Proof of Riemann hypothesis

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We make $(N+1)/2$ infinite series from one equation that gives $\zeta(s)$ analytic continuation and 2 formulas ($1/2+a+bi$, $1/2-a-bi$) that show non-trivial zero point of $\zeta(s)$. ($N = 1, 3, 5, 7, \cdots\cdots$) 2. We find that a cannot have any value but zero from the above infinite series by performing $N \to \infty$. 3. Therefore non-trivial zero point of $\zeta(s)$ must be $1/2 \pm bi$.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $Re(s) > 0$. "$+\cdots\cdots$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots\cdots = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$.

$$S_0 = 1/2 + a + bi \tag{2}$$

The range of a is $0 \le a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b > 14$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 (Conjugate complex number of S_0) $= 1/2 + a - bi$ is also non-trivial zero point of $\zeta(s)$. Therefore $b \ge 0$ is necessary and sufficient range for investigation.

1.2 The range of b of non-trivial zero points found until now is $b > 14$.

The following (3) also shows non-trivial zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a - bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

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We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. $(N+1)/2$ infinite series

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. And the value of (11) is always zero at any value of x.

$$\begin{aligned} 0 \equiv &\cos x\{\text{right side of } (9)\} + \sin x\{\text{right side of } (10)\} \\ =&\cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - \cdots\cdots\} \\ &+ \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - \cdots\cdots\} \\ =&f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\ &- f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots \end{aligned} \tag{11}$$

We have the following (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned} 0 =&f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\ &- f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots \end{aligned} \tag{12-1}$$

We have the following (12-3) by substituting $b\log 3$ for x in (11).

$$\begin{aligned} 0 =&f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3) \\ &- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \end{aligned} \tag{12-3}$$

We have the following (12-5) by substituting $b\log 5$ for x in (11).

$$0 =f(2)\cos(b\log 2 - b\log 5) - f(3)\cos(b\log 3 - b\log 5) + f(4)\cos(b\log 4 - b\log 5)$$

$$- f(5)\cos(b\log 5 - b\log 5) + f(6)\cos(b\log 6 - b\log 5) - \cdots\cdots \quad (12\text{-}5)$$

In the same way as above we can have the following (12-N) by substituting $b\log N$ for x in (11). $(N = 7, 9, 11, 13, \cdots\cdots)$

$$\begin{aligned} 0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N) \\ &- f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \end{aligned} \quad (12\text{-N})$$

3. Verification of $F(a) = 0$

We define $g(k, N)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots \quad N = 1, 3, 5, 7, \cdots\cdots)$

$$\begin{aligned} g(k,N) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 3) + \cos(b\log k - b\log 5) + \cdots + \cos(b\log k - b\log N) \\ &= \cos(b\log 1 - b\log k) + \cos(b\log 3 - b\log k) + \cos(b\log 5 - b\log k) + \cdots + \cos(b\log N - b\log k) \\ &= \cos(b\log 1/k) + \cos(b\log 3/k) + \cos(b\log 5/k) + \cdots + \cos(b\log N/k) \end{aligned} \quad (13)$$

We can have the following (14) from the equations of (12-1), (12-3), (12-5), $\cdots\cdots$, (12-N) with the method shown in item 1.4 of [Appendix 1].

$$\begin{aligned} 0 = & f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 3) + \cos(b\log 2 - b\log 5) + \cdots + \cos(b\log 2 - b\log N)\} \\ & - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 3) + \cos(b\log 3 - b\log 5) + \cdots + \cos(b\log 3 - b\log N)\} \\ & + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 3) + \cos(b\log 4 - b\log 5) + \cdots + \cos(b\log 4 - b\log N)\} \\ & - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 3) + \cos(b\log 5 - b\log 5) + \cdots + \cos(b\log 5 - b\log N)\} \\ & + \cdots\cdots \\ = & f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + f(6)g(6,N) - \cdots\cdots \end{aligned} \quad (14)$$

Here we define $F(a)$ as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \quad (15)$$

We can have the following (16) by dividing the above (14) by $g(2, N)$. Because $g(2, N) \neq 0$ is true in $(N_1 < N \;\; N : \text{odd number})$ as shown in [Appendix 2 : Proof of $g(2, N) \neq 0$]. N_1 is the odd number that holds (41) in item 2.2.5 of [Appendix 2].

$$0 = f(2) - \frac{f(3)g(3,N)}{g(2,N)} + \frac{f(4)g(4,N)}{g(2,N)} - \frac{f(5)g(5,N)}{g(2,N)} + \cdots\cdots \quad (N_1 < N \;\; N : \text{odd number}) \quad (16)$$

We can have the following (17) from the above (16) by performing $N \to \infty$. Because $\lim_{N\to\infty} \frac{g(k,N)}{g(2,N)} = 1 \quad (k = 3, 4, 5, 6, \cdots\cdots)$ is true as shown in [Appendix 4 : Proof of

$\lim_{N\to\infty} \frac{g(k,N)}{g(2,N)} = 1$].

$$0 = \lim_{N\to\infty} \{f(2) - \frac{f(3)g(3,N)}{g(2,N)} + \frac{f(4)g(4,N)}{g(2,N)} - \frac{f(5)g(5,N)}{g(2,N)} + \cdots\cdots\}$$
$$= f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots = F(a)$$
$$(N_1 < N \quad N : \text{odd number}) \tag{17}$$

4. Conclusion

$F(a) = 0$ has the only solution of $a = 0$ as shown in [Appendix 5 : Solution for $F(a) = 0$]. a has the range of $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. However, a cannot have any value but zero because a is the solution for $F(a) = 0$. Due to $a = 0$ non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by (2) and (3) must be $1/2 \pm bi$. Therefore Riemann hypothesis which says " All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $Re(s) = 1/2$. " is true.

Appendix 1. : Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

On condition that the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) are true.

$$\text{(Series 1)} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A$$
$$\text{(Series 2)} = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B$$
$$\text{(Series 3)} = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B$$
$$\text{(Series 4)} = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B$$

1.1. Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 1 \qquad (6)$$
$$\text{(Series 2)} = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 1 \qquad (4)$$
$$\text{(Series 4)} = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots = 1 - 1 = 0 \qquad (9)$$

Here $f(n)$ is defined as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \qquad (8)$$

1.2. Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 0 \qquad (7)$$
$$\text{(Series 2)} = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 0 \qquad (5)$$
$$\text{(Series 4)} = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots = 0 - 0 \qquad (10)$$

1.3. Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

$$\begin{aligned}\text{(Series 1)} &= \cos x\{\text{right side of (9)}\} \\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) \\ &\quad - f(5)\cos(b\log 5) + \cdots\cdots\} \equiv 0\end{aligned}$$

$$
\begin{aligned}
\text{(Series 2)} &= \sin x\{\text{right side of (10)}\}\\
&= \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4)\\
&\quad - f(5)\sin(b\log 5) + \cdots\cdots\} \equiv 0\\
\text{(Series 3)} &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x)\\
&\quad - f(5)\cos(b\log 5 - x) + \cdots\cdots \equiv 0 + 0 \qquad (11)
\end{aligned}
$$

1.4. Construction of (14)

1.4.1 We can have the following (12-1*3) as (Series 3) by regarding (12-1) and (12-3) as (Series 1) and (Series 2) respectively.

$$
\begin{aligned}
\text{(Series 1)} &= f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\
&\quad + f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\
&\quad + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0 \qquad \text{(12-1)}\\
\text{(Series 2)} &= f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3)\\
&\quad + f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3)\\
&\quad + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \qquad \text{(12-3)}\\
\text{(Series 3)} &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 3)\}\\
&\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 3)\}\\
&\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 3)\}\\
&\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 3)\}\\
&\quad + \cdots\cdots = 0 + 0 \qquad \text{(12-1*3)}
\end{aligned}
$$

1.4.2 We can have the following (12-1*5) as (Series 3) by regarding (12-1*3) and (12-5) as (Series 1) and (Series 2) respectively.

$$
\begin{aligned}
\text{(Series 2)} &= f(2)\cos(b\log 2 - b\log 5) - f(3)\cos(b\log 3 - b\log 5)\\
&\quad + f(4)\cos(b\log 4 - b\log 5) - f(5)\cos(b\log 5 - b\log 5)\\
&\quad + f(6)\cos(b\log 6 - b\log 5) - \cdots\cdots = 0 \qquad \text{(12-5)}
\end{aligned}
$$

$$
\begin{aligned}
&\text{(Series 3)}\\
&= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 3) + \cos(b\log 2 - b\log 5)\}\\
&\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 3) + \cos(b\log 3 - b\log 5)\}\\
&\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 3) + \cos(b\log 4 - b\log 5)\}\\
&\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 3) + \cos(b\log 5 - b\log 5)\}\\
&\quad + \cdots\cdots = 0 + 0 \qquad \text{(12-1*5)}
\end{aligned}
$$

1.4.3 We can have the following (12-1*7) as (Series 3) by regarding (12-1*5) and (12-7) as (Series 1) and (Series 2) respectively.

$$
\begin{aligned}
\text{(Series 2)} &= f(2)\cos(b\log 2 - b\log 7) - f(3)\cos(b\log 3 - b\log 7)\\
&\quad + f(4)\cos(b\log 4 - b\log 7) - f(5)\cos(b\log 5 - b\log 7)\\
&\quad + f(6)\cos(b\log 6 - b\log 7) - \cdots\cdots = 0 \qquad \text{(12-7)}
\end{aligned}
$$

(Series 3)

$$= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 3) + \cos(b\log 2 - b\log 5) + \cos(b\log 2 - b\log 7)\}$$
$$-f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 3) + \cos(b\log 3 - b\log 5) + \cos(b\log 3 - b\log 7)\}$$
$$+f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 3) + \cos(b\log 4 - b\log 5) + \cos(b\log 4 - b\log 7)\}$$
$$-f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 3) + \cos(b\log 5 - b\log 5) + \cos(b\log 5 - b\log 7)\}$$
$$+\cdots\cdots = 0 + 0 \qquad \text{(12-1*7)}$$

1.4.4 In the same way as above we can have the following (12-1*N)=(14) as (Series 3) by regarding (12-1*N-2) and (12-N) as (Series 1) and (Series 2) respectively. $(N = 9, 11, 13, 15, \cdots\cdots)$ $g(k, N)$ is defined in page 3. $(k = 2, 3, 4, 5, \cdots\cdots)$

$$f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 3) + \cos(b\log 2 - b\log 5) + \cdots + \cos(b\log 2 - b\log N)\}$$
$$-f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 3) + \cos(b\log 3 - b\log 5) + \cdots + \cos(b\log 3 - b\log N)\}$$
$$+f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 3) + \cos(b\log 4 - b\log 5) + \cdots + \cos(b\log 4 - b\log N)\}$$
$$-f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 3) + \cos(b\log 5 - b\log 5) + \cdots + \cos(b\log 5 - b\log N)\}$$
$$+\cdots\cdots$$

$$= f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + f(6)g(6,N) - \cdots\cdots$$
$$= 0 + 0 \qquad \text{(12-1*N)}$$

Appendix 2. : Proof of $g(2, N) \neq 0$

2.1. Investigation of $g(k, N)$

2.1.1 We define G and H as follows. $(N = 1, 3, 5, 7, \cdots\cdots)$

$$G = \lim_{N\to\infty} \frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{3}{N}) + \cos(b\log\frac{5}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$= \frac{1}{2}\int_0^1 \cos(b\log x)dx \qquad (20\text{-}1)$$
$$H = \lim_{N\to\infty} \frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{3}{N}) + \sin(b\log\frac{5}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$
$$= \frac{1}{2}\int_0^1 \sin(b\log x)dx \qquad (20\text{-}2)$$

We calculate G and H by Integration by parts.

$$2G = [x\cos(b\log x)]_0^1 + 2bH = 1 + 2bH$$
$$2H = [x\sin(b\log x)]_0^1 - 2bG = -2bG$$

Then we can have the values of G and H from the above equations as follows.

$$G = \frac{1}{2(1+b^2)} \qquad H = \frac{-b}{2(1+b^2)} \qquad (21)$$

2.1.2 We define as follows.

$$\frac{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{3}{N}) + \cos(b\log\frac{5}{N}) + \cdots + \cos(b\log\frac{N}{N})}{N} - G = E_c(N) \qquad (22\text{-}1)$$

$$\frac{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{3}{N}) + \sin(b\log\frac{5}{N}) + \cdots + \sin(b\log\frac{N}{N})}{N} - H = E_s(N) \qquad (22\text{-}2)$$

From (20-1), (20-2), (22-1) and (22-2) we have the following (23).

$$\lim_{N\to\infty} E_c(N) = 0 \qquad \lim_{N\to\infty} E_s(N) = 0 \qquad (23)$$

2.1.3 From (13) we can calculate $g(k, N)$ as follows. $(N = 1, 3, 5, 7, \cdots\cdots)$

$$g(k, N) = \cos(b\log 1/k) + \cos(b\log 3/k) + \cos(b\log 5/k) + \cdots + \cos(b\log N/k)$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{3}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{5}{N}\tfrac{N}{k}) + \cdots + \cos(b\log\tfrac{N}{N}\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N} + b\log\tfrac{N}{k}) + \cos(b\log\tfrac{3}{N} + b\log\tfrac{N}{k})$$
$$+ \cos(b\log\tfrac{5}{N} + b\log\tfrac{N}{k}) + \cdots\cdots + \cos(b\log\tfrac{N}{N} + b\log\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\cos(b\log\tfrac{N}{k})\{\cos(b\log\tfrac{1}{N}) + \cos(b\log\tfrac{3}{N}) + \cos(b\log\tfrac{5}{N}) + \cdots + \cos(b\log\tfrac{N}{N})\}$$
$$-N\tfrac{1}{N}\sin(b\log\tfrac{N}{k})\{\sin(b\log\tfrac{1}{N}) + \sin(b\log\tfrac{3}{N}) + \sin(b\log\tfrac{5}{N}) + \cdots + \sin(b\log\tfrac{N}{N})\}$$

$$= N\cos(b\log\tfrac{N}{k})G$$
$$+N\cos(b\log\tfrac{N}{k})\{\tfrac{\cos(b\log 1/N)+\cos(b\log 3/N)+\cos(b\log 5/N)+\cdots+\cos(b\log N/N)}{N}-G\}$$
$$-N\sin(b\log\tfrac{N}{k})H$$
$$-N\sin(b\log\tfrac{N}{k})\{\tfrac{\sin(b\log 1/N)+\sin(b\log 3/N)+\sin(b\log 5/N)+\cdots+\sin(b\log N/N)}{N}-H\} \tag{24-1}$$

$$= N\cos(b\log\frac{N}{k})G + N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})H$$
$$- N\sin(b\log\frac{N}{k})E_s(N) \tag{24-2}$$
$$= N\cos(b\log\frac{N}{k})\frac{1}{2(1+b^2)} + N\cos(b\log\frac{N}{k})E_c(N)$$
$$+ N\sin(b\log\frac{N}{k})\frac{b}{2(1+b^2)} - N\sin(b\log\frac{N}{k})E_s(N) \tag{24-3}$$
$$= \frac{N}{2\sqrt{1+b^2}}\{\cos(b\log\frac{N}{k})\frac{1}{\sqrt{1+b^2}} + \sin(b\log\frac{N}{k})\frac{b}{\sqrt{1+b^2}}\}$$
$$+ N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})E_s(N) \tag{24-4}$$
$$= \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{2\sqrt{1+b^2}}$$
$$+ N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})E_s(N) \tag{24-5}$$

2.1.4 From (22-1), (22-2) and (24-1) we have (24-2). From (21) and (24-2) we have (24-3).

2.2. Verification of $R_3 \neq 0$

We investigate the the condition of $R_3 = 0$ in the following 4 cases.

2.2.1 $\{E_c(N) \geq 0,\ E_s(N) \geq 0\}$ i.e. $\{E_c(N) = |E_c(N)|\,,\ E_s(N) = |E_s(N)|\}$

2.2.1.1 We have the followsing (25-1), (25-2), (25-3) and (25-4) from (24-5).

$$(24\text{-}5) = \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{2\sqrt{1+b^2}}$$
$$+ N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})E_s(N)$$
$$= \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{2\sqrt{1+b^2}}$$
$$+ N\cos(b\log\frac{N}{k})\,|E_c(N)| - N\sin(b\log\frac{N}{k})\,|E_s(N)|$$
$$= \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{2\sqrt{1+b^2}}$$
$$-N\sqrt{E_c(N)^2+E_s(N)^2}\{\sin(b\log\tfrac{N}{k})\tfrac{|E_s(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}} - \cos(b\log\tfrac{N}{k})\tfrac{|E_c(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}\}$$

(25-1)

$$= \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{2\sqrt{1+b^2}}$$

$$- N\sqrt{E_c(N)^2 + E_s(N)^2}\sin(b\log\frac{N}{k} - \tan^{-1}\left|\frac{E_c(N)}{E_s(N)}\right|) \quad (25\text{-}2)$$

$$= NR_1\sin(b\log N/k + \theta_1) - NR_2\sin(b\log N/k - \theta_2) \quad (25\text{-}3)$$

$$= NR_3\sin(b\log N/k + \theta_3) \quad (25\text{-}4)$$

We define as follows to have the above (25-3) from (25-2).

$$R_1 = \frac{1}{2\sqrt{1+b^2}} > 0 \quad (26\text{-}1)$$

$$\theta_1 = \tan^{-1}1/b \quad (26\text{-}2)$$

$$R_2 = \sqrt{E_c(N)^2 + E_s(N)^2} \geq 0 \quad (26\text{-}3)$$

$$\theta_2 = \tan^{-1}\left|\frac{E_c(N)}{E_s(N)}\right| \quad (26\text{-}4)$$

From (24-4) we have $\cos\theta_a = b/\sqrt{1+b^2} > 0$ and $\sin\theta_a = 1/\sqrt{1+b^2} > 0$. And from the above 2 equations we have the following (26-5). The range of θ_1 is given from $14 < b$ shown in page 1.

$$\theta_a = \tan^{-1}1/b = \theta_1 + 2n\pi \qquad (n = 0, \pm1, \pm2, \pm3, \cdots) \quad (26\text{-}5)$$

$$0 < \theta_1 < 0.023\pi = \tan^{-1}1/14$$

Even if we define the above (26-2), there is no contradiction in the above (25-3) because of the following (26-6).

$$\sin(b\log N/k + \tan^{-1}1/b) = \sin(b\log N/k + \theta_1 + 2n\pi)$$

$$= \sin(b\log N/k + \theta_1) \quad (26\text{-}6)$$

Similarly we have $\cos\theta_b = \frac{|E_s(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}} \geq 0$ and $\sin\theta_b = \frac{|E_c(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}} \geq 0$ from (25-1). And we have the following (26-7). The range of θ_2 is given from $0 \leq \left|\frac{E_c(N)}{E_s(N)}\right|$.

$$\theta_b = \tan^{-1}\left|\frac{E_c(N)}{E_s(N)}\right| = \theta_2 + 2n\pi \qquad (n = 0, \pm1, \pm2, \pm3, \cdots) \quad (26\text{-}7)$$

$$0 \leq \theta_2 < \pi/2$$

We can define (26-4) from the above (26-7).

2.2.1.2 If in the complex number $R_x\exp(\theta_x i)$, $R_y\exp(\theta_y i)$ and $R_z\exp(\theta_z i)$ the following (27-1) holds, the following (27-2) also holds.

$$R_x\exp(\theta_x i) \pm R_y\exp(\theta_y i) = R_z\exp(\theta_z i) \quad (27\text{-}1)$$

$$R_x\sin\theta_x \pm R_y\sin\theta_y = R_z\sin\theta_z \quad (27\text{-}2)$$

So we can calculate the following (28-1) and (28-2) from the following (Figure 1). R_3 can be calculated by Cosine theorem. We have the above (25-4) from (25-3), (28-1) and (28-2).

$$R_3 = \sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos(\theta_1 + \theta_2)} \tag{28-1}$$

$$\theta_3 = \tan^{-1}\frac{R_1\sin\theta_1 + R_2\sin\theta_2}{R_1\cos\theta_1 - R_2\cos\theta_2} \tag{28-2}$$

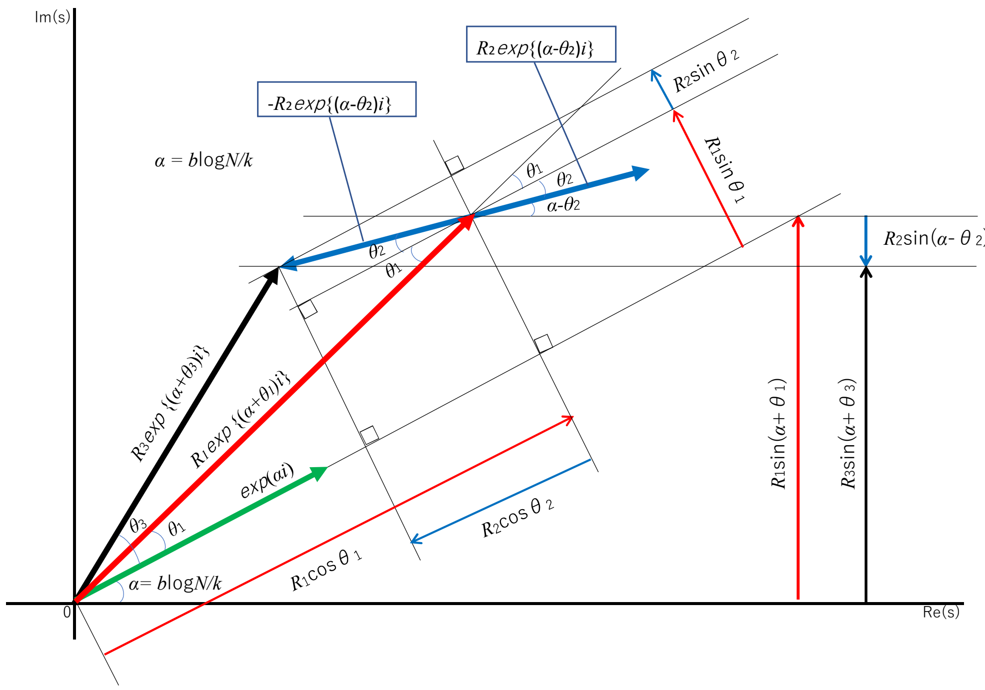


Figure 1 : $R_3\sin(b\log N/k + \theta_3)$ in $\{E_c(N) \geq 0,\ E_s(N) \geq 0\}$

2.2.1.3 From the above (28-1) we can confirm that $1 \geq \cos(\theta_1 + \theta_2) > 0$ must be true in order for $R_3 = 0$ to hold. Due to (Arithmetic mean)≥ (Geometric mean) we have the following (29).

$$R_1^2 + R_2^2 \geq 2R_1R_2 \geq 2R_1R_2\cos(\theta_1 + \theta_2) \tag{29}$$

In order for $R_3 = 0$ to hold the 2 equal signs in the above (29) must hold. Therefore the following (30-1) and (30-2) are the condition of $R_3 = 0$.

$$R_1 = R_2 \tag{30-1}$$

$$\theta_1 + \theta_2 = 0 \tag{30-2}$$

2.2.2 $\{E_c(N) \geq 0,\ E_s(N) \leq 0\}$ i.e. $\{E_c(N) = |E_c(N)|\,,\ E_s(N) = -|E_s(N)|\}$

2.2.2.1 We have the followsing (31-1), (31-2) and (31-3) from (24-5).

$$
\begin{aligned}
(24\text{-}5) = & \frac{N\sin(b\log N/k+\tan^{-1}1/b)}{2\sqrt{1+b^2}} \\
& + N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})E_s(N) \\
= & \frac{N\sin(b\log N/k+\tan^{-1}1/b)}{2\sqrt{1+b^2}} \\
& + N\cos(b\log\frac{N}{k})\,|E_c(N)| + N\sin(b\log\frac{N}{k})\,|E_s(N)| \\
= & \frac{N\sin(b\log N/k+\tan^{-1}1/b)}{2\sqrt{1+b^2}} \\
& + N\sqrt{E_c(N)^2+E_s(N)^2}\sin(b\log\frac{N}{k}+\tan^{-1}\left|\frac{E_c(N)}{E_s(N)}\right|) \quad (31\text{-}1) \\
= & NR_1\sin(b\log N/k+\theta_1) + NR_2\sin(b\log N/k+\theta_2) \quad (31\text{-}2) \\
= & NR_3\sin(b\log N/k+\theta_3) \quad (31\text{-}3)
\end{aligned}
$$

R_1, θ_1, R_2 and θ_2 are defined in item 2.2.1.1.

2.2.2.2 We can calculate the following (32-1) and (32-2) from the following (FIgure 2). We have the above (31-3) from (31-2), (32-1) and (32-2).

$$R_3 = \sqrt{R_1^2+R_2^2-2R_1R_2\cos(\pi+\theta_1-\theta_2)} \quad (32\text{-}1)$$

$$\theta_3 = \tan^{-1}\frac{R_1\sin\theta_1+R_2\sin\theta_2}{R_1\cos\theta_1+R_2\cos\theta_2} \quad (32\text{-}2)$$

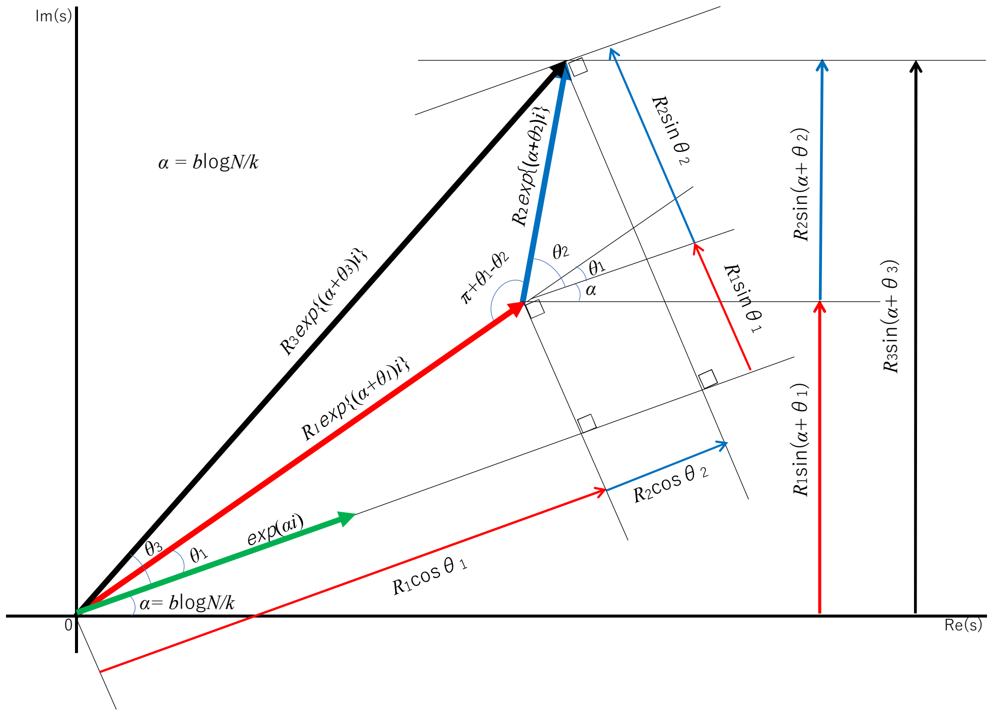


Figure 2：$R_3 \sin(b \log N/k + \theta_3)$ in $\{E_c(N) \geq 0,\ E_s(N) \leq 0\}$

2.2.2.3 Through the same discussion as in item 2.2.1.3 we can confirm the condition of $R_3 = 0$ as follows.

$$R_1 = R_2 \tag{34-1}$$

$$\theta_2 - \theta_1 = \pi \tag{34-2}$$

2.2.3 $\{E_c(N) \leq 0,\ E_s(N) \leq 0\}$ i.e. $\{E_c(N) = -|E_c(N)|,\ E_s(N) = -|E_s(N)|\}$

2.2.3.1 We have the followsing (35-1), (35-2) and (35-3) from (24-5).

$$\begin{aligned}
(24\text{-}5) &= \frac{N \sin(b \log N/k + \tan^{-1} 1/b)}{2\sqrt{1+b^2}} \\
&\quad + N \cos(b \log \frac{N}{k}) E_c(N) - N \sin(b \log \frac{N}{k}) E_s(N) \\
&= \frac{N \sin(b \log N/k + \tan^{-1} 1/b)}{2\sqrt{1+b^2}} \\
&\quad - N \cos(b \log \frac{N}{k}) |E_c(N)| + N \sin(b \log \frac{N}{k}) |E_s(N)| \\
&= \frac{N \sin(b \log N/k + \tan^{-1} 1/b)}{2\sqrt{1+b^2}} \\
&\quad + N\sqrt{E_c(N)^2 + E_s(N)^2} \sin(b \log \frac{N}{k} - \tan^{-1} \left| \frac{E_c(N)}{E_s(N)} \right|) && (35\text{-}1) \\
&= N R_1 \sin(b \log N/k + \theta_1) + N R_2 \sin(b \log N/k - \theta_2) && (35\text{-}2)
\end{aligned}$$

$$= NR_3 \sin(b \log N/k + \theta_3) \tag{35-3}$$

R_1, θ_1, R_2 and θ_2 are defined in item 2.2.1.1.

2.2.3.2 We can calculate the following (36-1) and (36-2) from the following (FIgure 3). We have the above (35-3) from (35-2), (36-1) and (36-2).

$$R_3 = \sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos(\pi - \theta_1 - \theta_2)} \tag{36-1}$$

$$\theta_3 = \tan^{-1}\frac{R_1\sin\theta_1 - R_2\sin\theta_2}{R_1\cos\theta_1 + R_2\cos\theta_2} \tag{36-2}$$

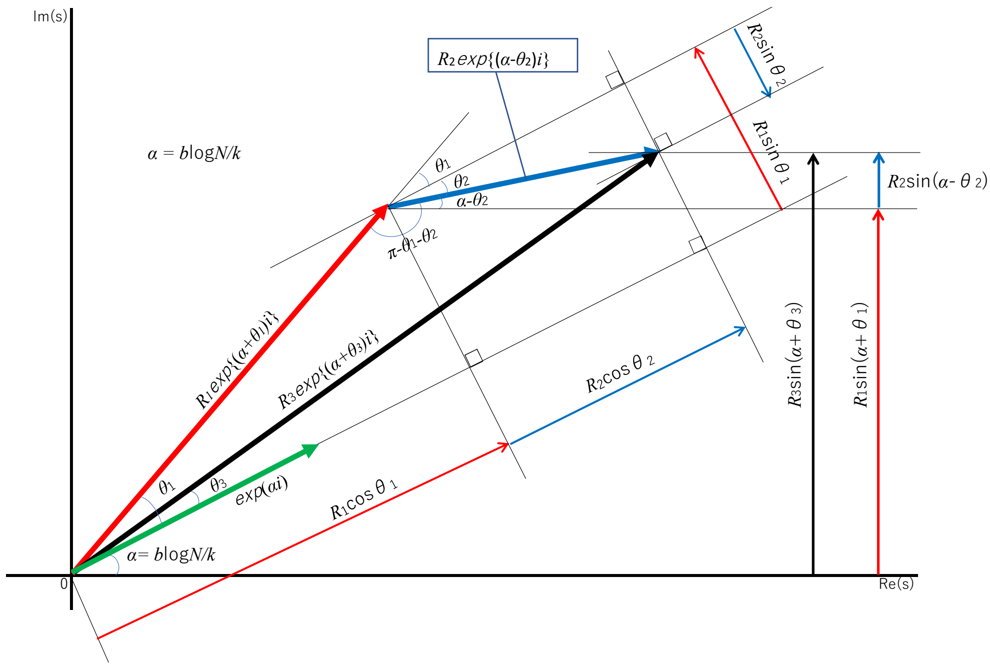


Figure 3 : $R_3 \sin(b \log N/k + \theta_3)$ in $\{E_c(N) \leq 0,\ E_s(N) \leq 0\}$

2.2.3.3 Through the same discussion as in item 2.2.1.3 we can confirm the condition of $R_3 = 0$ as follows.

$$R_1 = R_2 \tag{37-1}$$

$$\pi = \theta_1 + \theta_2 \tag{37-2}$$

2.2.4 $\{E_c(N) \leq 0,\ E_s(N) \geq 0\}$ i.e. $\{E_c(N) = -|E_c(N)|\,,\ E_s(N) = |E_s(N)|\}$

2.2.4.1 We have the followsing (38-1), (38-2) and (38-3) from (24-5).

$$(24\text{-}5) = \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{2\sqrt{1+b^2}}$$
$$+ N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})E_s(N)$$

$$= \frac{N\sin(b\log N/k + \tan^{-1} 1/b)}{2\sqrt{1+b^2}}$$
$$- N\cos(b\log\frac{N}{k})\,|E_c(N)| - N\sin(b\log\frac{N}{k})\,|E_s(N)|$$
$$= \frac{N\sin(b\log N/k + \tan^{-1} 1/b)}{2\sqrt{1+b^2}}$$
$$- N\sqrt{E_c(N)^2 + E_s(N)^2}\sin(b\log\frac{N}{k} + \tan^{-1}\left|\frac{E_c(N)}{E_s(N)}\right|) \tag{38-1}$$
$$= NR_1\sin(b\log N/k + \theta_1) - NR_2\sin(b\log N/k + \theta_2) \tag{38-2}$$
$$= NR_3\sin(b\log N/k + \theta_3) \tag{38-3}$$

R_1, θ_1, R_2 and θ_2 are defined in item 2.2.1.1.

2.2.4.2 We can calculate the following (36-1) and (36-2) from the following (FIgure 4). We have the above (38-3) from (38-2), (39-1) and (39-2).

$$R_3 = \sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos(\theta_2 - \theta_1)} \tag{39-1}$$
$$\theta_3 = \tan^{-1}\frac{R_1\sin\theta_1 - R_2\sin\theta_2}{R_1\cos\theta_1 - R_2\cos\theta_2} \tag{39-2}$$

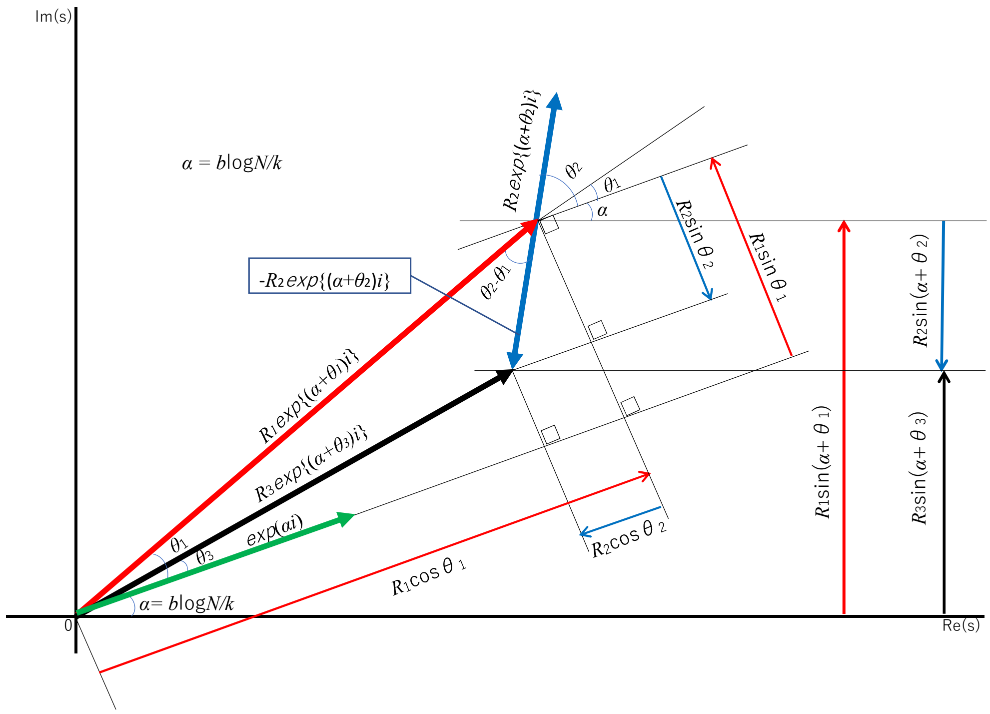


Figure 4 : $R_3\sin(b\log N/k + \theta_3)$ in $\{E_c(N) \leq 0,\ E_s(N) \geq 0\}$

2.2.4.3 Through the same discussion as in item 2.2.1.3 we can confirm the condition of $R_3 = 0$ as follows.

$$R_1 = R_2 \tag{40-1}$$
$$\theta_1 = \theta_2 \tag{40-2}$$

2.2.5 There is the odd number N_1 that holds the following (41) because $\lim_{N\to\infty} \sqrt{E_c(N)^2 + E_s(N)^2} = 0$ is true from (23) in item 2.1.2.

$$\frac{1}{2\sqrt{1+b^2}} = R_1 > R_2 = \sqrt{E_c(N)^2 + E_s(N)^2} \qquad (N_1 < N) \tag{41}$$

Therefore (30-1), (34-1), (37-1) and (40-1) do not hold in $(N_1 < N)$. Now we can confirm the following (42).

$$R_3 \neq 0 \qquad (N_1 < N) \tag{42}$$

2.3. Verification of $\sin(b\log N/2 + \theta_3) \neq 0$

2.3.1 If we assume the following (51) is true, the following (52) is also supposed to be true.

$$\sin(b\log N/2 + \theta_3) = 0 \qquad (N = 1, 3, 5, 7, \cdots\cdots) \tag{51}$$
$$b\log N/2 + \theta_3 = K\pi \qquad (K : \text{integer}) \tag{52}$$

2.3.2 We define as follows.

Type1 irrational number : Irrational number which consists of singular or plural irrational terms such as $2\sqrt{2}/e,\ \sqrt{2}/e + \sqrt{3}$, etc.

Type2 irrational number : Irrational number which has the formation of (rational number)+(type1 irrational number) such as $1+\sqrt{2},\ 2+2\sqrt{2}/e+\sqrt{3}$, etc.

2.3.3 The above (52) holds in the following cases.

Case 1 : The following (53-1), (53-2) and (53-3) holds.

$$b\log N/2 = A\pi \tag{53-1}$$
$$\theta_3 = B\pi \qquad (A,\ B : \text{rational number}) \tag{53-2}$$
$$A + B = K \qquad (K : \text{integer}) \tag{53-3}$$

Case 2 : The above (53-3), the following (53-4) and (53-5) holds.

$$b\log N/2 = (A+C)\pi \qquad (A+C : \text{type2 irrational number}) \tag{53-4}$$
$$\theta_3 = (B-C)\pi \qquad (C : \text{type1 irrational number}) \tag{53-5}$$

2.3.4 From $b\log N/2 = D\pi$ we have the following equation.

$$D = \frac{b\log N/2}{\pi}$$

The formation of D becomes (type1 irratioal number) regardless of the formation of b as follows.

Case 3 : b =(rational number)

$$D = (\text{rational number})\frac{\log N/2}{\pi} = (\text{type1 irratioal number} : Q_1)$$

$(N = 1, 3, 5, 7, \cdots\cdots)$

Case 4 : b =(type1 irrational number)

$$D = (\text{type1 irrational number} : Q_2)\frac{\log N/2}{\pi} = (\text{type1 irratioal number} : Q_3)$$

$(N = 1, 3, 5, 7, \cdots\cdots)$: When the following (conditoin 1) holds.
$(N = 1, 3, 5, 7, \cdots\cdots \quad N \neq N_2)$: When the following (condition 2) holds.

Condition 1 : b does not have the term of $\dfrac{A\pi}{\log N_2/2}$. Or b has the term of $\dfrac{A\pi}{\log N_2/2}$ and N_2 is an even number.
A : (rational number)

Condition 2 : b has the term of $\dfrac{A\pi}{\log N_2/2}$ and N_2 is an odd number.

Case 5 : b =(type2 irrational number)=(rational number)+(type1 irrational number)

$$\begin{aligned} D =& \{(\text{rational number}) + (\text{type1 irrational number} : Q_4)\}\frac{\log N/2}{\pi} \\ =& (\text{type1 irrational number} : Q_5) + (\text{type1 irratioal number} : Q_6) \\ =& (\text{type1 irrational number} : Q_7) \end{aligned}$$

$(N = 1, 3, 5, 7, \cdots\cdots)$: When (conditoin 1) holds.
$(N = 1, 3, 5, 7, \cdots\cdots \quad N \neq N_2)$: When (condition 2) holds.

2.3.5 As shown in the above item 2.3.4 D is not (rational number) or (type2 irrational number) but (type1 irrational number). Therefore (case 1) and (case 2) do not hold i.e. (52) does not hold in $(N = 1, 3, 5, 7, \cdots\cdots \quad N \neq N_2)$.

2.3.6 At $N = N_2$ (52) does not holds when (condition 2) holds as shown in [Appendix 3 : Proof of $b \log N_2/2 + \theta_3 \neq K\pi$].

2.3.7 Now we can confirm the following (54).

$$\sin(b \log N/2 + \theta_3) \neq 0 \qquad (N = 1, 3, 5, 7, \cdots\cdots) \tag{54}$$

2.4. Verification of $g(2, N) \neq 0$

We have the following (55) from (25-4) in item 2.2.1.1, (42) in item 2.2.5 and the above (54). We can confirm that $g(2, N)$ does not have the value of zero in $(N_1 < N \quad N$: odd number).

$$g(2, N) = NR_3 \sin(b \log N/2 + \theta_3) \neq 0 \qquad (N_1 < N \quad N : \text{odd number}) \tag{55}$$

Appendix 3. : Proof of $b\log N_2/2+\theta_3 \neq K\pi$

In this appendix we confim that the following (52) in item 2.3.1 does not hold at $N=N_2$ when (condition 2) holds.

$$b\log N/2+\theta_3 = K\pi \qquad (K: \text{integer}) \tag{52}$$

3.1 We confirm the value of θ_3 in the following 4 cases.

3.1.1 $\{E_c(N)\geq 0,\ E_s(N)\geq 0\}$ i.e. $\{E_c(N)=|E_c(N)|,\ E_s(N)=|E_s(N)|\}$

We have the following (61) from (21), (22-1) and (22-2) in item 2.1, (26-1), (26-3) and (28-2) in item 2.2 and the following (61-1) and (61-2).

$$\theta_3 = \tan^{-1}\frac{R_1\sin\theta_1+R_2\sin\theta_2}{R_1\cos\theta_1-R_2\cos\theta_2} \tag{28-2}$$

$$=\tan^{-1}\frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}}+\sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_c(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}}-\sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_s(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$=\tan^{-1}\frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}}+\sqrt{E_c(N)^2+E_s(N)^2}\frac{E_c(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}}-\sqrt{E_c(N)^2+E_s(N)^2}\frac{E_s(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$=\tan^{-1}\frac{G+E_c(N)}{-H-E_s(N)}$$

$$=\tan^{-1}\frac{\cos(b\log 1/N)+\cos(b\log 3/N)+\cos(b\log 5/N)+\cdots+\cos(b\log N/N)}{-\{\sin(b\log 1/N)+\sin(b\log 3/N)+\sin(b\log 5/N)+\cdots+\sin(b\log N/N)\}} \tag{61}$$

We have the following (61-1) and (62-2) from (26-2) and (26-4) in item 2.2.1.

$$\cos\theta_1 = b/\sqrt{1+b^2} \qquad \sin\theta_1 = 1/\sqrt{1+b^2} \tag{61-1}$$

$$\cos\theta_2 = \frac{|E_s(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}} \qquad \sin\theta_2 = \frac{|E_c(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}} \tag{61-2}$$

3.1.2 $\{E_c(N)\geq 0,\ E_s(N)\leq 0\}$ i.e. $\{E_c(N)=|E_c(N)|,\ E_s(N)=-|E_s(N)|\}$

Similarly we have the following (62) from (32-2) in item 2.2.2.2.

$$\theta_3 = \tan^{-1}\frac{R_1\sin\theta_1+R_2\sin\theta_2}{R_1\cos\theta_1+R_2\cos\theta_2} \tag{32-2}$$

$$=\tan^{-1}\frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}}+\sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_c(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}}+\sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_s(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$=\tan^{-1}\frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}}+\sqrt{E_c(N)^2+E_s(N)^2}\frac{E_c(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}}-\sqrt{E_c(N)^2+E_s(N)^2}\frac{E_s(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$=\tan^{-1}\frac{G+E_c(N)}{-H-E_s(N)}$$

$$= \tan^{-1} \frac{\cos(b\log 1/N)+\cos(b\log 3/N)+\cos(b\log 5/N)+\cdots+\cos(b\log N/N)}{-\{\sin(b\log 1/N)+\sin(b\log 3/N)+\sin(b\log 5/N)+\cdots+\sin(b\log N/N)\}} \tag{62}$$

3.1.3 $\{E_c(N) \le 0,\ E_s(N) \le 0\}$ i.e. $\{E_c(N) = -|E_c(N)|,\ E_s(N) = -|E_s(N)|\}$

Similarly we have the following (63) from (36-2) in item 2.2.3.2.

$$\theta_3 = \tan^{-1} \frac{R_1 \sin\theta_1 - R_2 \sin\theta_2}{R_1 \cos\theta_1 + R_2 \cos\theta_2} \tag{36-2}$$

$$= \tan^{-1} \frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}} - \sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_c(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}} + \sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_s(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$= \tan^{-1} \frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}} + \sqrt{E_c(N)^2+E_s(N)^2}\frac{E_c(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}} - \sqrt{E_c(N)^2+E_s(N)^2}\frac{E_s(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$= \tan^{-1} \frac{G + E_c(N)}{-H - E_s(N)}$$

$$= \tan^{-1} \frac{\cos(b\log 1/N)+\cos(b\log 3/N)+\cos(b\log 5/N)+\cdots+\cos(b\log N/N)}{-\{\sin(b\log 1/N)+\sin(b\log 3/N)+\sin(b\log 5/N)+\cdots+\sin(b\log N/N)\}} \tag{63}$$

3.1.4 $\{E_c(N) \le 0,\ E_s(N) \ge 0\}$ i.e. $\{E_c(N) = -|E_c(N)|,\ E_s(N) = |E_s(N)|\}$

Similarly we have the following (64) from (39-2) in item 2.2.4.2.

$$\theta_3 = \tan^{-1} \frac{R_1 \sin\theta_1 - R_2 \sin\theta_2}{R_1 \cos\theta_1 - R_2 \cos\theta_2} \tag{39-2}$$

$$= \tan^{-1} \frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}} - \sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_c(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}} - \sqrt{E_c(N)^2+E_s(N)^2}\frac{|E_s(N)|}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$= \tan^{-1} \frac{\frac{1}{2\sqrt{1+b^2}}\frac{1}{\sqrt{1+b^2}} + \sqrt{E_c(N)^2+E_s(N)^2}\frac{E_c(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}{\frac{1}{2\sqrt{1+b^2}}\frac{b}{\sqrt{1+b^2}} - \sqrt{E_c(N)^2+E_s(N)^2}\frac{E_s(N)}{\sqrt{E_c(N)^2+E_s(N)^2}}}$$

$$= \tan^{-1} \frac{G + E_c(N)}{-H - E_s(N)}$$

$$= \tan^{-1} \frac{\cos(b\log 1/N)+\cos(b\log 3/N)+\cos(b\log 5/N)+\cdots+\cos(b\log N/N)}{-\{\sin(b\log 1/N)+\sin(b\log 3/N)+\sin(b\log 5/N)+\cdots+\sin(b\log N/N)\}} \tag{64}$$

3.1.5 We have the following (65) from the above (61), (62), (63) and (64).

$$\theta_3 = \tan^{-1} \frac{\cos(b\log 1/N)+\cos(b\log 3/N)+\cos(b\log 5/N)+\cdots+\cos(b\log N/N)}{-\{\sin(b\log 1/N)+\sin(b\log 3/N)+\sin(b\log 5/N)+\cdots+\sin(b\log N/N)\}} \tag{65}$$

3.2 If we add 2 sine functions which have the common term β, the result becomes another sine function which has the common term β like the following (66). R_Z and θ_Z are culculated like the following (66-1) and (66-2) from the following (Figure 5).

$$R_X \sin(\beta - \theta_X) + R_Y \sin(\beta - \theta_Y) = R_Z \sin(\beta - \theta_Z) \tag{66}$$

$$R_Z = \sqrt{R_X^2 + R_Y^2 - 2R_X R_Y \cos(\pi + \theta_X - \theta_Y)} \tag{66-1}$$

$$\theta_Z = \tan^{-1} \frac{R_X \sin\theta_X + R_Y \sin\theta_Y}{R_X \cos\theta_X + R_Y \cos\theta_Y} \tag{66-2}$$

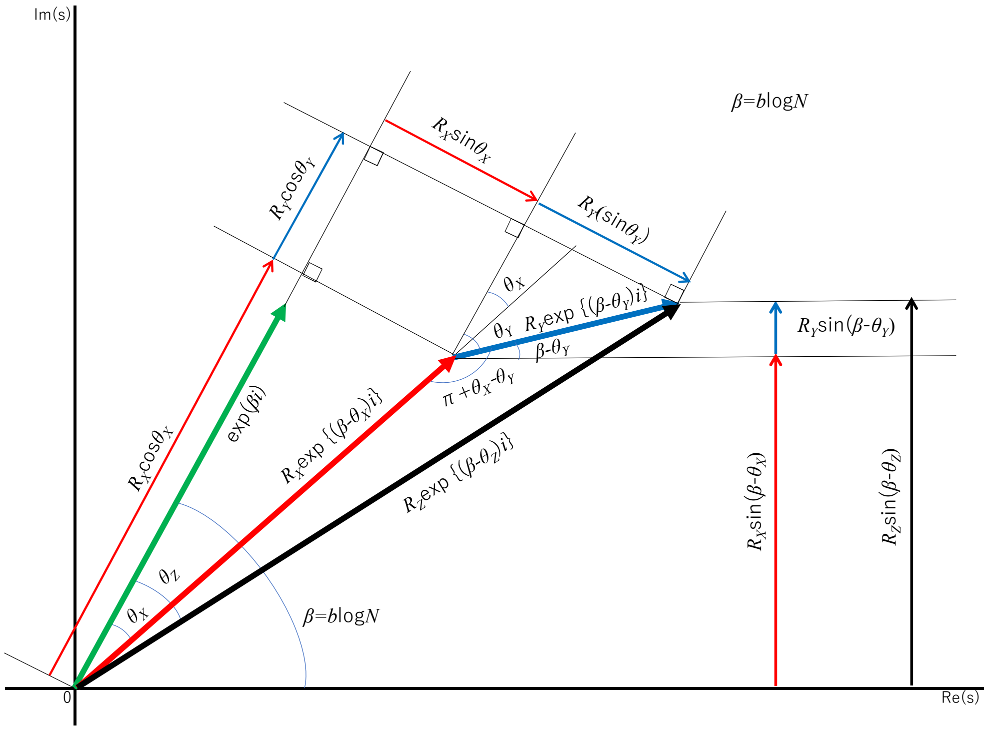


Figure 5： Sum of 2 sine functions

3.3 In the following (67-2) each sine function has the common term $\beta = b \log N$. And the sum of $(N+1)/2$ sine functions becomes one sine function which has β, L and M like the following (67-3). L and M do not depend on β because R_Z and θ_Z do not depend on β but on R_X, R_Y, θ_X and θ_Y as shown in the above (66-1) and (66-2).

$$-\{\sin(b \log 1/N) + \sin(b \log 3/N) + \sin(b \log 5/N) + \cdots + \sin(b \log N/N)\} \tag{67-1}$$

$$= \sin(b \log N - b \log 1) + \sin(b \log N - b \log 3) + \sin(b \log N - b \log 5)$$

$$+ \cdots + \sin(b \log N - b \log N) \tag{67-2}$$

$$= L \sin(b \log N - M) \tag{67-3}$$

3.4 In the following (68-3) each sine function has the common term $\gamma = b\log N + \pi/2$. And the sum of $(N+1)/2$ sine functions becomes one sine function which has γ, L and M like the following (68-4). Because L and M do not depend on common term β or γ but on R_X, R_Y, θ_X and θ_Y and each sine function in (67-2) has the same R_X, R_Y, θ_X and θ_Y as in (68-3).

$$\cos(b\log 1/N) + \cos(b\log 3/N) + \cos(b\log 5/N) + \cdots + \cos(b\log N/N) \quad (68\text{-}1)$$
$$= \cos(b\log N - b\log 1) + \cos(b\log N - b\log 3) + \cos(b\log N - b\log 5) +$$
$$+ \cdots + \cos(b\log N - b\log N) \quad (68\text{-}2)$$
$$= \sin(b\log N + \pi/2 - b\log 1) + \sin(b\log N + \pi/2 - b\log 3)$$
$$+ \sin(b\log N + \pi/2 - b\log 5) + \cdots + \sin(b\log N + \pi/2 - b\log N) \quad (68\text{-}3)$$
$$= L\sin(b\log N + \pi/2 - M) = L\cos(b\log N - M) \quad (68\text{-}4)$$

3.5 From the above (65), (67-1,2,3) and (68-1,2,3,4) we have the following (70).

$$\theta_3 = \tan^{-1}\frac{\cos(b\log 1/N)+\cos(b\log 3/N)+\cos(b\log 5/N)+\cdots+\cos(b\log N/N)}{-\{\sin(b\log 1/N)+\sin(b\log 3/N)+\sin(b\log 5/N)+\cdots+\sin(b\log N/N)\}}$$
$$= \tan^{-1}\frac{L\cos(b\log N - M)}{L\sin(b\log N - M)} = \tan^{-1}\cot(b\log N - M)$$
$$= \tan^{-1}\tan(\pi/2 + M - b\log N)$$
$$= \pi/2 + M - b\log N + K_1\pi \qquad (K_1 : \text{integer}) \quad (70)$$

3.6 We consider that b is (type2 irrational mumber) and has the term of $\frac{A\pi}{\log N_2/2}$ like the following (71). If b is (type1 irrational mumber), $E = 0$ holds.
b =(type2 irrational number)

=(rational number)+(type1 irrational number)

$$= E+\frac{A\pi}{\log N_2/2}+F$$
$$(E, A : \text{rational number} \quad F : \text{type1 irational number}) \quad (71)$$

3.7 From (52) and the above (70) and (71) we have the following (72) at $N = N_2$.

$$\text{left side of (52)} = b\log N_2/2 + \theta_3$$
$$= (E + \frac{A\pi}{\log N_2/2} + F)\log\frac{N_2}{2} + \frac{\pi}{2} + M - (E + \frac{A\pi}{\log N_2/2} + F)\log N_2 + K_1\pi$$
$$= A\pi - (E+F)\log 2 - \frac{A\pi\log N_2}{\log N_2/2} + \frac{\pi}{2} + M + K_1\pi$$
$$= \pi\{\frac{1}{2} + K_1 - \frac{A\log 2}{\log N_2/2} + \frac{M-(E+F)\log 2}{\pi}\} = J\pi \quad (72)$$

A, E : (rational number) F : (type1 irrational number) K_1 : (integer) M : the value of arctangent function which has the range of $-\pi/2 < M < \pi/2$ N_2 : (odd number)

3.8 In order for $J=K$ to hold in the above (72) the following (73-1) and (73-2) must hold.

$$J=\frac{1}{2}+K_1-\frac{A\log 2}{\log N_2/2}+\frac{M-(E+F)\log 2}{\pi}=K \quad (K:\text{integer}) \tag{73-1}$$

$$\frac{M-(E+F)\log 2}{\pi}-\frac{A\log 2}{\log N_2/2}=K-K_1-\frac{1}{2} \tag{73-2}$$

$\frac{A\log 2}{\log N_2/2}$ is (irrational number) and $(K-K_1-1/2)$ is (rational number).
If $\frac{M-(E+F)\log 2}{\pi}$ is (rational number), the above (73-2) becomes the following (73-3) and this equation does not hold.

$$(\text{rational number})-(\text{irrational number})=(\text{rational number}) \tag{73-3}$$

If $\frac{M-(E+F)\log 2}{\pi}$ is (irrational number), the following (74-1), (74-2) and (74-3) must hold in order for (73-2) to hold.

$$\frac{M-(E+F)\log 2}{\pi}=(\text{rational number}:P_1)+(\text{irrational number}:Q) \tag{74-1}$$

$$\frac{A\log 2}{\log N_2/2}=(\text{rational number}:P_2)+(\text{irrational number}:Q) \tag{74-2}$$

$$P_1-P_2=K-K_1-1/2 \tag{74-3}$$

But $\frac{A\log 2}{\log N_2/2}$ cannot be divided into (rational number) and (irrational number) like the above (74-2).
Then the above (73-3) and (74-2) do not hold i.e. (73-2) does not hold. Therefor (73-1) i.e. $J=K$ does not hold.

3.9 Now we can confim that the following (52) in item 2.3.1 does not hold at $N=N_2$ when (condition 2) holds.

$$b\log N/2+\theta_3=K\pi \qquad (K:\text{integer}) \tag{52}$$

Appendix 4. : Proof of $\lim_{N\to\infty}\dfrac{g(k,N)}{g(2,N)}=1$

From (24-5) in item 2.1.3 we have the following (75).

$$
\begin{aligned}
&\frac{g(k,N)}{g(2,N)} \\
&= \frac{\frac{N\sin(b\log N/k+\tan^{-1}1/b)}{2\sqrt{1+b^2}} + N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})E_s(N)}{\frac{N\sin(b\log N/2+\tan^{-1}1/b)}{2\sqrt{1+b^2}} + N\cos(b\log\frac{N}{2})E_c(N) - N\sin(b\log\frac{N}{2})E_s(N)} \\
&= \frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b}) + 2\sqrt{1+b^2}\{\cos(b\log\frac{N}{k})E_c(N)-\sin(b\log\frac{N}{k})E_s(N)\}}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b}) + 2\sqrt{1+b^2}\{\cos(b\log\frac{N}{2})E_c(N)-\sin(b\log\frac{N}{2})E_s(N)\}}
\end{aligned}
$$

$$
= \frac{\sin\{\frac{b\log N/k+\tan^{-1}1/b}{b\log N/2+\tan^{-1}1/b}(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})\}+2\sqrt{1+b^2}\{\cos(b\log\frac{N}{k})E_c(N)-\sin(b\log\frac{N}{k})E_s(N)\}}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})+2\sqrt{1+b^2}\{\cos(b\log\frac{N}{2})E_c(N)-\sin(b\log\frac{N}{2})E_s(N)\}} \tag{75}
$$

We can confirm that the following (76) holds from the above (75), the following (77) and the following (23) shown in item 2.1.2.

$$
\lim_{N\to\infty}\frac{g(k,N)}{g(2,N)} = \frac{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})} = 1 \qquad (N_1 < N \quad N: \text{odd number}) \tag{76}
$$

$$
\lim_{N\to\infty}\frac{b\log\frac{N}{k}+\tan^{-1}\frac{1}{b}}{b\log\frac{N}{2}+\tan^{-1}\frac{1}{b}} = \lim_{N\to\infty}\frac{1-\frac{\log k}{\log N}+\frac{\tan^{-1}1/b}{b\log N}}{1-\frac{\log 2}{\log N}+\frac{\tan^{-1}1/b}{b\log N}} = 1 \tag{77}
$$

$$
\lim_{N\to\infty}E_c(N) = 0 \qquad \lim_{N\to\infty}E_s(N) = 0 \tag{23}
$$

Appendix 5. : Solution for $F(a) = 0$

5.1. Preparation for verification of $F(a) > 0$

5.1.1. Investigation of $f(n)$

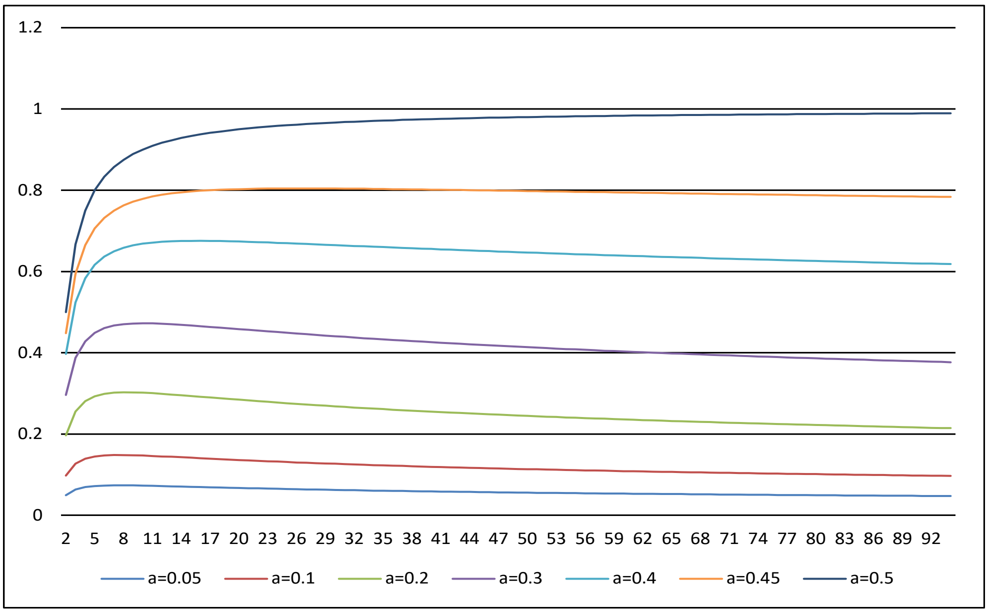
$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \tag{15}$$

$a = 0$ is the solution for $F(a) = 0$ due to $f(n) \equiv 0$ at $a = 0$. Hereafter we define the range of a as $0 < a < 1/2$ to verify $F(a) > 0$. The alternating series $F(a)$ converges due to $\lim_{n\to\infty} f(n) = 0$.
We have the following (81) by differentiating $f(n)$ regarding n.

$$\frac{df(n)}{dn} = \frac{1/2+a}{n^{a+3/2}} - \frac{1/2-a}{n^{3/2-a}} = \frac{1/2+a}{n^{a+3/2}}\{1 - (\frac{1/2-a}{1/2+a})n^{2a}\} \tag{81}$$

The value of $f(n)$ increases with increase of n and reaches the maximum value $f(n_{max})$ at $n = n_{max}$. Afterward $f(n)$ decreases to zero with $n \to \infty$. n_{max} is one of the 2 consecutive natural numbers that sandwich $(\frac{1/2+a}{1/2-a})^{\frac{1}{2a}}$. (Graph 1) shows $f(n)$ in various value of a. At $a = 1/2$ $f(n)$ does not have $f(n_{max})$ and increases to 1 with $n \to \infty$ due to $n_{max} = \infty$.



Graph 1 : $f(n)$ in various a

5.1.2. Verification method for $F(a) > 0$

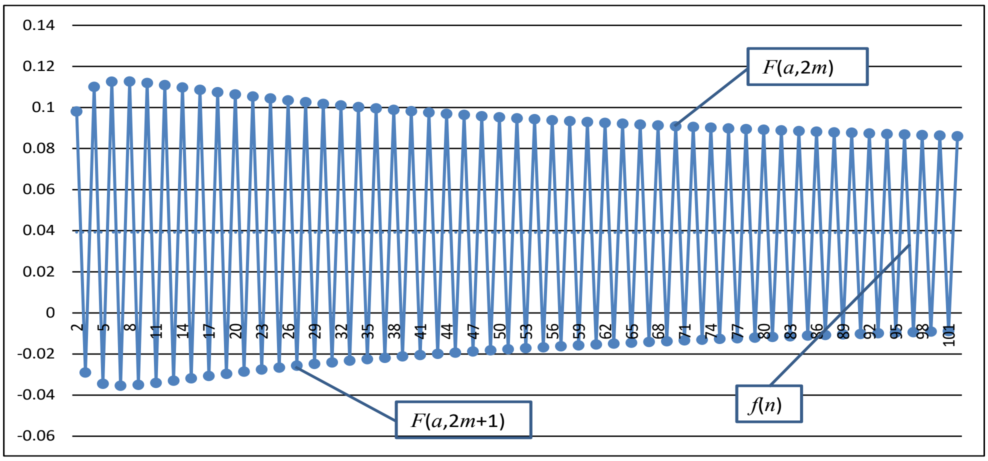
We define $F(a, n)$ as the following (82).

$$F(a, n) = f(2) - f(3) + f(4) - f(5) + \cdots + (-1)^n f(n) \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{82}$$

$$\lim_{n\to\infty} F(a,n) = F(a) \tag{83}$$

$F(a)$ is an alternating series. So $F(a,n)$ repeats increase and decrease by $f(n)$ with increase of n as shown in (Graph 2). In (Graph 2) upper points mean $F(a,2m)$ $(m = 1,2,3,\cdots\cdots)$ and lower points mean $F(a,2m+1)$. $F(a,2m)$ decreases and converges to $F(a)$ with $m\to\infty$. $F(a,2m+1)$ increases and also converges to $F(a)$ with $m\to\infty$ due to $\lim_{n\to\infty} f(n) = 0$. From the above (83) we have the following (84).

$$\lim_{m\to\infty} F(a,2m) = \lim_{m\to\infty} F(a,2m+1) = F(a) \tag{84}$$



Graph 2 : $F(0.1,n)$ from 1st to 100th term

We define $F1(a)$ and $F1(a,2m+1)$ as follws.

$$F1(a) = \{f(2)-f(3)\} + \{f(4)-f(5)\} + \{f(6)-f(7)\} + \cdots\cdots \tag{85}$$

$$F1(a,2m+1) = \{f(2)-f(3)\} + \{f(4)-f(5)\} + \cdots + \{f(2m)-f(2m+1)\}$$
$$= f(2)-f(3)+f(4)-f(5)+\cdots+f(2m)-f(2m+1) = F(a,2m+1) \tag{86}$$

$$\lim_{m\to\infty} F1(a,2m+1) = F1(a) \tag{87}$$

From the above (84), (86) and (87) we have $F(a) = F1(a)$. We can use $F1(a)$ instead of $F(a)$ to verify $F(a) > 0$.

We enclose 2 terms of $F(a)$ each from the first term with { } as follows. If n_{max} is p or $p+1$ (p: odd number) , the inside sum of { } from $f(2)$ to $f(p)$ has negative value and the inside sum of { } after $f(p+1)$ has positive value.

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)-f(7)+\cdots\cdots$$
$$= \{f(2)-f(3)\} + \{f(4)-f(5)\} + \cdots + \{f(p-1) - f(p)\} + \{f(p+1) - f(p+2)\} + \cdots\cdots$$

(inside sum of { }) < 0 ⟵|⟶(inside sum of { }) > 0
(total sum of { }) $= -B$ ⟵|⟶(total sum of { }) $= A$

We define as follows.

[the partial sum from $f(2)$ to $f(p)$] $= -B < 0$
[the partial sum from $f(p+1)$ to $f(\infty)$] $= A > 0$
$$F(a) = A - B \tag{88}$$
So we can verify $F(a) > 0$ by verifying $A > B$.

5.1.3. Investigation of $\{f(n) - f(n+1)\}$

We have the following (89) by differentiating $\{f(n) - f(n+1)\}$ regarding n.

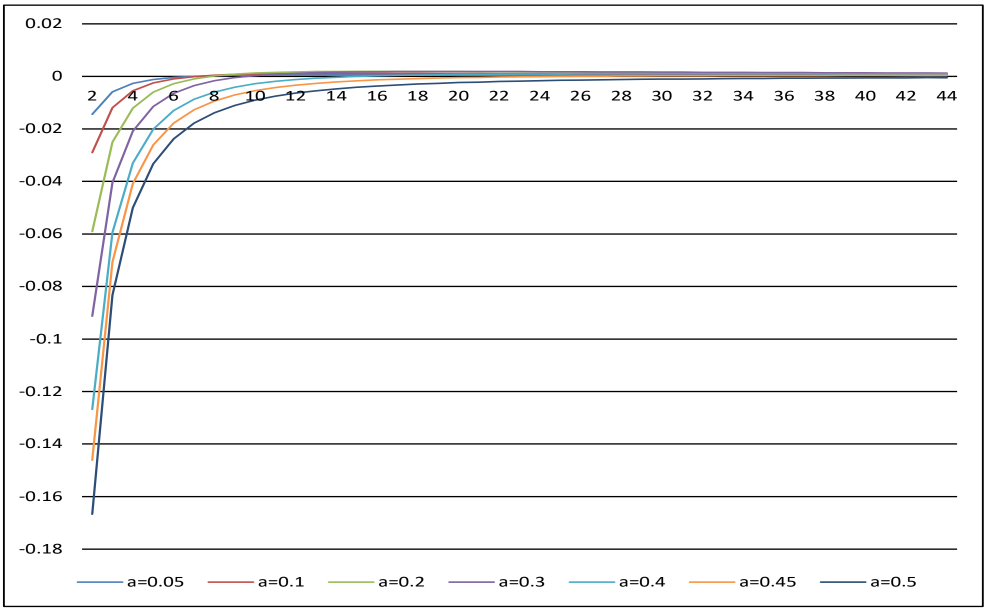
$$\frac{df(n)}{dn} - \frac{df(n+1)}{dn} = \frac{1/2+a}{n^{3/2+a}}\{1-(\frac{n}{n+1})^{3/2+a}\} - \frac{1/2-a}{n^{3/2-a}}\{1-(\frac{n}{n+1})^{3/2-a}\}$$
$$= C(n) - D(n) \tag{89}$$

When n is a small natural number the value of $\{f(n)-f(n+1)\}$ increases with increase of n due to $C(n) > D(n)$. With increase of n the value reaches the maximum value $\{q_{max}\}$ at $C(n) \fallingdotseq D(n)$. (n is a natural number. The situation cannot be $C(n) = D(n)$.) After that the situation changes to $C(n) < D(n)$ and the value decreases to zero with $n \to \infty$. (Graph 3) shows the value of $\{f(n) - f(n+1)\}$ in various value of a. (Graph 4) shows the value of $\{f(n) - f(n+1)\}$ at $a = 0.1$. We can find the following from (Graph 3) and (Graph 4).

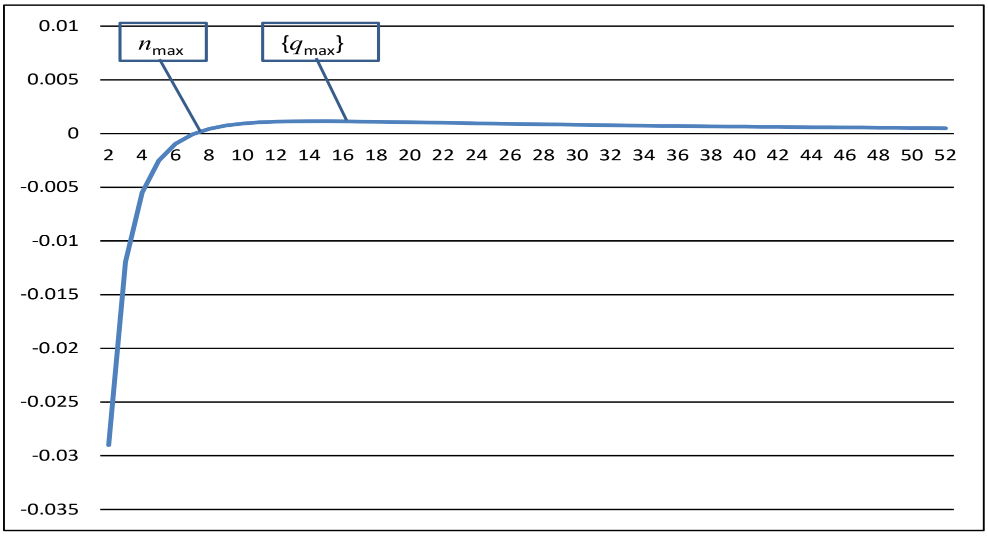
5.1.3.1 When $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value $|f(n) - f(n+1)|$ also becomes the maximum value at same value of a. From (Graph 1) we can find that $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value at $n = 2$. Therefore the maximum value of $|f(n) - f(n+1)|$ is $\{f(3) - f(2)\}$ at same value of a as shown in (Graph 3).

5.1.3.2 With increase of n the sign of $\{f(n) - f(n+1)\}$ changes from minus to plus at $n = n_{max}$ ($n = n_{max} + 1$) when n_{max} is even(odd) number as shown in (Graph 4).

5.1.3.3 After that the value reaches the maximum value $\{q_{max}\}$ and the value decreases to zero with $n \to \infty$ as shown in (Graph 4).



Graph 3 : $\{f(n) - f(n+1)\}$ in various a



Graph 4 : $\{f(n) - f(n+1)\}$ at $a = 0.1$

5.2. Verification of $A > B$ (n_{max} is odd number.)

n_{max} is odd number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$

$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 2)\} + \{f(n_{max} - 1) - f(n_{max})\}$$

$$+\{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} + \{f(n_{max} + 5) - f(n_{max} + 6)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 2) - f(n_{max} - 3)\} + \{f(n_{max}) - f(n_{max} - 1)\}$$

$$A = \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} + \{f(n_{max} + 5) - f(n_{max} + 6)\} + \cdots\cdots$$

5.2.1. Condition for B

We define as follows.

{ (yellow) } : the term which is included within B.

{ (gray) } : the term which is not included within B.

We have the following (90).

$$f(n_{max}) - f(2) = \{f(n_{max}) - f(n_{max} - 1)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\} + \{f(n_{max} - 2) - f(n_{max} - 3)\} + \cdots + \{f(7) - f(6)\} + \{f(6) - f(5)\} + \{f(5) - f(4)\} + \{f(4) - f(3)\} + \{f(3) - f(2)\} \quad (90)$$

And we have the following inequalities from (Graph 3) and (Graph 4).

$$\{f(3) - f(2)\} > \{f(4) - f(3)\} > \{f(5) - f(4)\} > \{f(6) - f(5)\} > \{f(7) - f(6)\} > \cdots\cdots > \{f(n_{max} - 2) - f(n_{max} - 3)\} > \{f(n_{max} - 1) - f(n_{max} - 2)\} > \{f(n_{max}) - f(n_{max} - 1)\} > 0$$

From the above (90) we have the following (91).

$$f(n_{max}) - f(2) + \{f(3) - f(2)\}$$

$$= \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 2) - f(n_{max} - 3)\} + \{f(n_{max}) - f(n_{max} - 1)\}$$

$$\quad \| \qquad \wedge \qquad \wedge \qquad \wedge \quad \leftarrow\text{Value comparison}\rightarrow \quad \wedge$$

$$+\{f(3) - f(2)\} + \{f(4) - f(3)\} + \{f(6) - f(5)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 4)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\}$$

$$> 2B \quad (91)$$

Due to [Total sum of upper row of the above (91) $= B <$ Total sum of lower row of (91)] we have the following (92).

$$f(n_{max}) - f(2) + \{f(3) - f(2)\} > 2B \quad (92)$$

5.2.2. Condition for A ($\{q_{max}\}$ is included within A.)

We abbreviate $\{f(n_{max} + q) - f(n_{max} + q + 1)\}$ to $\{q\}$ for easy description. $(q = 0, 1, 2, 3, \cdots\cdots)$ All $\{q\}$ has positive value as shown in item 5.1.2.

We define as follows.

{ (yellow) } : the term which is included within A.

{ (gray) } : the term which is not included within A.

$\{q_{max}\}$ has the maximum value in all $\{q\}$. And $\{q_{max}\}$ is included within A. Then value comparison of $\{q\}$ is as follows.

$$\{1\} < \{2\} < \{3\} < \cdots < \{q_{max} - 3\} < \{q_{max} - 2\} < \{q_{max} - 1\} < \{q_{max}\} > \{q_{max} + 1\} > \{q_{max} + 2\} > \{q_{max} + 3\} > \cdots\cdots$$

We have the following (93).

$$f(n_{max} + 1) = \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{93}$$

From the above (93) we have the following (94).

$$f(n_{max} + 1) - \{q_{max} - 1\}$$

$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{94}$$

← ··········· Range 1 ············ →|← ············ Range 2 ············

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{1\} < \{2\} < \{3\} < \{4\} < \cdots < \{q_{max} - 4\} < \{q_{max} - 3\} < \{q_{max} - 2\}$$

And we can find the following.

Total sum of { } = {1} + {3} + {5} + {7} + ⋯ + {$q_{max} - 4$} + {$q_{max} - 2$}

∨ ∨ ∨ ∨ ∨ ←Value comparison

Total sum of { } = {2} + {4} + {6} + ⋯ + {$q_{max} - 5$} + {$q_{max} - 3$}

Therefore [Total sum of { } > Total sum of { }] holds.
In (Range 2) value comparison is as follows.

$$\{q_{max}\} > \{q_{max} + 1\} > \{q_{max} + 2\} > \{q_{max} + 3\} > \{q_{max} + 4\} > \{q_{max} + 5\} > \{q_{max} + 6\} > \cdots\cdots$$

And we can find the following.

Total sum of { } = {q_{max}} + {$q_{max} + 2$} + {$q_{max} + 4$} + {$q_{max} + 6$} + ⋯⋯

∨ ∨ ∨ ∨ ←Value comparison

Total sum of { } = {$q_{max} + 1$} + {$q_{max} + 3$} + {$q_{max} + 5$} + {$q_{max} + 7$} + ⋯⋯

Therefore [Total sum of { } > Total sum of { }] holds.
In (Range 1)+(Range 2) we have [Total sum of { } $= A >$ Total sum of { }].
We have the following (95).

$$f(n_{max} + 1) - \{q_{max} - 1\} < 2A \tag{95}$$

5.2.3. Condition for A ($\{q_{max}\}$ is not included within A.)

We have the following (96). $\{q_{max}\}$ is not included within A.

$$f(n_{max} + 1) = \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\}$$

$$+ \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{96}$$

From the above (96) we have the following (97).

$$f(n_{max} + 1) - \{q_{max}\}$$

$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{97}$$

← ················· Range 1 ··············· →|← ········· Range 2 ·········

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{\colorbox{yellow}{1}\} < \{\colorbox{lightgray}{2}\} < \{\colorbox{yellow}{3}\} < \{\colorbox{lightgray}{4}\} < \cdots < \{\colorbox{lightgray}{$q_{max}-4$}\} < \{\colorbox{yellow}{$q_{max}-3$}\} < \{\colorbox{lightgray}{$q_{max}-2$}\} < \{\colorbox{yellow}{$q_{max}-1$}\}$$

And we can find the following.

$$\begin{array}{lcl} \text{Total sum of } \{\colorbox{yellow}{}\} = & \{\colorbox{yellow}{1}\} + \{\colorbox{yellow}{3}\} + \{\colorbox{yellow}{5}\} + \{\colorbox{yellow}{7}\} + \cdots + \{\colorbox{yellow}{$q_{max}-3$}\} + \{\colorbox{yellow}{$q_{max}-1$}\} & \\ & \vee \quad \vee \quad \vee \quad \vee \quad \vee & \leftarrow \text{Value comparison} \\ \text{Total sum of } \{\colorbox{lightgray}{}\} = & \{\colorbox{lightgray}{2}\} + \{\colorbox{lightgray}{4}\} + \{\colorbox{lightgray}{6}\} + \cdots + \{\colorbox{lightgray}{$q_{max}-4$}\} + \{\colorbox{lightgray}{$q_{max}-2$}\} & \end{array}$$

Therefore [Total sum of $\{\colorbox{yellow}{}\}$ > Total sum of $\{\colorbox{lightgray}{}\}$] holds.
In (Range 2) value comparison is as follows.

$$\{\colorbox{yellow}{$q_{max}+1$}\} > \{\colorbox{lightgray}{$q_{max}+2$}\} > \{\colorbox{yellow}{$q_{max}+3$}\} > \{\colorbox{lightgray}{$q_{max}+4$}\} > \{\colorbox{yellow}{$q_{max}+5$}\} > \{\colorbox{lightgray}{$q_{max}+6$}\} > \{\colorbox{yellow}{$q_{max}+7$}\} > \cdots\cdots$$

And we can find the following.

$$\begin{array}{lcl} \text{Total sum of } \{\colorbox{yellow}{}\} = & \{\colorbox{yellow}{$q_{max}+1$}\} + \{\colorbox{yellow}{$q_{max}+3$}\} + \{\colorbox{yellow}{$q_{max}+5$}\} + \{\colorbox{yellow}{$q_{max}+7$}\} + \cdots\cdots & \\ & \vee \quad \vee \quad \vee \quad \vee & \leftarrow \text{Value comparison} \\ \text{Total sum of } \{\colorbox{lightgray}{}\} = & \{\colorbox{lightgray}{$q_{max}+2$}\} + \{\colorbox{lightgray}{$q_{max}+4$}\} + \{\colorbox{lightgray}{$q_{max}+6$}\} + \{\colorbox{lightgray}{$q_{max}+8$}\} + \cdots\cdots & \end{array}$$

Therefore [Total sum of $\{\colorbox{yellow}{}\}$ > Total sum of $\{\colorbox{lightgray}{}\}$] holds.
In (Range 1)+(Range 2) we have [Total sum of $\{\colorbox{yellow}{}\} = A >$ Total sum of $\{\colorbox{lightgray}{}\}$].
We have the following (98).

$$f(n_{max}+1) - \{q_{max}\} < 2A \tag{98}$$

5.2.4. Condition for $A > B$

From (95) and (98) we have the following inequality.

$$f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] < 2A$$

As shown in item 5.1.3.1 $\{f(3) - f(2)\}$ is the maximum in all $|f(n) - f(n+1)|$. Then the following holds.

$$\begin{aligned} \{f(3) - f(2)\} >& [\{q_{max}\} \text{ or } \{q_{max}-1\}] \\ \{f(3) - f(2)\} >& f(n_{max}) - f(n_{max}+1) \end{aligned}$$

We have the following inequality from the above 3 inequalities.

$$\begin{aligned} 2A >& f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] > f(n_{max}+1) - \{f(3) - f(2)\} \\ >& f(n_{max}) - \{f(3) - f(2)\} - \{f(3) - f(2)\} = f(n_{max}) - 2\{f(3) - f(2)\} \end{aligned} \tag{99}$$

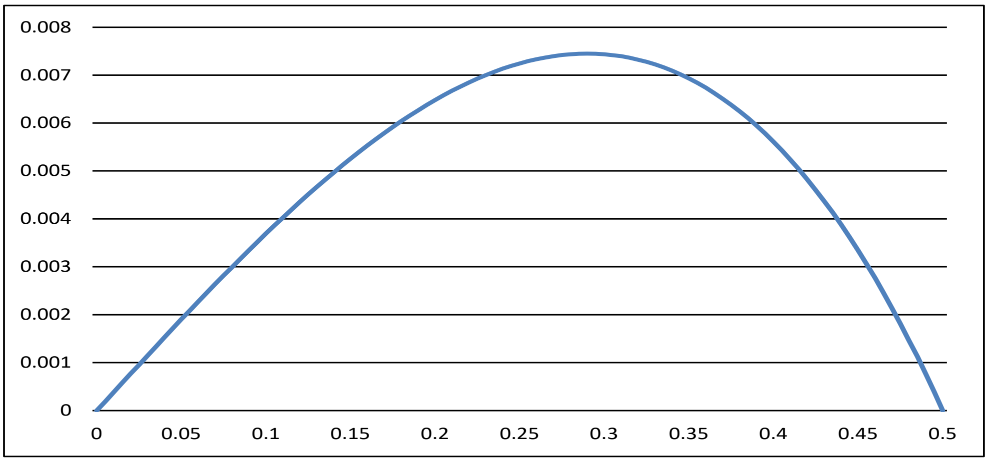
We have the following (100) for $A > B$ from (92) and (99).

$$2A > f(n_{max}) - 2\{f(3) - f(2)\} > f(n_{max}) - f(2) + \{f(3) - f(2)\} > 2B \tag{100}$$

From the above (100) we can have the final condition for $A > B$ as follows.

$$(4/3)f(2) > f(3) \tag{101}$$

(Graph 5) shows $(4/3)f(2) - f(3) = (4/3)(\frac{1}{2^{1/2-a}} - \frac{1}{2^{1/2+a}}) - (\frac{1}{3^{1/2-a}} - \frac{1}{3^{1/2+a}})$.

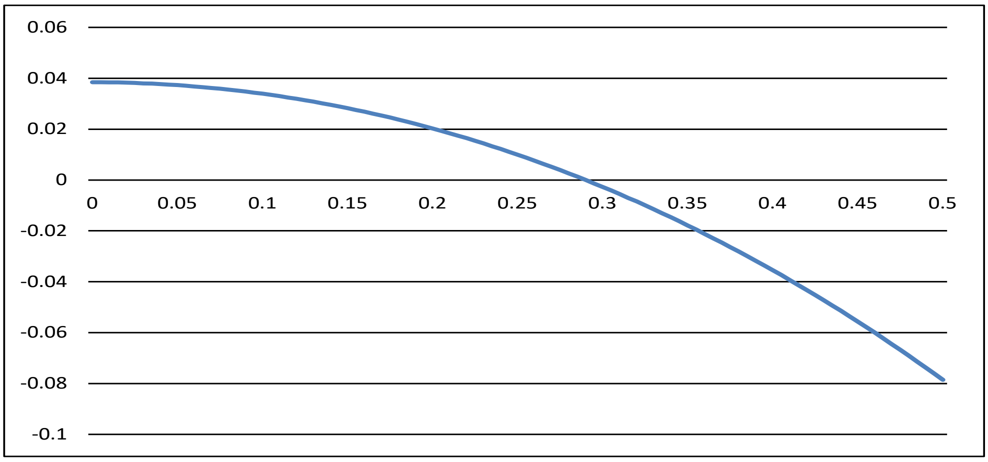


Graph 5 : $(4/3)f(2) - f(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f(2)-f(3)	0	0.001903	0.003694	0.005257	0.00648	0.007246	0.007437	0.006933	0.005611	0.003343	0

Table 1 : The values of $(4/3)f(2) - f(3)$

(Graph 6) shows [differentiated $\{(4/3)f(2) - f(3)\}$ regarding a] i.e. $(4/3)f'(2) - f'(3) = (4/3)\{\log 2(\frac{1}{2^{1/2-a}} + \frac{1}{2^{1/2+a}})\} - \{\log 3(\frac{1}{3^{1/2-a}} + \frac{1}{3^{1/2+a}})\}$.



Graph 6 : $(4/3)f'(2) - f'(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f'(2)−f'(3)	0.038443	0.037313	0.033921	0.02825	0.020277	0.009967	−0.00272	−0.01785	−0.03547	−0.05567	−0.07852

Table 2：The values of $(4/3)f'(2) - f'(3)$

From (Graph 5) and (Graph 6) we can find $[(4/3)f(2) - f(3) > 0$ in $0 < a < 1/2]$ that means $A > B$ i.e. $F(a) > 0$ in $0 < a < 1/2$.

5.3. Verification of $A > B$ (n_{max} is even number.)

n_{max} is even number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$

$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 4) - f(n_{max} - 3)\} + \{f(n_{max} - 2) - f(n_{max} - 1)\}$$

$$+\{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\}$$

$$+ \cdots + \{f(n_{max} - 3) - f(n_{max} - 4)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\}$$

$$A = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

$$f(n_{max}) = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\}$$

$$+\{f(n_{max} + 3) - f(n_{max} + 4)\} + \cdots\cdots$$

$$= \{0\} + \{1\} + \{2\} + \{3\} + \{4\}$$

$$+ \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots$$

After the same process as in item 5.2.1 we can have the following (102).

$$f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \tag{102}$$

As shown in item 5.1.3.1 $\{f(3) - f(2)\}$ is the maximum in all $|f(n) - f(n + 1)|$. Then the following holds.

$$\{f(3) - f(2)\} > [\{q_{max}\} \text{ or } \{q_{max} - 1\}]$$
$$f(n_{max}) > f(n_{max} - 1)$$

We have the following (103) from the above inequalities and the same process as in item 5.2.2 and item 5.2.3.

$$\begin{aligned} 2A >& f(n_{max}) - [\{q_{max}\} \text{ or } \{q_{max} - 1\}] > f(n_{max}) - \{f(3) - f(2)\} \\ >& f(n_{max} - 1) - \{f(3) - f(2)\} \end{aligned} \tag{103}$$

We have the following (104) for $A > B$ from (102) and (103).

$$2A > f(n_{max} - 1) - \{f(3) - f(2)\} > f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \tag{104}$$

From (104) we can have the final condition for $A > B$ as follows.

$$(3/2)f(2) > f(3) \tag{105}$$

In the inequality of $[(3/2)f(2) > (4/3)f(2) > f(3) > 0]$, $(3/2)f(2) > (4/3)f(2)$ is true self-evidently and in item 5.2.4 we already confirmed that the following (101) was true in $0 < a < 1/2$.

$$(4/3)f(2) > f(3) \tag{101}$$

Therefore the above (105) is true in $0 < a < 1/2$. Now we can confirm $F(a) > 0$ in $0 < a < 1/2$.

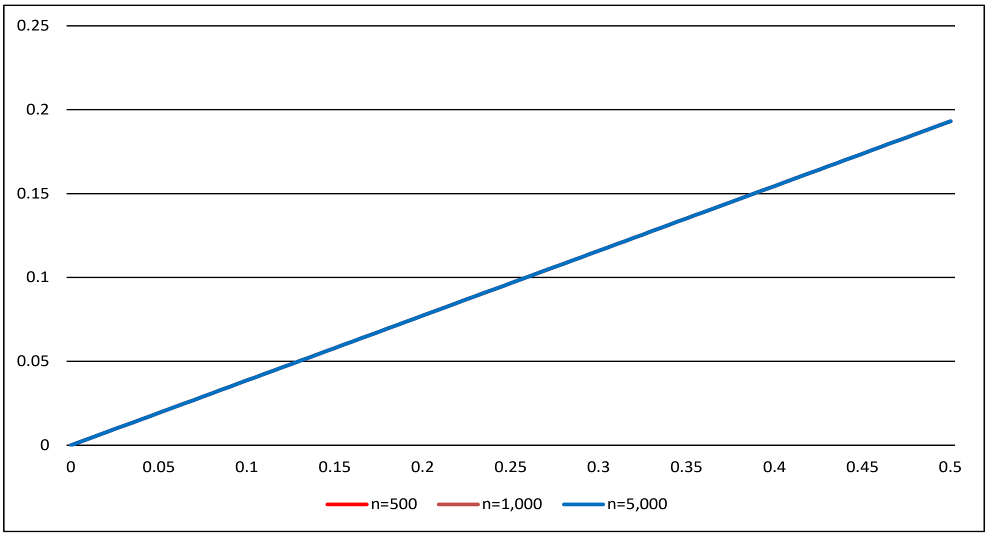
5.4. Conclusion

As shown in item 5.2 and item 5.3 $[F(a) > 0$ in $0 < a < 1/2]$ is true. Therefore $F(a) = 0$ has the only solution of $a = 0$ from $[0 \leq a < 1/2]$ and $[F(0) = 0]$.

5.5. Graph of $F(a)$

We can approximate $F(a)$ with the average of $\{F(a, n-1) + F(a, n)\}/2$. But we approximate $F(a)$ by the following (106) for better accuracy. (Graph 7) shows $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$.

$$\frac{\frac{F(a,n-1)+F(a,n)}{2} + \frac{F(a,n)+F(a,n+1)}{2}}{2} = F(a)_n \tag{106}$$



Graph 7 : $F(a)_n$ at 3 cases

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
n=500	0	0.01932876	0.03865677	0.05798326	0.0773074	0.09662832	0.11594507	0.13525658	0.15456168	0.17385904	0.19314718
n=1,000	0	0.01932681	0.03865282	0.05797725	0.0772993	0.09661821	0.11593325	0.13524382	0.15454955	0.17385049	0.19314743
n=5,000	0	0.01932876	0.03865676	0.05798324	0.07730738	0.09662829	0.11594504	0.13525655	0.15456165	0.17385902	0.19314718

Table 3 : The values of $F(a)_n$ at 3 cases

3 line graphs overlapped. Because $F(a)_n$ calculated at 3 cases of n =500, 1000, 5000 are equal to 4 digits after the decimal point. The range of a is $0 \leq a < 1/2$. $a = 1/2$ is not included in the range. But we added $F(1/2)_n$ to calculation due to the following reason. $[f(n)$ at $a = 1/2]$ is $(1-1/n)$ and $F(1/2)$ fluctuates due to $\lim_{n\to\infty} f(n) = 1$. But the value of the above (106) converges to the fixed value on the condition of $\lim_{n\to\infty} \{f(n+1) - f(n)\} = 0$. The condition holds due to $f(n+1) - f(n) = 1/(n + n^2)$.
$F(a)$ is a monotonically increasing function as shown in (Graph 7). So $F(a) = 0$ has the only solution and the solution must be $a = 0$ due to the following facts. Therefore Riemann hypothesis must be true.

5.5.1 In 1914 G. H.Hardy proved that there are infinite non-trivial zero points on the line of $Re(s) = 1/2$.

5.5.2 All non-trivial zero points found until now exist on the line of $Re(s) = 1/2$.

Data availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

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