need to obtain the limit of $HP(n) - H(n)$, with $2b$ a non-integer complex number. Since this limit can also be attained by means of the digamma function, $\psi(n)$, this is just a new, more interesting way of deriving that limit.

In section (3), we review the limits of the integrals that appear in the expressions of $HP_k(n)$ as n tends to infinity, which are central to this solution.

The process of obtaining the limits of $C_{2k}^m(b, n)$ and $S_{2k+1}^m(b, n)$ is much simpler than that of $C_{2k+1}^m(b, n)$ and $S_{2k}^m(b, n)$, since the latter involve the limit of $HP(n)$, which is not finite.

2 The partial Fourier sums

The subsequent expressions are the partial sums of the Fourier series associated with the generalized harmonic progressions from [3], and hold for all complex m, a and b and for all integer $n \geq 1$.

By definition $HP_0(n) = 0$ for all positive integer n, so they actually have no effect in the sums. If $b = 0$, we can discard any term that has a null denominator and the equation still holds (technically, we take the limit as b tends to 0, as we see in section (6.2)).

2.1 $C_{2k}^m(a, b, n)$ and $S_{2k+1}^m(a, b, n)$

For all integer $k \geq 1$:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} \cos\frac{2\pi(aj+b)}{m} = -\frac{1}{2b^{2k}}\left(\cos\frac{2\pi b}{m} - \sum_{j=0}^{k}\frac{(-1)^j}{(2j)!}\left(\frac{2\pi b}{m}\right)^{2j}\right)$$
$$+\frac{1}{2(an+b)^{2k}}\left(\cos\frac{2\pi(an+b)}{m} - \sum_{j=0}^{k}\frac{(-1)^j}{(2j)!}\left(\frac{2\pi(an+b)}{m}\right)^{2j}\right) + \sum_{j=1}^{k}\frac{(-1)^{k-j}}{(2k-2j)!}\left(\frac{2\pi}{m}\right)^{2k-2j} HP_{2j}(n)$$
$$+\frac{(-1)^k}{2(2k-1)!}\left(\frac{2\pi}{m}\right)^{2k}\int_0^1 (1-u)^{2k-1}\left(\sin\frac{2\pi(an+b)u}{m} - \sin\frac{2\pi bu}{m}\right)\cot\frac{\pi au}{m}\,du$$

For all integer $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} \sin\frac{2\pi(aj+b)}{m} = -\frac{1}{2b^{2k+1}}\left(\sin\frac{2\pi b}{m} - \sum_{j=0}^{k}\frac{(-1)^j}{(2j+1)!}\left(\frac{2\pi b}{m}\right)^{2j+1}\right)$$
$$+\frac{1}{2(an+b)^{2k+1}}\left(\sin\frac{2\pi(an+b)}{m} - \sum_{j=0}^{k}\frac{(-1)^j}{(2j+1)!}\left(\frac{2\pi(an+b)}{m}\right)^{2j+1}\right) + \sum_{j=1}^{k}\frac{(-1)^{k-j}}{(2k+1-2j)!}\left(\frac{2\pi}{m}\right)^{2k+1-2j} HP_{2j}(n)$$
$$+\frac{(-1)^k}{2(2k)!}\left(\frac{2\pi}{m}\right)^{2k+1}\int_0^1 (1-u)^{2k}\left(\sin\frac{2\pi(an+b)u}{m} - \sin\frac{2\pi bu}{m}\right)\cot\frac{\pi au}{m}\,du$$

2.1.1 The limits of $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$

For comparison purposes, let's review some limits that we derived previously for the particular cases $C_{2k}^m(n)$ and $S_{2k+1}^m(n)$ (that is, $a = 1$ and $b = 0$). We expect the limits of the more general expressions to coincide with them.

At infinity these particular cases become Fourier series (denoted here by C_{2k}^m and S_{2k+1}^m), which have limits given by:

$$C_{2k}^m = \sum_{j=1}^{\infty}\frac{1}{j^{2k}}\cos\frac{2\pi j}{m} = \sum_{j=0}^{k}\frac{(-1)^{k-j}}{(2k-2j)!}\left(\frac{2\pi}{m}\right)^{2k-2j}\zeta(2j) + \frac{(-1)^k|m|}{4(2k-1)!}\left(\frac{2\pi}{m}\right)^{2k} \quad (\forall \text{ integer } k \geq 1)$$

$$S_{2k+1}^m = \sum_{j=1}^{\infty}\frac{1}{j^{2k+1}}\sin\frac{2\pi j}{m} = \sum_{j=0}^{k}\frac{(-1)^{k-j}}{(2k+1-2j)!}\left(\frac{2\pi}{m}\right)^{2k+1-2j}\zeta(2j) + \frac{(-1)^k|m|}{4(2k)!}\left(\frac{2\pi}{m}\right)^{2k+1} \quad (\forall \text{ integer } k \geq 0)$$

These limits only hold for real $|m| \geq 1$ ($k = 0$ and $|m| = 1$ are exceptions and also trivial cases). So for $S_1^1 = 0$ the formula breaks down (see section (3) to know why). Both these results are known in the literature, they're rewrites of equations that feature in [1] (page 805).

2.2 $C_{2k+1}^m(a,b,n)$ and $S_{2k}^m(a,b,n)$

For all integer $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} \cos\frac{2\pi(aj+b)}{m} = -\frac{1}{2b^{2k+1}}\left(\cos\frac{2\pi b}{m} - \sum_{j=0}^{k}\frac{(-1)^j}{(2j)!}\left(\frac{2\pi b}{m}\right)^{2j}\right)$$
$$+\frac{1}{2(an+b)^{2k+1}}\left(\cos\frac{2\pi(an+b)}{m} - \sum_{j=0}^{k}\frac{(-1)^j}{(2j)!}\left(\frac{2\pi(an+b)}{m}\right)^{2j}\right) + \sum_{j=0}^{k}\frac{(-1)^{k-j}}{(2k-2j)!}\left(\frac{2\pi}{m}\right)^{2k-2j} HP_{2j+1}(n)$$
$$+\frac{(-1)^k}{2(2k)!}\left(\frac{2\pi}{m}\right)^{2k+1}\int_0^1 (1-u)^{2k}\left(\cos\frac{2\pi(an+b)u}{m} - \cos\frac{2\pi bu}{m}\right)\cot\frac{\pi au}{m}\,du$$

For all integer $k \geq 1$:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} \sin\frac{2\pi(aj+b)}{m} = -\frac{1}{2b^{2k}}\left(\sin\frac{2\pi b}{m} - \sum_{j=0}^{k-1}\frac{(-1)^j}{(2j+1)!}\left(\frac{2\pi b}{m}\right)^{2j+1}\right)$$
$$+\frac{1}{2(an+b)^{2k}}\left(\sin\frac{2\pi(an+b)}{m} - \sum_{j=0}^{k-1}\frac{(-1)^j}{(2j+1)!}\left(\frac{2\pi(an+b)}{m}\right)^{2j+1}\right) - \sum_{j=0}^{k-1}\frac{(-1)^{k-j}}{(2k-1-2j)!}\left(\frac{2\pi}{m}\right)^{2k-1-2j} HP_{2j+1}(n)$$
$$-\frac{(-1)^k}{2(2k-1)!}\left(\frac{2\pi}{m}\right)^{2k}\int_0^1 (1-u)^{2k-1}\left(\cos\frac{2\pi(an+b)u}{m} - \cos\frac{2\pi bu}{m}\right)\cot\frac{\pi au}{m}\,du$$

2.2.1 The limits of $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$

The limits of $C_{2k+1}^m(n)$ and $S_{2k}^m(n)$ for real $|m| \geq 1$ are given by:

$$C_{2k+1}^m = \sum_{j=1}^{\infty}\frac{1}{j^{2k+1}}\cos\frac{2\pi j}{m} = \sum_{j=1}^{k}\frac{(-1)^{k-j}}{(2k-2j)!}\left(\frac{2\pi}{m}\right)^{2k-2j}\zeta(2j+1) + \frac{(-1)^k}{(2k)!}\left(\frac{2\pi}{m}\right)^{2k}\log|m|$$
$$-\frac{(-1)^k}{2(2k)!}\left(\frac{2\pi}{m}\right)^{2k+1}\int_0^1 (1-u)^{2k}\cot\frac{\pi u}{m} - m(1-u)\cot\pi u\,du \;(\forall \text{ integer } k \geq 0)$$

$$S_{2k}^m = \sum_{j=1}^{\infty}\frac{1}{j^{2k}}\sin\frac{2\pi j}{m} = -\sum_{j=1}^{k-1}\frac{(-1)^{k-j}}{(2k-1-2j)!}\left(\frac{2\pi}{m}\right)^{2k-1-2j}\zeta(2j+1) - \frac{(-1)^k}{(2k-1)!}\left(\frac{2\pi}{m}\right)^{2k-1}\log|m|$$
$$+\frac{(-1)^k}{2(2k-1)!}\left(\frac{2\pi}{m}\right)^{2k}\int_0^1 (1-u)^{2k-1}\cot\frac{\pi u}{m} - m(1-u)\cot\pi u\,du \;(\forall \text{ integer } k \geq 1)$$

The exception is $C_1^1 = \infty$, since integral $\int_0^1 \cot\pi u - (1-u)\cot\pi u\,du$ diverges, which means that $H(n)$ diverges. These results are probably original.

3 The limits of the integrals

In [2] we introduced the following theorems, whose validity we now fully extend. For all real $k \geq 0$ and real m:

Theorem 1 $\displaystyle\lim_{n\to\infty} \int_0^1 (1-u)^k \sin\frac{2\pi n u}{m} \cot\frac{\pi u}{m}\, du = \begin{cases} 1, & \text{if } k=0 \text{ and } |m|=1 \\ |\frac{m}{2}|, & \text{if } |m| \geq 1 \end{cases}$

Another result we need is in the following theorem, which holds for all real $k \geq 0$ and real $|m| \geq 1$ (except $k = 0$ and $|m| = 1$, for which the integral doesn't converge):

Theorem 2 $\displaystyle\lim_{n\to\infty} \int_0^1 (1-u)^k \cos\frac{2\pi n u}{m} \cot\frac{\pi u}{m} - m\,(1-u)\cos 2\pi n u \cot \pi u\, du = \frac{m \log|m|}{\pi}$

A direct consequence of theorem 2 and of the various possible formulae for $H(n)$ (see [2]), the below result is useful to handle the half-integers in (5.1) and (6.3):

$$\lim_{n\to\infty} \int_0^1 (1-u)^k \cos\frac{2\pi n u}{m} \cot\frac{\pi u}{m} - \frac{m}{2}\,(1-u)\cos \pi n u \cot \frac{\pi u}{2}\, du = \frac{m}{\pi}\log\frac{|m|}{2}$$

We don't provide a proof for these results due to the scope, but they should be simple.

Though these limits shouldn't converge for non-real complex m, when they are linearly combined like $l_1 + \boldsymbol{i}\, l_2$ their infinities cancel out giving a finite value. This property is what allows our final formula from section (6.3) to converge nearly always, even when the parameters are not real.

4 $HP(n)$ asymptotic behavior

Here we figure out the relationship between $HP(n)$ and $H(n)$.

For this exercise, we make use of the sine-based $HP(n)$ formula from [4], which is:

$$\sum_{j=1}^{n} \frac{1}{j+b} = -\frac{1}{2b} + \frac{1}{2(n+b)} + \frac{\pi}{\sin 2\pi b} \int_0^1 (\sin 2\pi(n+b)u - \sin 2\pi b u) \cot \pi u\, du$$

We can "expand" the sine (that is, use the identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$), getting:

$$\sum_{j=1}^{n} \frac{1}{j+b} = -\frac{1}{2b} + \frac{1}{2(n+b)} + \frac{\pi}{\sin 2\pi b} \int_0^1 (\cos 2\pi b u \sin 2\pi n u + \sin 2\pi b u \cos 2\pi n u - \sin 2\pi b u) \cot \pi u\, du$$

We take the first part of the integrand, change the variables, expand $\cos 2\pi b(1-u)$ (with identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$), and by means of the theorem 1 we can conclude that:

$$\lim_{n\to\infty} \frac{\pi}{\sin 2\pi b} \int_0^1 \cos 2\pi b(1-u) \sin 2\pi n(1-u) \cot \pi(1-u)\, du = \frac{\pi}{2} (\cot 2\pi b + \csc 2\pi b) \tag{1}$$

Now we need to work out the second part of the integrand. We make a change of variables, expand $\sin 2\pi b(1-u)$, and when using theorem 2 we need to avoid the case $k=0$ and $m=1$ (since that integral doesn't converge), which leads us to:

$$\begin{aligned} &\int_0^1 (\sin 2\pi b(1-u) \cos 2\pi n(1-u) - \sin 2\pi b(1-u)) \cot \pi(1-u)\, du = \\ &\quad \sin 2\pi b \int_0^1 (\cos 2\pi bu - 1 - u(\cos 2\pi b - 1)) \cos 2\pi n(1-u) \cot \pi(1-u)\, du \\ &\quad\quad - \cos 2\pi b \int_0^1 (\sin 2\pi bu - u \sin 2\pi b) \cos 2\pi n(1-u) \cot \pi(1-u)\, du \\ &\quad\quad\quad + \int_0^1 (-\sin 2\pi b(1-u) + (\sin 2\pi b)(1-u) \cos 2\pi n(1-u)) \cot \pi(1-u)\, du \end{aligned}$$

The two first integrals on the right-hand side cancel out, per theorem 2, when n goes to infinity, leaving only the third integral to be figured.

But if we look back at the expression for $H(n)$ from [2], we notice it matches part of the last integral:

$$\sum_{j=1}^{n} \frac{1}{j} = \frac{1}{2n} + \pi \int_0^1 u\,(1 - \cos 2\pi n(1-u)) \cot \pi(1-u)\, du, \tag{2}$$

which means the last integral can be further split:

$$\begin{aligned} &\frac{\pi}{\sin 2\pi b} \int_0^1 (-\sin 2\pi b(1-u) + (\sin 2\pi b)(1-u) \cos 2\pi n(1-u)) \cot \pi(1-u)\, du = \\ &\quad \frac{\pi}{\sin 2\pi b} \int_0^1 (-\sin 2\pi b(1-u) - u \sin 2\pi b + \sin 2\pi b \cos 2\pi n(1-u)) \cot \pi(1-u)\, du \\ &\quad\quad + \pi \int_0^1 u\,(1 - \cos 2\pi n(1-u)) \cot \pi(1-u)\, du \end{aligned}$$

At this point, there's only the limit of the first integral on the right-hand side left to figure out, but fortunately that integral is constant for all integer n[1]. Therefore, after simplifying (1) further, we conclude that for sufficiently large n:

$$\sum_{j=1}^{n} \frac{1}{j+b} \sim -\frac{1}{2b} + \frac{\pi}{2} \cot \pi b - \pi \int_0^1 \left(\frac{\sin 2\pi bu}{\sin 2\pi b} - u \right) \cot \pi u\, du + H(n) \tag{3}$$

[1] It stems from $\int_0^1 (1 - \cos 2\pi n\, u) \cot \pi\, u\, du = 0$ for all integer n.

Coincidentally, the above integral is identical to the generating function of the zeta function at the odd integers, that we've seen in [2]:

$$\sum_{k=1}^{\infty} \zeta(2k+1)x^{2k+1} = -\pi x \int_0^1 \left(\frac{\sin 2\pi x u}{\sin 2\pi x} - u \right) \cot \pi u \, du$$

That means that for sufficiently large n and $0 < |b| < 1$ we can write the interesting approximation:

$$\sum_{j=1}^{n} \frac{1}{j+b} \sim H(n) - \sum_{k=2}^{\infty} (-1)^k \zeta(k) b^{k-1}$$

Now, since formula (3) clearly doesn't hold at the half-integers, for such b we can resort to a different integral representation for the $\zeta(2k+1)$ generating function[2], which leads to:

$$\sum_{j=1}^{n} \frac{1}{j+b} \sim -\frac{1}{2b} + \frac{\pi}{2} \int_0^1 (-1 + u + \cos \pi b u) \cot \frac{\pi u}{2} \, du + H(n),$$

since $\cot \pi b$ is zero for all half-integer b.

5 The full Fourier series

Although the expressions of $C_k^m(a, b, n)$ and $S_k^m(a, b, n)$ hold for all positive integers k and n and complex m, a and b, the limits that we find next are constrained by the requirements of the theorems 1 and 2 from section (3).

Without loss of generality, let's set $a = 1$ to simplify the calculations:

$$C_k^m(b, n) = \sum_{j=1}^{n} \frac{1}{(j+b)^k} \cos \frac{2\pi(j+b)}{m} \quad \text{and} \quad S_k^m(b, n) = \sum_{j=1}^{n} \frac{1}{(j+b)^k} \sin \frac{2\pi(j+b)}{m}$$

And since $k = 0$ and $|m| = 1$ leads to trivial cases, we're not going to account for them in the following reasoning (so remember the final formulae may not be true for $k = 0$ and $|m| = 1$).

5.1 The limit of $C_{2k+1}^m(b, n)$

The limit of $C_{2k+1}^m(b, n)$ is much harder to figure out than the limit of $S_{2k+1}^m(b, n)$, which should come as no surprise given the limits we've seen in sections (2.1.1) and (2.2.1) for the particular cases. So, without further ado, let's see how to go about it:

$$\lim_{n\to\infty} C_{2k+1}^m(b, n) = -\frac{1}{2b^{2k+1}} \left(\cos \frac{2\pi b}{m} - \sum_{j=0}^{k} \frac{(-1)^j (\frac{2\pi b}{m})^{2j}}{(2j)!} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j} (\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!} \zeta(2j+1, b+1)$$
$$+ \lim_{n\to\infty} \frac{(-1)^k}{2(2k)!} \left(\frac{2\pi}{m} \right)^{2k+1} \left(\frac{m}{\pi} \sum_{j=1}^{n} \frac{1}{j+b} + \int_0^1 (1-u)^{2k} \left(\cos \frac{2\pi(n+b)u}{m} - \cos \frac{2\pi b u}{m} \right) \cot \frac{\pi u}{m} \, du \right)$$

Now, if we recall the approximation we found for $HP(n)$ in (3), $HP(n) \sim c + H(n)$ for large n (where c is the part that doesn't depend on n), we only need to solve the limit:

$$\lim_{n\to\infty} \left(\frac{m}{\pi}(c + H(n)) + \int_0^1 (1-u)^{2k} \left(\cos\frac{2\pi(n+b)u}{m} - \cos\frac{2\pi bu}{m} \right) \cot\frac{\pi u}{m}\, du \right)$$

We can expand the cosine in the last integral:

$$\int_0^1 (1-u)^{2k} \left(\cos\frac{2\pi bu}{m} \left(-1 + \cos\frac{2\pi nu}{m} \right) - \sin\frac{2\pi bu}{m} \sin\frac{2\pi nu}{m} \right) \cot\frac{\pi u}{m}\, du$$

But due to theorem 1, the below limit is 0 (we just need to expand the first sine):

$$\lim_{n\to\infty} \int_0^1 u^{2k} \sin\frac{2\pi b(1-u)}{m} \sin\frac{2\pi n(1-u)}{m} \cot\frac{\pi(1-u)}{m}\, du = 0$$

Now, by replacing $H(n)$ with its equation (2) and adding it up to what's left in the integral:

$$\int_0^1 (1-u)^{2k} \cos\frac{2\pi bu}{m} \left(-1 + \cos\frac{2\pi nu}{m} \right) \cot\frac{\pi u}{m} + m(1-u)\,(1 - \cos 2\pi nu) \cot \pi u\, du$$

Looking at theorem 2, we can recombine the terms conveniently into an integral that converges as n goes to infinity:

$$\lim_{n\to\infty} \int_0^1 (1-u)^{2k} \cos\frac{2\pi bu}{m} \cos\frac{2\pi nu}{m} \cot\frac{\pi u}{m} - m(1-u) \cos 2\pi nu \cot \pi u\, du = \frac{m \log|m|}{\pi},$$

which is justified by the following (only one piece shown):

$$\cos\frac{2\pi b}{m} \int_0^1 u^{2k} \cos\frac{2\pi bu}{m} \cos\frac{2\pi n(1-u)}{m} \cot\frac{\pi(1-u)}{m} - m \cos\frac{2\pi b}{m}\, u \cos 2\pi n(1-u) \cot \pi(1-u)\, du$$
$$\to \left(\cos\frac{2\pi b}{m} \right)^2 \frac{m \log|m|}{\pi},$$

whereas the remaining integral converges on its own.

Let's summarize the result. The below limit doesn't change if we pick different formulae for $H(n)$, which is useful to figure out how the formula changes for the half-integers b:

$$\begin{aligned}\lim_{n\to\infty} &\left(\frac{m}{\pi} H(n) + \int_0^1 (1-u)^{2k} \left(\cos\frac{2\pi(n+b)u}{m} - \cos\frac{2\pi bu}{m} \right) \cot\frac{\pi u}{m}\, du \right) \\ &= \frac{m \log|m|}{\pi} - \int_0^1 (1-u)^{2k} \cos\frac{2\pi bu}{m} \cot\frac{\pi u}{m} - m(1-u) \cot \pi u\, du \\ &= \frac{m}{\pi} \log\frac{|m|}{2} - \int_0^1 (1-u)^{2k} \cos\frac{2\pi bu}{m} \cot\frac{\pi u}{m} - \frac{m}{2}(1-u) \cot\frac{\pi u}{2}\, du\end{aligned}$$

5.1.1 Non-integer $2b$

After we put everything together, the conclusion is that for all integer $k \geq 0$ and real $|m| \geq 1$:

$$\sum_{j=1}^{\infty} \frac{\cos \frac{2\pi(j+b)}{m}}{(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} \left(\cos \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j)!} \left(\frac{2\pi b}{m}\right)^{2j} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k-2j)!} \left(\frac{2\pi}{m}\right)^{2k-2j} \zeta(2j+1, b+1)$$
$$+ \frac{(-1)^k \pi}{2(2k)!} \left(\frac{2\pi}{m}\right)^{2k} (\cot 2\pi b + \csc 2\pi b) + \frac{(-1)^k}{(2k)!} \left(\frac{2\pi}{m}\right)^{2k} \log|m|$$
$$- \frac{(-1)^k}{2(2k)!} \left(\frac{2\pi}{m}\right)^{2k+1} \int_0^1 (1-u)^{2k} \cos \frac{2\pi bu}{m} \cot \frac{\pi u}{m} + m \left(-1 + \frac{\sin 2\pi bu}{\sin 2\pi b}\right) \cot \pi u \, du$$

As we can see, it takes a really convoluted function to generate this simple Fourier series.

5.1.2 Half-integer b

At the half-integers b, the formula reduces to:

$$\sum_{j=1}^{\infty} \frac{\cos \frac{2\pi(j+b)}{m}}{(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} \left(\cos \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j)!} \left(\frac{2\pi b}{m}\right)^{2j} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k-2j)!} \left(\frac{2\pi}{m}\right)^{2k-2j} \zeta(2j+1, b+1)$$
$$+ \frac{(-1)^k}{(2k)!} \left(\frac{2\pi}{m}\right)^{2k} \log \frac{|m|}{2} - \frac{(-1)^k}{2(2k)!} \left(\frac{2\pi}{m}\right)^{2k+1} \int_0^1 (1-u)^{2k} \cos \frac{2\pi bu}{m} \cot \frac{\pi u}{m} - \frac{m}{2} \cos \pi bu \cot \frac{\pi u}{2} \, du$$

5.1.3 Integer b

For integer b:

$$\sum_{j=1}^{\infty} \frac{\cos \frac{2\pi(j+b)}{m}}{(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} \left(\cos \frac{2\pi b}{m} - \sum_{j=0}^{k} \frac{(-1)^j}{(2j)!} \left(\frac{2\pi b}{m}\right)^{2j} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k-2j)!} \left(\frac{2\pi}{m}\right)^{2k-2j} \zeta(2j+1, b+1)$$
$$- \frac{(-1)^k}{(2k)!} \left(\frac{2\pi}{m}\right)^{2k} H(b) + \frac{(-1)^k}{(2k)!} \left(\frac{2\pi}{m}\right)^{2k} \log|m|$$
$$- \frac{(-1)^k}{2(2k)!} \left(\frac{2\pi}{m}\right)^{2k+1} \int_0^1 (1-u)^{2k} \cos \frac{2\pi bu}{m} \cot \frac{\pi u}{m} - m(1-u) \cot \pi u \, du$$

5.2 The limit of $S_{2k+1}^m(b, n)$

In the case of $S_{2k+1}^m(b, n)$, regardless of integer or half-integer we have:

$$\lim_{n\to\infty} S_{2k+1}^m(b,n) = -\frac{1}{2b^{2k+1}} \left(\sin \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \left(\frac{2\pi b}{m}\right)^{2j+1} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k+1-2j)!} \left(\frac{2\pi}{m}\right)^{2k+1-2j} \zeta(2j, b+1)$$
$$+ \lim_{n\to\infty} \frac{(-1)^k}{2(2k)!} \left(\frac{2\pi}{m}\right)^{2k+1} \int_0^1 (1-u)^{2k} \left(\sin \frac{2\pi(n+b)u}{m} - \sin \frac{2\pi bu}{m} \right) \cot \frac{\pi u}{m} \, du$$

This one is much simpler and we can easily deduce the limit of the integral by means of the theorem 1, without even having to expand the sine in the integrand. Thus, for all integer $k \geq 0$ and real $|m| \geq 1$:

$$\sum_{j=1}^{\infty} \frac{\sin \frac{2\pi(j+b)}{m}}{(j+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} \left(\sin \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \left(\frac{2\pi b}{m} \right)^{2j+1} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k+1-2j)!} \left(\frac{2\pi}{m} \right)^{2k+1-2j} \zeta(2j, b+1)$$
$$+ \frac{(-1)^k |m|}{4(2k)!} \left(\frac{2\pi}{m} \right)^{2k+1} - \frac{(-1)^k}{2(2k)!} \left(\frac{2\pi}{m} \right)^{2k+1} \int_0^1 (1-u)^{2k} \sin \frac{2\pi b u}{m} \cot \frac{\pi u}{m} \, du$$

5.3 $C_{2k}^m(b)$ and $S_{2k}^m(b)$

The next two formulae, $C_{2k}^m(b)$ and $S_{2k}^m(b)$, are analogs and don't require further explanations.

$$\sum_{j=1}^{\infty} \frac{\cos \frac{2\pi(j+b)}{m}}{(j+b)^{2k}} = -\frac{1}{2b^{2k}} \left(\cos \frac{2\pi b}{m} - \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j)!} \left(\frac{2\pi b}{m} \right)^{2j} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(2k-2j)!} \left(\frac{2\pi}{m} \right)^{2k-2j} \zeta(2j, b+1)$$
$$+ \frac{(-1)^k |m|}{4(2k-1)!} \left(\frac{2\pi}{m} \right)^{2k} - \frac{(-1)^k}{2(2k-1)!} \left(\frac{2\pi}{m} \right)^{2k} \int_0^1 (1-u)^{2k-1} \sin \frac{2\pi b u}{m} \cot \frac{\pi u}{m} \, du$$

$$\sum_{j=1}^{\infty} \frac{\sin \frac{2\pi(j+b)}{m}}{(j+b)^{2k}} = -\frac{1}{2b^{2k}} \left(\sin \frac{2\pi b}{m} - \sum_{j=0}^{k-2} \frac{(-1)^j}{(2j+1)!} \left(\frac{2\pi b}{m} \right)^{2j+1} \right) - \sum_{j=1}^{k-1} \frac{(-1)^{k-j}}{(2k-1-2j)!} \left(\frac{2\pi}{m} \right)^{2k-1-2j} \zeta(2j+1, b+1)$$
$$- \frac{(-1)^k \pi}{2(2k-1)!} \left(\frac{2\pi}{m} \right)^{2k-1} (\cot 2\pi b + \csc 2\pi b) - \frac{(-1)^k \log |m|}{(2k-1)!} \left(\frac{2\pi}{m} \right)^{2k-1}$$
$$+ \frac{(-1)^k}{2(2k-1)!} \left(\frac{2\pi}{m} \right)^{2k} \int_0^1 (1-u)^{2k-1} \cos \frac{2\pi b u}{m} \cot \frac{\pi u}{m} + m \left(-1 + \frac{\sin 2\pi b u}{\sin 2\pi b} \right) \cot \pi u \, du$$

6 Lerch's Φ at the positive integers

In this section we find out the values of the Lerch transcendent function, $\Phi(e^m, k, b)$, at the positive integers k.

6.1 Partial Lerch's Φ sums, $E_k^m(b, n)$

It's straightforward to derive an expression for the partial sums of Lerch's Φ function using the formulae from (2.1) and (2.2). If $\boldsymbol{i}$ is the imaginary unit, we just make:

$$E_k^{2\pi \boldsymbol{i}/m}(b, n) = \sum_{j=1}^{n} \frac{e^{2\pi \boldsymbol{i}(j+b)/m}}{(j+b)^k} = C_k^m(b, n) + \boldsymbol{i} S_k^m(b, n)$$

Omitting the calculations and making a simple transformation ($m := 2\pi \boldsymbol{i}/m$) (to bring the variables into the domain of the real numbers, which are easier to understand), we can produce a single formula for both the odd and even powers:

$$\sum_{j=1}^{n} \frac{e^{m(j+b)}}{(j+b)^k} = -\frac{e^{mb}}{2b^k} + \frac{e^{m(n+b)}}{2(n+b)^k} + \frac{1}{2b^k}\sum_{j=0}^{k} \frac{(mb)^j}{j!} - \frac{1}{2(n+b)^k}\sum_{j=0}^{k} \frac{(m(n+b))^j}{j!}$$
$$+ \sum_{j=1}^{k} \frac{m^{k-j}}{(k-j)!} HP_j(n) + \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}\left(e^{m(n+b)u} - e^{mbu}\right)\coth\frac{mu}{2}\,du$$

From this new equation, it's easy to see that as n goes to infinity, the sum on the left-hand side converges only if $\Re(m) < 0$. However, we can obtain an analytic continuation for this sum, by removing the second term on the right-hand side, which explodes out to infinity if $\Re(m) > 0$. Perhaps not surprisingly, this analytic continuation coincides with the Lerch Φ function.

6.2 Partial polylogarithm sums, $E_k^m(0, n)$

When $b = 0$, we have one interesting particular case:

$$\sum_{j=1}^{n} \frac{e^{mj}}{j^k} = \frac{e^{mn}}{2n^k} - \frac{1}{2n^k}\sum_{j=0}^{k} \frac{(mn)^j}{j!} + \sum_{j=1}^{k} \frac{m^{k-j}}{(k-j)!} H_j(n)$$
$$+ \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}\left(e^{mnu} - 1\right)\coth\frac{mu}{2}\,du \quad (4)$$

To obtain this expression, we take the limit as b tends to 0:

$$\lim_{b\to 0} -\frac{e^{mb}}{2b^k} + \frac{1}{2b^k}\sum_{j=0}^{k} \frac{(mb)^j}{j!} = 0$$

6.3 Lerch's Φ

The limits we found in section (5) allow us to find the below infinite sum:

$$\sum_{j=1}^{\infty} \frac{e^{\boldsymbol{i}2\pi(j+b)/m}}{(j+b)^k} = \lim_{n\to\infty} C_k^m(b, n) + \boldsymbol{i}S_k^m(b, n)$$

6.3.1 Non-integer $2b$

After we carry out all the algebraic calculations, we find that for all integer $k \geq 1$ and all complex m (except m such that $\Re(m) >= 0$ and $|\Im(m)| > 2\pi$):

$$\sum_{j=1}^{\infty} \frac{e^{m(j+b)}}{(j+b)^k} = -\frac{1}{2b^k}\left(e^{mb} - \sum_{j=0}^{k-2}\frac{(mb)^j}{j!}\right) + \sum_{j=2}^{k}\frac{m^{k-j}}{(k-j)!}\zeta(j,b+1)$$
$$+ \frac{\pi\, m^k}{2(k-1)!}\cot\pi b - \frac{m^{k-1}}{(k-1)!}\log\left(-\frac{m}{2\pi}\right)$$
$$- \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1} e^{m\,b\,u}\coth\frac{mu}{2} + \frac{2\pi}{m}\left(-1 + \frac{\sin 2\pi bu}{\sin 2\pi b}\right)\cot\pi u\, du$$

The infinite sum on the left-hand side converges whenever $\Re(m) < 0$, whereas the expression on the right-hand size, $E_k^m(b)$, is well defined always, except when $2b$ is an integer. At $m = 0$, although improper, the expression has a limit.

This sum is related to the Lerch function by the below relation:

$$\Phi(e^m, k, b) = \frac{1}{b^k} + e^{-m\,b}\sum_{j=1}^{\infty}\frac{e^{m(j+b)}}{(j+b)^k}$$

6.3.2 Half-integer b

For half-integer b, the formula is only slightly different. For all integer $k \geq 1$ and all complex m (except m such that $\Re(m) >= 0$ and $|\Im(m)| > 2\pi$):

$$\sum_{j=1}^{\infty} \frac{e^{m(j+b)}}{(j+b)^k} = -\frac{1}{2b^k}\left(e^{mb} - \sum_{j=0}^{k-2}\frac{(mb)^j}{j!}\right) + \sum_{j=2}^{k}\frac{m^{k-j}}{(k-j)!}\zeta(j,b+1)$$
$$- \frac{m^{k-1}}{(k-1)!}\log\left(-\frac{m}{\pi}\right) - \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1} e^{m\,b\,u}\coth\frac{mu}{2} - \frac{\pi}{m}\cos\pi bu\cot\frac{\pi u}{2}\, du$$

6.3.3 Integer b

When b is a positive integer, $E_k^m(b)$ becomes an incomplete polylogarithm series, which we cover next. Therefore it's very simple to derive its formula, we just need to subtract the missing part from the full polylogarithm. A similar reasoning is used if b is a negative integer.

Nonetheless, the formula when b is a positive integer is:

$$\sum_{j=1}^{\infty} \frac{e^{m(j+b)}}{(j+b)^k} = -\frac{1}{2b^k}\left(e^{mb} - \sum_{j=0}^{k-2}\frac{(mb)^j}{j!}\right) + \sum_{j=2}^{k}\frac{m^{k-j}}{(k-j)!}\zeta(j,b+1)$$
$$- \frac{m^{k-1}}{(k-1)!}\log\left(-\frac{m}{2\pi}\right) - \frac{m^{k-1}}{(k-1)!}\left(H(b) - \frac{1}{2b}\right)$$
$$- \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}e^{m\,b\,u}\coth\frac{mu}{2} - \frac{2\pi}{m}(1-u)\cot\pi u\,du$$

6.4 The polylogarithm, $\mathrm{Li}_k(e^m)$

The limit of $E_k^m(0,n)$ when n tends to infinity is the limit of the expression we just found when b tends to 0, and it relies on the following two notable limits:

$$\lim_{b\to 0} -\frac{1}{2b^k}\left(e^{mb} - \sum_{j=0}^{k-2}\frac{(mb)^j}{j!}\right) + \frac{\pi\, m^k}{2(k-1)!}\cot\pi b = -\frac{m^k}{2\,k!}, \text{ and } \lim_{b\to 0}\frac{\sin 2\pi b u}{\sin 2\pi b} = u$$

Therefore, for all integer $k \geq 1$ and all complex m (except m such that $\Re(m) >= 0$ and $|\Im(m)| > 2\pi$):

$$\sum_{j=1}^{\infty}\frac{e^{mj}}{j^k} = -\frac{m^{k-1}}{(k-1)!}\log\left(-\frac{m}{2\pi}\right) + \sum_{\substack{j=0\\ j\neq 1}}^{k}\frac{m^{k-j}}{(k-j)!}\zeta(j)$$
$$- \frac{m^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}\coth\frac{mu}{2} - \frac{2\pi}{m}(1-u)\cot\pi u\,du$$

This infinite sum is known as the polylogarithm, $\mathrm{Li}_k(e^m)$, and the formula on the right-hand side provides an analytic continuation for it for when $\Re(m) > 0$.

Note how the first limit fit perfectly into the second sum (together with the other $\zeta(j)$ values, except for the singularity). And it's easy to show that when m goes to 0, the formula we found goes to $\zeta(k)$, if $k \geq 2$.

The formula we found for $\mathrm{Li}_k(e^m)$ allows us to deduce the following power series for e^m:

$$\lim_{k\to\infty}\sum_{j=2}^{k}\frac{m^{k-j}}{(k-j)!}\zeta(j) = e^m$$

6.5 The Hurwitz zeta function, $\zeta(-k,b)$

The literature teaches us that the Hurwitz zeta function is related to the polylogarithm function by means of a relatively simple relation:

$$\frac{(2\pi)^k}{(k-1)!}\zeta(1-k,b) = \boldsymbol{i}^{-k}\,\mathrm{Li}_k(e^{2\pi\boldsymbol{i}\,b}) + \boldsymbol{i}^k\,\mathrm{Li}_k(e^{-2\pi\boldsymbol{i}\,b}),$$

which holds roughly speaking for $|\Re(b)| \leq 1$.

Since the formula we have for $\mathrm{Li}_k(e^m)$ holds at the positive integers k, this relation allows us to obtain a formula for $\zeta(-k,b)$ that holds at the negative integers $-k$. Without showing the simple but long calculations involved, we conclude that despite the constraints of the initial relation, the below formula holds for every b:

$$\zeta(-k,b) = \frac{b^k}{2} + 2\,k!\,b^{k+1} \sum_{j=0}^{\lfloor (k+1)/2 \rfloor} \frac{(-1)^j (2\pi\, b)^{-2j} \zeta(2j)}{(k+1-2j)!} = -\frac{B_{k+1}(b)}{k+1},$$

where $B_{k+1}(b)$ are the Bernoulli polynomials, whose relation with the Hurwitz zeta is also known from literature. So we conclude we found an alternative expression for said Bernoulli polynomials.

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