Proof of a Combinatorial Identity

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December 7, 2021

Abstract

In this present paper we will show you some interesting identity involving combinatorial symbols and a proof of it as a theorem. The theorem was a discovery from the times when I was studying Calculus at USAC/CUNOC University in Quetzaltenango, Guatemala around 2004 year.

Theorem 1. (Danilo Chavez 2004) If \( n, k \in (N \cup 0) \), \( n \geq 0 \), \( k \geq 0 \), \( n \geq k \) then

\[
\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}
\]

Proof. By mathematical induction. If \( n=0 \) then \( k=0 \)

\[
\binom{0}{0} = \frac{0!}{0!0!} = 1 = \frac{1!}{1!0!} = \binom{1}{1}
\]

If \( n=1 \), we have two possibilities: \( k=0 \) or \( k=1 \)

\[
\binom{0}{0} + \binom{1}{0} = \frac{0!}{0!0!} + \frac{1!}{0!1!} = 2 = \frac{2!}{1!1!} = \binom{2}{1}
\]

\[
\binom{1}{1} = \frac{1!}{1!0!} = 1 = \frac{2!}{2!0!} = \binom{2}{2}
\]
So, \( n=0 \) and \( n=1 \) are covered. Supposing \( n=r \) we have

\[
\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{r-1}{k} + \binom{r}{k} = \binom{r+1}{k+1}
\]

It must be for \( n=r+1 \)

\[
\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{r-1}{k} + \binom{r}{k} + \binom{r+1}{k} = \binom{r+1}{k+1} + \binom{r+1}{k}
\]

We are looking for an expression like this

\[
\binom{r+1}{k+1} + \binom{r+1}{k} = \binom{r+2}{k+1}
\]

Let’s start

\[
\binom{r+1}{k+1} + \binom{r+1}{k} = \frac{(r+1)!}{(k+1)!(r-k)!} + \frac{(r+1)!}{k!(r-k+1)!}
\]

\[
= (r+1)!\left(\frac{1}{(k+1)!(r-k)!} + \frac{1}{k!(r-k+1)!}\right)
\]

\[
= (r+1)!\left(\frac{k!(r-k+1)! + (k+1)!(r-k)!}{(k+1)!(r-k)!k!(r-k+1)!}\right)
\]

\[
= \frac{(r+1)!}{(k+1)!(r-k)!k!(r-k+1)!}((r-k+1) + (k+1))
\]

\[
= \frac{(r+1)!k!(r-k)!}{(k+1)!(r-k)!k!(r-k+1)!}((r-k+1) + (k+1))
\]

\[
= \frac{(r+1)!(r+2)}{(k+1)!(r-k+1)!} = \frac{(r+2)!}{(k+1)!(r-k+1)!} = \binom{r+2}{k+1}
\]

Quod erat demonstrandum. \( \Box \)