The Extremal Nature of Membrane Newton-Cartan Formulations with Exotic Supergravity Theories

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Abstract

We construct a non-relativistic limit of eleven and ten-dimensional supergravity theories from the point of view of the fundamental symmetries, the higher-dimensional effective action, and the equations of motion. This fundamental limit can only be realized in a supersymmetric way provided we impose by hand a set of geometric constraints, invariant under all the symmetries of the non-relativistic theory, that define a so-called Dilatation-invariant Superstring Newton-Cartan geometry and Membrane Newton-Cartan expansion. In order to obtain a finite fundamental limit, the field strength of the eleven-dimensional four-form is required to obey a transverse self-duality constraint, ultimately due to the presence of the Chern-Simons term in eleven dimensions. The present research consider a non-relativistic fundamental limit of the bosonic sector of eleven-dimensional supergravity, leading to a theory based on a Covariant Membrane Newton-Cartan Supergeometry. We further show that the Membrane Newton-Cartan theory can be embedded in the U-duality symmetric formulation of exceptional field theory, demonstrating that it shares the same exceptional Lie algebraic symmetries as the relativistic supergravity, and providing an alternative derivation of the extra Poisson equation.

Contents

1 Introduction

There has been a growing interest in exploring Newtonian supergravity theories due to their use in strongly coupled membrane systems and relativistic effective field theories. The construction of Newtonian gravity, describing the physical gravitational force at non-relativistic level, requires to consider the so-called Newton-Cartan geometry. Such geometrical framework is necessary to covariantize the Poisson equation of Newtonian gravity. Nevertheless, a principle action for Newtonian gravity was recently presented which has required to extend the Bargmann algebra by including three additional generators. Subsequently, a three-dimensional Chern-Simons (CS) action has been constructed in the current literature which is invariant under a central extension of the symmetry group that leaves the recently constructed Newtonian gravity action invariant. The novel symmetry has been denoted as extended Newtonian algebra and can be recovered by means of a contraction of a bi-metric model being the sum of Einstein gravity in the Lorentzian and Euclidean signatures. Interestingly, the matter coupling of the extended Newtonian gravity theory admits backgrounds with non-trivial curvature whenever matter is present, similarly to the matter-coupled extended Bargmann gravity. The introduction of a cosmological constant in non-relativistic gravity theories is done considering the Newton-Hooke symmetry. However, an extension of the extended Newton-Hooke algebra is needed to include a cosmological constant to the extended Newtonian gravity theory. The novel symmetry is denoted as exotic Newtonian algebra and can be seen as an enhanced Bargmann-Newton-Hooke algebra. Both extended and exotic Newtonian gravity theories can be recovered as the non-relativistic limit of the coadjoint Poincaré $\oplus\ \mathfrak{u}(1)^2$ and coadjoint AdS $\oplus\ \mathfrak{u}(1)^2$ gravity theories. Supersymmetric extensions of three-dimensional non-relativistic gravity models have been recently approached and subsequently studied in supergravity. In particular, a CS action based on the supersymmetric extension of the extended Newtonian algebra has been presented. Although a cosmological constant has been accommodated in a non-relativistic supergravity theory through the extended Newton-Hooke superalgebra, the possible supersymmetric extensions of the exotic Newtonian gravity remain unexplored. Unlike bosonic non-relativistic gravity, the construction of an action based on a non-relativistic superalgebra is non-trivial and requires the introduction of additional bosonic generators. Furthermore, the non-relativistic limit is often ambiguous when supercharges are present. One way to circumvent this difficulty is through the expansion method based on Maurer-Cartan forms and semigroups, which have proved to be useful to obtain known and new non-relativistic supergravity theories from relativistic ones. We considered exotic branes as a particular class of non-geometric solutions that can be described within ExFT. Here we consider a rather different class of backgrounds, namely non-Riemannian backgrounds. Whereas the solutions in the previous section were characterised by either a lack of a global geometric description, owing to requiring duality transformations to patch correctly, or a lack of a local geometric description, due to a dependence of coordinates outside of the physical spacetime, the solutions we consider here are exotic in that they do not admit even local descriptions in terms of an invertible Riemannian metric. The definition is rather broad and includes various singular limits of the metric that obstruct its inversion. The key to describing such backgrounds is realising that fact that the generalised metric can remain regular in such backgrounds, even if the spacetime metric becomes singular, due to the presence of the off-diagonal terms in the generalised metric that can compensate for it. This fact was already appreciated where it appeared in the context of the doubled sigma model. Their work was then extended to a full characterisation of the possible backgrounds that one can obtain in DFT by solving the O(D, D) constraints on the generalised metric in generality n terms of the conventional supergravity description, exotic branes generate backgrounds with non-trivial monodromies. This means that they are not globally well defined solutions in supergravity and one needs to view the supergravity as being embedded in a larger theory where the duality group is used to patch together solutions via duality transformations. One may consider the approach as one which allows us to generate backgrounds of exotic branes. The exotic duality of a *single* brane was suggested from the viewpoint of the string duality groups and their representations. This was also analyzed in by virtue of the E_{11} supergravity technique. Furthermore, in the framework of other extended supergravity such as β-supergravity and its extended version, the exotic duality was further investigated. In this work, we would like to confirm the validity of the exotic duality from the viewpoint of the supersymmetry projection rules, and apply it to new brane configurations that involve *multiple* non-parallel exotic branes. They are also called the higher Kaluza-Klein branes, since the quadratic dependence on the radii in the isometry directions is similar to the case of the Kaluza-Klein monopole, KK5= 5^1_2. For the special case of $p = 7$, we frequently denote it by NS7 instead of 7_3. The duality relation between the standard branes and the exotic branes is summarized in Superstring theory contains various extended objects such as fundamental strings, solitonic five-branes, and Dp-branes. These objects are known to couple to the standard background fields; the B-field or the Ramond-Ramond fields. If we consider a compactification on a seven-torus, $T^7_{3\cdots9}$, there arise additional objects, called *exotic branes*. The exotic branes can exist only in the presence of compact isometry directions, just like the Kaluza-Klein monopoles, and have the tension proportional to g_s^α with $\alpha = -2, -3, -4$. Among them, a 5^2_2-brane, which has two isometry directions, has been well-studied recently. Since the 5^2_2 background has a non-vanishing (magnetic) *Q-flux*, we can identify the 5^2_2-brane as an object that magnetically couples to a bi-vector field β^{ij} whose derivative gives the Q-flux. This can be shown more explicitly by writing down the worldvolume effective action of the 5^2_2-brane. In this paper, assuming the existence of some isometry directions, we construct effective actions for various mixed-symmetry tensors that couple to exotic branes. We consider the cases of the exotic 5^2_2-brane, the 1^6_4-brane, and the Dp_{7-p}-brane, and argue that these exotic branes are the magnetic sources of the non-geometric fluxes associated with polyvectors β^{ij}, $\beta^{i_1\cdots i_6}$, and $\gamma^{i_1\cdots i_{7-p}}$, respectively. As it is well-known, an exotic-brane background written in terms of the usual background fields is not single-valued and has a U-duality monodromy. However, with a suitable redefinition of the background fields, the U-duality monodromy of the exotic-brane background simply becomes a gauge transformation associated with a shift in a polyvector (which corresponds to a natural extension of the *β-transformation* known in the generalized geometry). This kind of field redefinition and the rewriting of the action in terms of the new background fields are the main tasks of this paper. In spite of the presence of a symmetric structure between the exotic branes and the usual branes, little is known about the exotic branes, the background fields which couple to the exotic branes have not been studied in detail, other than the case of the 5^2_2-brane. There exists an $SL(2,\mathbb{Z})$ duality group under which the standard branes of $n = 0, 1$ are mapped to the exotic branes of $n = 4, 3$ and vice versa, and the solitonic branes of $n = 2$ are mapped to other solitonic branes. This duality group is a subgroup of the U-duality group in each dimension. This is referred to as the *exotic duality*. Even though the U-duality group in a certain spacetime dimension is different from that of a different

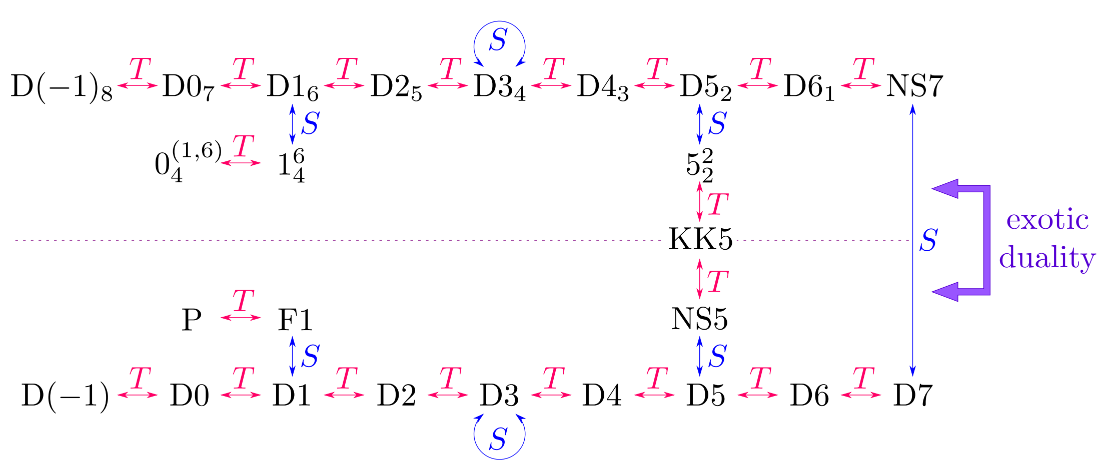
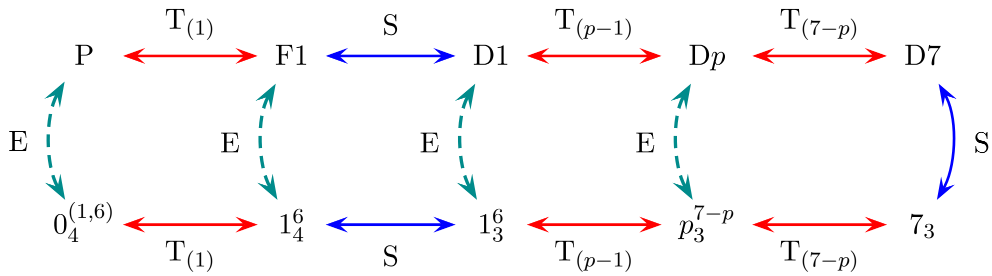
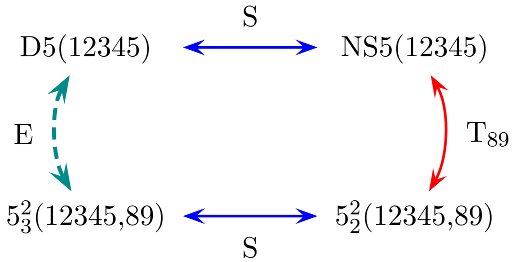


Figure 1: A family of exotic branes and the duality web.

dimension, any exotic duality is described by $SL(2,\mathbb{Z})$. This duality is illustrated in



Exotic duality, S-duality and T-duality along k directions among the standard branes and the exotic branes.



Exotic duality between D5-brane and 5^2_3-brane. It is remarked that this T_{89}-duality between NS5-brane and 5^2_2-brane is also interpreted as an exotic duality.

The main research interest in this article is to identify the background fields which couple to the exotic branes and to write down the effective supergravity action for the background fields. For the 5^2_2-brane, the relevant background field is a bi-vector β^{ij} which is a function of the standard NS-NS fields. The effective theory for the β-field has been constructed in a series of works and is called the *β-supergravity*. On the other hand, for the $\mathrm{D}p_{7-p}$-brane, the relevant background field is expected to be a $(7-p)$-vector $\gamma^{i_1\cdots i_{7-p}}$ whose derivative is called the non-geometric *P-flux*, where the γ-fields are introduced in the study of the exceptional generalized geometry, where the relation between mixed-symmetry tensors and exotic branes is discussed, the effective $\mathrm{D}5_2$-brane action is written down and the $\mathrm{D}5_2$-brane is found to couple to a bi-vector γ^{ij} magnetically, and where a possible relation between the polyvectors γ and exotic branes is discussed. However, the definition of the γ-fields and the effective action for the γ-fields are still not fully understood. Our results and those obtained could be extended to other relativistic superalgebras. Indeed, it seems that the $S_E^{(4)}$ semigroup allows to obtain the respective Newtonian version of a relativistic (super)algebra. In particular, the procedure used here could be useful in presence of supersymmetry, where the study of the non-relativistic limit is highly non-trivial. It is interesting to notice that the exotic Newtonian superalgebra can alternatively be recovered by expanding the enhanced Nappi-Witten superalgebra. Although both methods are based on the semigroup expansion method, they present subtle differences which could lead to diverse extensions of our results. Indeed, to obtain diverse Newtonian superalgebras from an enhanced Nappi-Witten (super)algebra, we need to consider diverse semigroups. On the other hand, the derivation of various Newtonian (super)algebras by expanding a relativistic superalgebra requires to consider different original algebras without modifying the semigroup. We first exhibit the supersymmetry projection rules on the standard branes in type II superstring theories and M-theory. Following the rules, we introduce the superstring dualities acting on the supersymmetry parameters. Using the superstring dualities, we write down the rules on various exotic branes. To avoid complications, we do not write down the concrete derivation of each exotic brane in this section. Next, we apply the supersymmetry projection rules to certain brane configurations derived from an F-string ending on a D3-brane. Analogous to the superstring dualities on the mass formulae of branes, we do not seriously consider their global structures.

2 Generalised Metrics, Projectors and the Extremal $E_{8(8)}$ Vacua

2.1 Generalised metrics and diffeomorphisms

The local symmetries of general relativity, double field theory and exceptional field theory can all be treated in same manner, by defining (generalised) diffeomorphisms associated to a group G. For general relativity, this group is $G = \mathrm{GL}(d)$, for DFT, it is $G = \mathrm{O}(d, d)$, and for ExFT, it is $E_{d(d)}$. We work with coordinates (X^μ, Y^M), where $\mu = 1, \ldots, n$ and Y^M transform in what we call the R_1 representation of G. In DFT and ExFT, we will call the X^μ coordinates "external" and the Y^M "internal" or "extended", mimicking the language we would use if we reduced to an n-dimensional theory (however no compactification is assumed or needed to formulate these theories). The R_1 representation is the d-dimensional fundamental of $\mathrm{GL}(d)$ in the case of general relativity, the $2d$-dimensional fundamental in the case of $\mathrm{O}(d, d)$, and for $E_{d(d)}$ the representations are listed with the rule is that R_1 is the representation whose highest weight is the fundamental weight associated to the rightmost node on the Dynkin diagram.

We define diffeomorphisms associated to the transformation of the coordinates $\delta Y^{\mathcal{M}} = -\Lambda^{\mathcal{M}}$ in terms of a Lie derivative acting on vectors $\delta_\Lambda V^{\mathcal{M}} = \mathcal{L}_\Lambda V^{\mathcal{M}}$ by

$$\mathcal{L}_\Lambda V^{\mathcal{M}} = \Lambda^{\mathcal{N}}\partial_{\mathcal{N}} V^{\mathcal{M}} - \alpha \mathbb{P}_{adj}{}^{\mathcal{M}}{}_{\mathcal{K}}{}^{\mathcal{N}}{}_{\mathcal{L}} \partial_{\mathcal{N}}\Lambda^{\mathcal{L}} V^{\mathcal{K}} + \lambda \partial_{\mathcal{K}}\Lambda^{\mathcal{K}} V^{\mathcal{M}} \,, \tag{2.1}$$

where $\mathbb{P}_{adj}{}^{\mathcal{M}}{}_{\mathcal{K}}{}^{\mathcal{N}}{}_{\mathcal{L}}$ denotes the projector from $R_1 \otimes \bar{R}_1$ onto the adjoint representation, α is a constant which depends on the group under consideration and λ denotes the weight of $V^{\mathcal{M}}$. It is often useful to expand the projector to obtain an equivalent form of the generalised Lie derivative:

$$\mathcal{L}_\Lambda V^{\mathcal{M}} = \Lambda^{\mathcal{N}}\partial_{\mathcal{N}} V^{\mathcal{M}} - V^{\mathcal{N}}\partial_{\mathcal{N}}\Lambda^{\mathcal{M}} + Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}\mathcal{L}} \partial_{\mathcal{N}}\Lambda^{\mathcal{K}} V^{\mathcal{L}} + (\lambda + \omega)\partial_{\mathcal{K}}\Lambda^{\mathcal{K}} V^{\mathcal{M}} \,, \tag{2.2}$$

which makes apparent how the structure differs from the ordinary Lie derivative (which is given by the first two terms). The modification involves the so-called Y-tensor, which is constructed out of group invariants (for instance, for $\mathrm{O}(d, d)$, $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}\mathcal{L}} = \eta^{\mathcal{M}\mathcal{N}}\eta_{\mathcal{K}\mathcal{L}}$), and also a constant ω which can be thought of as an intrinsic weight. When $G = \mathrm{GL}(d)$, clearly $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}\mathcal{L}} = 0$ and $\omega = 0$.

We could define the ordinary Lie derivative involving two ten-dimensional generalised vectors, but this would give a $\mathrm{GL}(10)$ Lie derivative and not capture the symmetries we want. Instead, let's think about the group $\mathrm{SL}(5)$. This has the totally antisymmetric invariants $\epsilon_{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}}$ and $\epsilon^{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}}$. A generalised Lie derivative which preserves these invariants is defined by

$$\mathcal{L}_\Lambda W^{\mathcal{M}} = \frac{1}{2}\Lambda^{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{P}\mathcal{Q}} W^{\mathcal{M}} - W^{\mathcal{P}}\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{M}\mathcal{Q}} + \frac{1}{5}\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{P}\mathcal{Q}} W^{\mathcal{M}} \,, \tag{2.3}$$

acting on a field $W^{\mathcal{M}}$ carrying a single five-dimensional index. The factor of 1/2 in the first term is inserted to prevent overcounting. Using the Leibniz rule, this implies on a second generalised vector $V^{\mathcal{M}\mathcal{N}}$ we have:

$$\begin{aligned}\mathcal{L}_\Lambda V^{\mathcal{M}\mathcal{N}} = &\frac{1}{2}\Lambda^{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{P}\mathcal{Q}} V^{\mathcal{M}\mathcal{N}} - \frac{1}{2} V^{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{M}\mathcal{N}} \\ &+ \frac{1}{8}\epsilon^{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{T}}\epsilon_{\mathcal{K}\mathcal{L}\mathcal{R}\mathcal{S}\mathcal{T}}\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{K}\mathcal{L}} V^{\mathcal{R}\mathcal{S}} - \frac{1}{5}\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{P}\mathcal{Q}} V^{\mathcal{M}\mathcal{N}} \,,\end{aligned} \tag{2.4}$$

or in terms of a single 10-dimensional index $M \equiv [\mathcal{M}\mathcal{N}]$, letting $V^M \equiv V^{\mathcal{M}\mathcal{N}}$, $\Lambda^M \equiv \Lambda^{\mathcal{M}\mathcal{N}}$, we can write

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - V^N \partial_N \Lambda^M + Y^{MN}{}_{PQ}\partial_N \Lambda^P V^Q - \frac{1}{5}\partial_N \Lambda^N V^M \,. \tag{2.5}$$

letting $Y^{MN}{}_{PQ} \equiv \epsilon^{MN\mathcal{K}}\epsilon_{PQ\mathcal{K}}$. The final term with the $\frac{1}{5}$ coefficient is a consequence of choosing to define an $\mathrm{SL}(5)$ rather than $\mathrm{GL}(5)$ Lie derivative. In practice it is convenient to eliminate this from many expressions by declaring all generalised vectors to have weight $+\frac{1}{5}$ (note this means ∂_M has weight $-\frac{1}{5}$).

The consistency of the theory, in particular closure of the algebra generated by generalised Lie derivatives, again requires a section condition, which this time takes the form:

$$\epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\partial_{\mathcal{PQ}}\Psi = 0\,, \quad \epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\Psi\partial_{\mathcal{PQ}}\Psi' = 0\,, \tag{2.6}$$

where again Ψ, Ψ' stand for any quantities in the theory.

The solution of this constraint which returns us to the d-dimensional theory is $\tilde{\partial}^{ij} = 0$. Again, though the requirement $\tilde{\partial}^{ij} = 0$ appears to be of the same nature as the Kaluza-Klein truncation condition, this is really a more stringent condition.

The geometry of general relativity is, of course, described by a metric. Similarly the generalised, or "extended", geometry of DFT/ExFT will be described by a generalised metric. We define this to be a symmetric matrix, $\mathcal{M}_{MN}$, which is an element of G and so preserves the appropriate invariant tensors. The generalised Lie derivative of the generalised metric follows from (2.1) or (2.2) using the Leibniz property. It takes the form:

$$\delta_\Lambda \mathcal{M}_{MN} = \Lambda^P\partial_P\mathcal{M}_{MN} + 2\alpha P_{MN}{}^{KL}\partial_K\Lambda^P\mathcal{M}_{LP}\,, \tag{2.7}$$

in which the following projector appears:

$$P_{MN}{}^{KL} = \frac{1}{\alpha}\left(\delta_M^{(K}\delta_N^{L)} - \omega\mathcal{M}_{MN}\mathcal{M}^{KL} - \mathcal{M}_{MQ}Y^{Q(K}{}_{RN}\mathcal{M}^{L)R}\right)\,, \tag{2.8}$$

or in terms of the adjoint projector,

$$P_{MN}{}^{KL} = \mathcal{M}_{MQ}\mathbb{P}_{adj}{}^Q{}_N{}^{(K}{}_R\mathcal{M}^{L)R}\,. \tag{2.9}$$

Note that as the Y-tensor, or equivalently the adjoint projector, is a group invariant it is preserved by the simultaneous action of $\mathcal{M}$ and $\mathcal{M}^{-1}$ on all four indices, which can be used to check that $P_{MN}{}^{KL}$ is actually symmetric in both its upper and lower pairs of indices. We can think of equation (2.7) as expressing the variation of the generalised metric, in terms of a parameter $\partial_{(K}\Lambda^P\mathcal{M}_{L)P}$, which is then projected from the symmetric tensor product of R_1 with itself into the space in which $\mathcal{M}_{MN}$ lives by means of $P_{MN}{}^{KL}$. Generically, $\mathcal{M}_{MN}$ is in fact valued in a coset G/H.

We can calculate the trace of the projector to compute the number of independent components of the generalised metric, i.e. the dimension of the coset G/H in which it lives. In general, we find:

$$P_{MN}{}^{MN} = \frac{1}{2\alpha}\left(\dim R_1(\dim R_1 + 1 - 2\omega) - Y^{MN}{}_{MN} - \mathcal{M}_{MN}Y^{MN}{}_{PQ}\mathcal{M}^{PQ}\right)\,. \tag{2.10}$$

Evidently, in general relativity we have $\alpha = 1$, and the terms in (2.8) involving ω and the Y-tensor do not appear. Hence we find $P_{MN}{}^{MN} = \frac{1}{2}d(d+1)$ which is the number of independent components of a symmetric matrix and also the dimension of the coset $\mathrm{GL}(d)/\mathrm{SO}(d)$.

In DFT and ExFT the situation is rather more interesting. Part of the trace (2.10) is independent of the generalised metric and follows from representation theory as the Y-tensor can be related to the projector onto the R_2 representation. For $d = 4$ to $d = 6$ it is directly proportional to this projector, and we find that its trace is $Y^{MN}{}_{MN} = 2(d-1)\dim R_2$. For $d = 7$, an additional term appears in the Y-tensor involving the antisymmetric invariant of $E_{7(7)}$ (i.e. a projector onto also the trivial representation) and in this case $Y^{MN}{}_{MN} = 2(d-1)\dim R_2 - \dim R_1/2$. For $d = 8$, the situation changes again and the trace does not have quite such a simple expression.

The crucial information about the coset then appears in the very final term in (2.10), which we may single out and define as

$$r \equiv \frac{1}{2\alpha}\mathcal{M}_{MN}Y^{MN}{}_{KL}\mathcal{M}^{KL}\,. \tag{2.11}$$

One finds, as summarised, that for all groups except $E_{8(8)}$ the trace of the projector gives exactly the dimension of the usual G/H coset minus r. For $E_{8(8)}$ we obtain the dimension of $E_{8(8)}/\mathrm{SO}(16)$ plus $2/15$ minus r. It follows that non-zero r, if possible, generically corresponds to parametrisations in which there are fewer independent components of the generalised metric, signalling a coset G/H of lower dimension. Information about H can be introduced in the form of a generalised vielbein, $E_M{}^{\mathcal{A}}$, with a flat index $\mathcal{A}$ transforming under H. The generalised metric is then given $\mathcal{M}_{MN} = E_M{}^{\mathcal{A}} E_N{}^{\mathcal{B}} \mathcal{H}_{AB}$, with the flat metric $\mathcal{H}_{AB}$ which is left invariant by local H transformations. Using the group properties of the generalised vielbein (it must preserve the Y-tensor), it is then possible to explicitly evaluate r, as we will see below for $E_{8(8)}$ in section 2.2.

We can write the action most compactly by introducing a ten-by-ten representation of the generalised metric

$$\mathcal{M}_{MN} \equiv \mathcal{M}_{\mathcal{M}\mathcal{M}',\mathcal{N}\mathcal{N}'} = 2m_{\mathcal{M}[\mathcal{N}} m_{\mathcal{N}']\mathcal{M}'} \,. \tag{2.12}$$

The parametrisation of $\mathcal{M}_{MN}$ resulting is

$$\mathcal{M}_{MN} = |g|^{\frac{1}{5}} \begin{pmatrix} g_{ik} + \frac{1}{2} C_{imn} C_k{}^{mn} & \frac{1}{2} C_i{}^{mn} \epsilon_{klmn} \\ \frac{1}{2} C_k{}^{mn} \epsilon_{ijmn} & 2|g| g_{i[k} g_{l]j} \end{pmatrix} , \tag{2.13}$$

where indices on the three-form are raised using g^{ij}.

This is the direct generalisation of the $O(d,d)$ generalised metric. Using $\mathcal{M}_{MN}$ and Δ, we can then search for a quantity quadratic in derivatives which is a scalar under generalised diffeomorphisms (up to terms vanishing by the section condition). The result leads to the action

$$\begin{aligned} S = \int \mathrm{d}^{10} X \, e^{-2\Delta} \Big(& \frac{1}{12} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} - \frac{1}{2} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_K \mathcal{M}_{LN} \\ & + \frac{24}{7} \left(\partial_M \mathcal{M}^{MN} \partial_N \Delta - \mathcal{M}^{MN} \partial_M \Delta \partial_N \Delta + \mathcal{M}^{MN} \partial_M \partial_N \Delta \right) \\ & - \partial_M \partial_N \mathcal{M}^{MN} \Big) \,. \end{aligned} \tag{2.14}$$

However, this does not rule out the possibility of finding alternative parametrisations of the generalised metric which correspond to new cosets G/H of lower dimension. Indeed, this underlies the non-Riemannian parametrisations of [35], which we will review from the perspective of the projector $P_{MN}{}^{KL}$ in section 4, and will appear below in an interesting context for the $E_{8(8)}$ ExFT.

Let us now discuss the dynamics of the generalised metric. Its equations of motion follow from the ExFT action, which is constructed using the requirement of invariance under the local symmetries of ExFT. These include not only generalised diffeomorphisms but also external diffeomorphisms associated to transformations of the coordinates X^μ, and various generalised gauge transformations of gauge fields that also appear in the theory. The projector then plays a vital role in the equations of motion for the generalised metric. (Here we are thinking only of the bosonic part of the action: if we include fermions then we will have to use a projector onto the variation of the generalised vielbein. We will comment more on this later.) In fact, it was in this context that the projector was first written down where it was obtained for the groups SL(5) and SO(5,5) by explicitly varying known parametrisations of the generalised metric. When one varies the action with respect to $\mathcal{M}_{MN}$, one naively obtains an expression of the form

$$\delta S = \int \delta \mathcal{M}^{MN} \mathcal{K}_{MN} \,, \qquad \mathcal{K}_{MN} \equiv \frac{\delta S}{\delta \mathcal{M}^{MN}} \tag{2.15}$$

but the true equations of motion are

$$P_{MN}{}^{KL} \mathcal{K}_{KL} = 0 \,. \tag{2.16}$$

The reason for this is that one must insist that the variations of the generalised metric $\delta\mathcal{M}^{MN}$ are still compatible with G and so we impose this by a projector. In the standard formulation of ExFT, the actions do not explicitly impose this and so one needs to include these projectors by hand though it is equivalent to just calculating the variations of the action subject to G-compatibility.

Now, recalling that the projector depends on $\mathcal{M}_{MN}$, we might consider whether it is possible to find a generalised metric such that the projector vanishes:

$$P_{MN}{}^{KL} = 0\,, \tag{2.17}$$

meaning the equations of motion (2.16) are trivially obeyed. This is evidently a very special possibility. It corresponds to changing the structure of the theory such that the coset is G/G. Furthermore, as *any* variation of the generalised metric must be projected, $\delta\mathcal{M}_{MN} = P_{MN}{}^{KL}\delta\mathcal{M}_{KL}$, there can be no fluctuations about such a background.

For $\mathrm{O}(d,d)$, the "maximally non-Riemannian" background $\mathcal{H}_{MN} = \eta_{MN}$ is of this type [35]. This background is invariant under $\mathrm{O}(d,d)$, i.e. it corresponds to a symmetric invariant tensor of the group. This characterisation is easy to search for in ExFT, where the symmetric product of R_1 with itself does not contain the trivial representation for any $E_{d(d)}$ except for $d = 8$. For $E_{8(8)}$ we have $R_1 = \mathbf{248}$, which is the adjoint representation and there is an obvious symmetric quadratic invariant given by the Killing form. We will now discuss this ExFT and what one can say about the non-Riemannian background where the generalised metric is proportional to the Killing form.

2.2 The $E_{8(8)}$ ExFT and its topological phase

Generalised Diffeomorphisms and the Action

The $E_{8(8)}$ ExFT [44] is based on an extended geometry parametrised by 248 coordinates Y^M valued therefore in the adjoint of $E_{8(8)}$. Denoting its generators as T^M, we define structure constants $f^{MN}{}_K$ with the convention $[T^M, T^N] = -f^{MN}{}_K T^K$, and the Killing form by

$$\kappa^{MN} \equiv \frac{1}{60}\,\mathrm{Tr}(T^M T^N) = \frac{1}{60} f^{MP}{}_Q f^{NQ}{}_P\,. \tag{2.18}$$

We freely raise and lower all indices using κ^{MN} and its inverse κ_{MN}.

The generalised Lie derivative of an adjoint vector of weight λ is explicitly given by

$$\mathcal{L}_\Lambda V^M = \Lambda^K\partial_K V^M - 60(\mathbb{P}_{\mathbf{248}})^M{}_K{}^N{}_L\partial_N\Lambda^L V^K + \lambda(V)\partial_N\Lambda^N V^M \tag{2.19}$$

in which we have used the projector onto the adjoint representation $(\mathbb{P}_{\mathbf{248}})^M{}_K{}^N{}_L$ defined by

$$(\mathbb{P}_{\mathbf{248}})^M{}_K{}^N{}_L = \frac{1}{60} f^M{}_{KP} f^{PN}{}_L. \tag{2.20}$$

Alternatively, one can write the part of this transformation involving Λ^M in the form (2.2) involving the Y-tensor, given here by

$$Y^{MN}{}_{KL} = -f^M{}_{LP} f^{PN}{}_K + 2\delta_K^{(M}\delta_L^{N)}\,. \tag{2.21}$$

A special feature of the $E_{8(8)}$ ExFT is that it includes additional gauge transformations which appear alongside the conventional generalised Lie derivative. Under this extra gauge symmetry, generalised vectors transform as

$$\delta_\Sigma V^M = -\Sigma_L f^{LM}{}_N V^N\,, \tag{2.22}$$

where the gauge parameter Σ_M is not an arbitrary covector but is constrained as part of the section condition of the $E_{8(8)}$ ExFT. This section condition applies to any two quantities F_M, F'_M which are said to be "covariantly constrained" meaning that they vanish when their tensor product is projected into the $\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875} \subset \mathbf{248} \otimes \mathbf{248}$, i.e.

$$\kappa^{MN} F_M \otimes F'_N = 0\,, \quad f^{MNK} F_N \otimes F'_K = 0\,, \quad (\mathbb{P}_{\mathbf{3875}})^{KL}{}_{MN} F_K \otimes F'_L = 0\,. \tag{2.23}$$

These quantities include derivatives, ∂_M, as usual, the gauge parameters Σ_M, and a number of other gauge parameters and field [44].

This section condition guarantees closure of the algebra of the combined action of generalised diffeomorphisms and constrained Σ_M transformations, which we denote by

$$\mathbb{L}_{(\Lambda,\Sigma)} \equiv \mathcal{L}_\Lambda + \delta_\Sigma \,. \tag{2.24}$$

The inclusion of the Σ_M transformations is in fact necessary for closure: the algebra based on the ordinary generalised Lie derivative (2.19) alone cannot be made to close on its own. The underlying physical reason for the extra gauge transformation (2.22) is the appearance of dual graviton degrees of freedom in the generalised metric of the $E_{8(8)}$ ExFT. For further details on these subtleties, we refer the reader to the original paper [44] or the recent review.

We proceed to discuss the field content of the theory. This consists of the generalised metric, $\mathcal{M}_{MN}$, an external metric, $g_{\mu\nu}$, and a pair of gauge fields $(\mathcal{A}_\mu{}^M, \mathcal{B}_{\mu M})$, with $\mathcal{B}_{\mu M}$covariantly constrained as in (2.23). These gauge fields have field strengths $(\mathcal{F}_{\mu\nu}{}^M, \mathcal{G}_{\mu\nu M})$ whose precise forms can be found in [44]. All these fields depend on the three-dimensional coordinates X^μ as well as the 248-dimensional coordinates Y^M, subject to the section condition. The gauge field $\mathcal{A}_\mu{}^M$ can be thought of as serving as a gauge field for generalised diffeomorphisms while $\mathcal{B}_{\mu M}$ is a gauge field for the constrained Σ_M transformations. We define an improved derivative $D_\mu \equiv \partial_\mu - \mathbb{L}_{(\mathcal{A}_\mu,\mathcal{B}_\mu)}$ which is used in place of ∂_μ. The action for the $E_{8(8)}$ ExFT is constructed in [44] and is given by

$$S = \int \mathrm{d}^3x \mathrm{d}^{248}Y \sqrt{|g|}\left(\hat{R}[g] + \frac{1}{240} g^{\mu\nu} D_\mu \mathcal{M}_{MN} D_\nu \mathcal{M}^{MN} - V(\mathcal{M},g) + \frac{1}{\sqrt{|g|}}\mathcal{L}_{CS}\right) \tag{2.25}$$

where $\hat{R}[g]$ is the usual Ricci scalar for the metric $g_{\mu\nu}$, except constructed in terms of D_μ instead of ∂_μ. The two terms at the end are:

$$\begin{aligned} V(\mathcal{M},g) = &-\frac{1}{240}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} \\ &+ \frac{1}{7200} f^{NQ}{}_P f^{MS}{}_R \mathcal{M}^{PK}\partial_M\mathcal{M}_{QK}\mathcal{M}^{RL}\partial_N\mathcal{M}_{SL} \\ &- \frac{1}{2}\partial_M \ln|g| \partial_N \mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}\left(\partial_M \ln|g|\partial_N \ln|g| + \partial_M g^{\mu\nu}\partial_N g_{\mu\nu}\right) , \end{aligned} \tag{2.26}$$

which is usually referred to as the "potential", taking the point of view of the external three-dimensional space, and the Chern-Simons term:

$$S_{\text{CS}} \sim \int_{\Sigma^4} \mathrm{d}^4x \int \mathrm{d}^{248}Y \left(\mathcal{F}^M \wedge \mathcal{G}_M - \frac{1}{2} f_{MN}{}^K \mathcal{F}^M \wedge \partial_K \mathcal{G}^N\right) \tag{2.27}$$

written here in a manifestly gauge invariant form using the usual construction of an auxiliary space Σ^4 whose boundary $\partial\Sigma^4$ is the physical three-dimensional space, and where $\wedge$ denotes the usual product with respect to the external indices, $\mu, \nu, \ldots$.

Generalised metric and projector

Conventionally, we view the generalised metric as being an element of $E_{8(8)}/H$, with $H = \mathrm{SO}(16)$, and then this coset is parametrised in terms of a spacetime metric and p-form fields. Instead, following the intuition from the DFT approach of [35] where the generalised metric was defined as a symmetric two index object obeying the $\mathrm{O}(d,d)$ compatibility condition, we will define the $E_{8(8)}$ generalised metric by the properties that are needed in [44] to ensure the invariance of the action (2.25). Thus we define the $E_{8(8)}$ generalised metric to be the symmetric two index object that obeys the constraints:

$$\mathcal{M}_{MK}\mathcal{M}_{NL}\mathcal{M}_{PQ}f^{KLQ} = -f_{MNP}\,, \quad \mathcal{M}_{MK}\kappa^{KL}\mathcal{M}_{LN} = \kappa_{MN}\,. \tag{2.28}$$

One can check that the conventional coset parametrisation of $\mathcal{M}_{MK}$ obeys these constraints but new results will follow from a solution to these constraints that does not obey the coset parametrisation. The full generalised Lie derivative (including the additional transformations involving Σ_M) of the generalised metric takes the form

$$\mathbb{L}_{(\Lambda,\Sigma)}\mathcal{M}_{MN} = \Lambda^P\partial_P\mathcal{M}_{MN} + 2\cdot 60 P_{MN}{}^{KL}\left(\partial_K\Lambda^P + \frac{1}{60}f^{QP}{}_K\Sigma_Q\right)\mathcal{M}_{PL}\,, \tag{2.29}$$

with the projector given simply by

$$P_{MN}{}^{KL} = \frac{1}{60}\mathcal{M}_{MQ}f^Q{}_{NP}f^{P(K}{}_R\mathcal{M}^{L)R}\,. \tag{2.30}$$

The trace is

$$P_{MN}{}^{MN} = \frac{1}{2}\left(\kappa^{MN}\mathcal{M}_{MN} + 248\right)\,. \tag{2.31}$$

Now, for the usual $E_{8(8)}/\mathrm{SO}(16)$ coset, we introduce a generalised vielbein $E_M{}^{\mathcal{A}}$ such that

$$E_M{}^{\mathcal{A}} \equiv (E_M{}^A, E_M{}^{IJ})\,, \quad \kappa^{MN}E_M{}^AE_N{}^B = \delta^{AB}\,, \quad \kappa^{MN}E_M{}^{IJ}E_N{}^{KL} = -2\delta^{I[K}\delta^{L]J}\,, \tag{2.32}$$

where A is a spinor index corresponding to the **128** of SO(16), and I the 16-dimensional vector representation, with $E_M{}^{IJ} = -E_M{}^{JI}$ in the **120** of SO(16). The generalised metric is then given by $\mathcal{M}_{MN} = E_M{}^AE_N{}^B\delta_{AB} + \frac{1}{2}E_M{}^{IJ}E_N{}^{KL}\delta_{IK}\delta_{JL}$ and it follows from the defining properties of the vielbein that $\kappa^{MN}\mathcal{M}_{MN} = 128 - 120 = 8$. Thus we find $P_{MN}{}^{MN} = 128$ as expected.

Now we can consider whether there are alternative parametrisations of $\mathcal{M}_{MN}$ such that $P_{MN}{}^{MN} \neq 128$. Remarkably, we can immediately write down a choice of $\mathcal{M}_{MN}$ such that $P_{MN}{}^{KL}$ vanishes identically, given by

$$\mathcal{M}_{MN} = -\kappa_{MN}\,. \tag{2.33}$$

This is easily checked to be compatible with the defining constraints (2.28) for $\mathcal{M}_{MN}$ (no other multiple of the Killing form is). The projector then vanishes as $f^{P(KL)} = 0$.

Restricting to the "topological phase"

Now let us consider what this implies for the equations of motion. On general grounds, as we have explained, the equations of motion of $\mathcal{M}_{MN}$ itself will be of the form $P_{MN}{}^{KL}\mathcal{K}_{KL} = 0$, where $\mathcal{K}_{MN}$ is the result of varying the action with respect to $\mathcal{M}^{MN}$. As the projector vanishes for $\mathcal{M}_{MN} = -\kappa_{MN}$, the equations of motion are trivially obeyed.

Now consider the variation of the other fields in the action. For instance, the equation of motion of the external metric is:

$$\begin{aligned}0 = \hat{R}_{\mu\nu} &- \frac{1}{2}g_{\mu\nu}\left(\hat{R}[g] + \frac{1}{240}g^{\rho\sigma}D_\rho\mathcal{M}_{MN}D_\sigma\mathcal{M}^{MN} - V(\mathcal{M},g)\right)\\ &+ \frac{1}{240}D_\mu\mathcal{M}_{MN}D_\nu\mathcal{M}^{MN} + \frac{1}{2}\sqrt{|g|}^{-1}g_{\mu\nu}\partial_M\left(\sqrt{|g|}(\partial_N\mathcal{M}^{MN} + \mathcal{M}^{MN}\partial_N\ln|g|)\right)\\ &- \frac{1}{2}\sqrt{|g|}^{-1}\partial_M(\sqrt{|g|}\mathcal{M}^{MN})\partial_N g_{\mu\nu} - \frac{1}{2}\mathcal{M}^{MN}g_{\mu\rho}\partial_M g^{\rho\sigma}\partial_N g_{\sigma\nu} - \frac{1}{2}\mathcal{M}^{MN}\partial_M\partial_N g_{\mu\nu}\,.\end{aligned} \tag{2.34}$$

Here $\hat{R}_{\mu\nu}$ is defined to be the result of varying $\hat{R}[g]$ with respect to $g_{\mu\nu}$. Now, when $\mathcal{M}_{MN} = -\kappa_{MN}$ all terms involving the generalised metric vanish identically, either because $D_\mu \kappa_{MN} = 0$ (as the generalised Lie derivative appearing in the definition of D_μ preserves the Killing form) or because of the section condition $\kappa^{MN}\partial_M \otimes \partial_N = 0$. Similarly, the equations of motion of the gauge fields $\mathcal{A}_\mu{}^M$, $\mathcal{B}_{\mu M}$ will involve $\mathcal{M}_{MN}$ only in the form of (derivatives of) $D_\mu \mathcal{M}_{MN}$, and so the contribution of the generalised metric to these equations of motion also vanishes identically.

We can conclude that the equations of motion for $(g_{\mu\nu}, \mathcal{A}_\mu{}^M, \mathcal{B}_{\mu M})$ when $\mathcal{M}_{MN} = -\kappa_{MN}$ are those that are obtained from the truncation of the ExFT action obtained by setting $\mathcal{M}_{MN} = -\kappa_{MN}$ within the action, i.e. in this background the dynamics of the resulting fields are governed by:

$$S = \int \mathrm{d}^3x\, \mathrm{d}^{248}Y \sqrt{|g|}\hat{R}[g] + \int_{\Sigma^4} \mathrm{d}^4x\, \mathrm{d}^{248}Y \left(\mathcal{F}^M \wedge \mathcal{G}_M - \frac{1}{2} f_{MN}{}^K \mathcal{F}^M \wedge \partial_K \mathcal{G}^N \right) . \tag{2.35}$$

Let us make a short comment about the fermions of the $E_{8(8)}$ ExFT. We would expect that after truncating the generalised metric degrees of freedom that we should also truncate out the internal fermions. At this point the supersymmetry of the non-Riemannian background is a little mysterious since usually in ExFT the fermions should transform in a representation of H. What this means when $H = E_{8(8)}$ is uncertain but what is apparent is that one cannot just naively insert the condition $\mathcal{M}_{MN} = -\kappa_{MN}$ into the generalised Killing spinor equations. The realisation of fermions in the non-Riemannian background has yet to be determined. Note that the variation of the action with respect to the generalised vielbein, $E_M{}^A$, requires a projector to ensure that $\delta E_M{}^A$ is not arbitrary. Evidently this projector will depend explicitly on the precise form of H (whereas the projector $P_{MN}{}^{KL}$ acting on variations of the generalised metric only knew about H implicitly, through the term $\mathcal{M}_{MN} Y^{MN}{}_{KL} \mathcal{M}^{KL}$) and so must be constructed on a case-by-case basis when starting from a particular non-Riemannian parametrisation of $\mathcal{M}_{MN}$.

A related technical comment is to note that setting $\mathcal{M}_{MN} = -\kappa_{MN}$ is consistent with the invariance of the ExFT action under external diffeomorphisms with parameter $\xi^\mu(X, Y)$, which includes a generalised metric dependent transformation of $\mathcal{A}_\mu{}^M$, namely

$$\delta_\xi \mathcal{A}_\mu{}^M \supset \mathcal{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu . \tag{2.36}$$

Normally, this requires cross-cancellation between the scalar potential and the other parts of the action. If this vanishes, $V(\mathcal{M} = -\kappa, g) = 0$, then one might be concerned whether the action is still invariant. However, when one inspects the calculation in [44] of the variation of the action under these transformations, one finds that all possible terms that could spoil invariance vanish by the section condition on setting $\mathcal{M}^{MN} = -\kappa^{MN}$.

3 Gauge Invariance of the Pseudo-Lagrangian

We now show that the E_{11} exceptional field theory pseudo-Lagrangian given is gauge-invariant. For this we calculate the variation of each term in the pseudo-Lagrangian under generalised diffeomorphisms and then demonstrate that the combination of these variations vanishes. As always in these checks in exceptional field theory it is sufficient to show that the non-covariant gauge variation Δ_ξ vanishes up to total derivatives. Our proof proceeds in two steps. In order to underline the necessity of including the fields ζ_M, we first consider the pseudo-Lagrangian for $\zeta_M = 0$ and computes its non-covariant gauge variation. As we shall see there are already many cancellations but some terms are left over. Then we shall show that these terms are exactly cancelled by the ζ_M-dependent terms.

3.1 Gauge variation at $\zeta = 0$

As explained above, we compute first the non-covariant gauge variation of all the pieces of $\mathcal{L}$ at $\zeta_M = 0$.

First potential term

The first potential term, does not depend on ζ_M and we can immediately calculate the full non-covariant gauge variation. A standard exceptional field theory calculation involving the definition of the current ${J_M}^{\alpha}$ and the section constraint gives the first step

$$\begin{aligned}\Delta_\xi\Big[\mathcal{L}_{\text{pot}_1}\Big] &= \Big[-{T_\beta}^R{}_Q\mathcal{M}^{MN} + {T_\beta}^M{}_P\delta^N_Q\mathcal{M}^{PR} + f_{\beta\alpha\gamma}T^{\gamma M}{}_PT^{\alpha R}{}_Q\mathcal{M}^{NP}\Big]\partial_M\partial_R\xi^Q{J_N}^\beta \\ &= \mathcal{M}C_{P\widehat{\alpha}}C_{Q\beta}T^{\widehat{\alpha}R}{}_S\mathcal{M}^{QM}\mathcal{M}^{PN}\partial_M\partial_R\xi^S{J_N}^\beta - 2\partial_{[M}\Big(\partial_{N]}\partial_P\xi^M\mathcal{M}^{NP}\Big)\,.\end{aligned} \tag{3.1}$$

In the second step we have used the identity and simplified the terms with a single representation matrix T_β and a single inverse $\mathcal{M}^{MN}$ into a total derivative.

It is worthwhile to remark that the E_{11}-representation with index has as lowest component $R(\Lambda_3)$. When decomposing E_{11} with respect to $GL(11-n)\times E_n$ the first time this representation enters the scalar sector is for E_8 which is in agreement with the fact that this is the first time the potential term is not gauge-invariant and also the first time ancillary transformations are needed. We shall show next how the failure of gauge-invariance of the first potential term involving the index is accounted for by the second potential term.

Second potential term

The second potential term does not depend on ζ_M either. Calculating the full non-covariant gauge transformation yields

$$\begin{aligned}\Delta_\xi\Big[\mathcal{L}_{\text{pot}_2}\Big] &= -\mathcal{M}C_{P\widehat{\alpha}}C_{Q\widehat{\beta}}\mathcal{M}^{QM}\mathcal{M}^{PN}T^{\widehat{\alpha}R}{}_S\Big(\partial_M\partial_R\xi^S + \mathcal{M}_{RU}\mathcal{M}^{ST}\partial_M\partial_T\xi^U\Big){J_N}^{\widehat{\beta}} \\ &\quad - \mathcal{M}C_{P\tilde{\alpha}}C_{Q\widehat{\beta}}\mathcal{M}^{QM}\mathcal{M}^{PN}\Pi^{\tilde{\alpha}}{}_{RS}\mathcal{M}^{TR}\partial_M\partial_T\xi^S{J_N}^{\widehat{\beta}} \\ &= -\mathcal{M}C_{P\widehat{\alpha}}C_{Q\beta}T^{\widehat{\alpha}R}{}_S\mathcal{M}^{QM}\mathcal{M}^{PN}\partial_M\partial_R\xi^S{J_N}^\beta \\ &\quad - \mathcal{M}_{IJ}C^{IM}{}_{\tilde{\alpha}}C^{JN}{}_{\widehat{\beta}}\Pi^{\tilde{\alpha}}{}_{QP}\mathcal{M}^{QR}\partial_M\partial_R\xi^P{J_N}^{\widehat{\beta}}\,,\end{aligned} \tag{3.2}$$

where we have first written out the non-covariant variation $\Delta_\xi {J_M}^{\widehat{\alpha}}$. In the next step we have distributed the parenthesis on the first line and used the identity to cancel the second contribution

$$\begin{aligned}&\big(\mathcal{M}C_{P\widehat{\alpha}}T^{\widehat{\alpha}R}{}_S\,\mathcal{M}^{PN}\mathcal{M}^{ST}\mathcal{M}_{RU}\big)\,C_{Q\widehat{\beta}}\mathcal{M}^{QM}\partial_M\partial_T\xi^U{J_N}^{\widehat{\beta}} \\ =\;&\big(\eta C_{Q\widehat{\alpha}}T^{\widehat{\alpha}P}{}_R\,\eta^{QM}\eta^{RN}\eta_{PS}\big)\,C_{T\widehat{\beta}}\mathcal{M}^{TU}\partial_U\partial_N\xi^S{J_M}^{\widehat{\beta}} = 0\,,\end{aligned} \tag{3.3}$$

where we split the $\widehat{\beta}$ index on the first contribution and used the identities to remove the ${\chi_N}^{\tilde{\beta}}$ component.

The first term we obtain in (3.1) cancels precisely the contribution from the first potential term. This cancelation is the same one that ensures the invariance of the potential for any finite-dimensional simply laced groups. Consistently, the identity that was used in this cancelation is proved using a construction that generalises to the Kac–Moody algebra e_{11}. Here, we obtain the combined non-covariant gauge variation

$$\Delta_\xi\Big[\mathcal{L}_{\text{pot}_1} + \mathcal{L}_{\text{pot}_2}\Big] = -\mathcal{M}_{IJ}C^{IM}{}_{\tilde{\alpha}}C^{JN}{}_{\widehat{\beta}}\Pi^{\tilde{\alpha}}{}_{QP}\mathcal{M}^{QR}\partial_M\partial_R\xi^P{J_N}^{\widehat{\beta}} - 2\partial_{[M}\Big(\partial_{N]}\partial_P\xi^M\mathcal{M}^{NP}\Big)\,. \tag{3.4}$$

Thus, compared to present results where no $\Pi^{\tilde{\alpha}}{}_{MN}$ appears, the combination for E_{11} is not gauge-invariant and we shall invoke an additional ingredient to arrive at a gauge-invariant pseudo-Lagrangian.

Kinetic term at $\zeta = 0$

In order to determine the non-covariant gauge variation of the kinetic term we break it up into the parts that contain the constrained fields ζ_M (before variation) and those that do not, beginning with the latter:

$$\begin{aligned}
&\Delta_\xi \Big[\mathcal{L}_{\text{kin}}\big|_{\zeta=0} \Big] \\
&= \frac{1}{2} \mathcal{M}_{IJ} \Big(C^{JM}{}_{\hat{\alpha}} T^{\hat{\alpha}S}{}_Q \mathcal{M}_{SP} \mathcal{M}^{QR} + C^{JM}{}_{\tilde{\alpha}} \Pi^{\tilde{\alpha}}{}_{QP} \mathcal{M}^{QR} \Big) C^{IN}{}_{\hat{\beta}} J_N{}^{\hat{\beta}} \partial_M \partial_R \xi^P
\end{aligned} \tag{3.5}$$

where we have used the identity to cancel the term in $T^{\hat{\alpha}N}{}_P \partial_M \partial_N \xi^P$ from the non-covariant gauge variations.

Topological term at $\zeta = 0$

We first compute the non-covariant gauge transformation at $\zeta_M = 0$. An important first observation is that the total derivative $\Pi_{\tilde{\alpha}}{}^{MN} \partial_M \chi_N{}^{\tilde{\alpha}}$ is not invariant under its non-covariant gauge transformation up to a total derivative. To compute $\Delta_\xi = \delta_\xi - \mathcal{L}_\xi$ of $\Pi_{\tilde{\alpha}}{}^{MN} \partial_M \chi_N{}^{\tilde{\alpha}}$ we need to determine the Lie derivative of the combined object $\partial_M \chi_N{}^{\tilde{\alpha}}$ which is given by

$$\begin{aligned}
\mathcal{L}_\xi (\partial_M \chi_N{}^{\tilde{\alpha}}) = \xi^P \partial_P (\partial_M \chi_N{}^{\tilde{\alpha}}) + \partial_M \xi^P \partial_P \chi_N{}^{\tilde{\alpha}} + \partial_N \xi^P \partial_M \chi_P{}^{\tilde{\alpha}} \\
- T_\alpha{}^P{}_Q \partial_P \xi^Q (T^{\alpha\tilde{\alpha}}{}_{\tilde{\beta}} \partial_M \chi_N{}^{\tilde{\beta}} + K^{\alpha\tilde{\alpha}}{}_\beta \partial_M J_N{}^\beta) \,.
\end{aligned} \tag{3.6}$$

This not a total derivative. Therefore the non-covariant gauge variation is

$$\begin{aligned}
\Delta_\xi \Big[\Pi_{\tilde{\alpha}}{}^{MN} \partial_M \chi_N{}^{\tilde{\alpha}} \Big] &= \Pi_{\tilde{\alpha}}{}^{MN} \Big[\partial_M (\delta_\xi \chi_N^{\tilde{\alpha}}) - \mathcal{L}_\xi (\partial_M \chi_N{}^{\tilde{\alpha}}) \Big] \\
&= \Pi_{\tilde{\alpha}}{}^{MN} \Big[- T_\alpha{}^R{}_P T^{\alpha\tilde{\alpha}}{}_{\tilde{\beta}} \partial_M \partial_R \xi^P \chi_N{}^{\tilde{\beta}} \\
&\quad + \Big(- T^{\alpha R}{}_P K_\alpha{}^{\tilde{\alpha}}{}_\beta - T_\beta{}^U{}_Q T^{\tilde{\alpha}Q}{}_S \mathcal{M}_{UP} \mathcal{M}^{SR} + T^{\tilde{\alpha}U}{}_Q T_\beta{}^Q{}_S \mathcal{M}_{UP} \mathcal{M}^{SR} \\
&\qquad + \Pi^{\tilde{\alpha}}{}_{QP} T_\beta{}^Q{}_S \mathcal{M}^{SR} \Big) \partial_M \partial_R \xi^P J_N{}^\beta \Big]
\end{aligned} \tag{3.7}$$

where we used the section constraint on $\mathcal{L}_\xi \chi_M{}^{\tilde{\alpha}}$. The three last terms come from $\partial_M (\Delta_\xi \chi_N^{\tilde{\alpha}})$ and therefore do combine into a total derivative, but it will be convenient to distribute the derivative as above.

The remaining terms in $\Theta_{MN}{}^{\tilde{\alpha}}$ just pick up their non-covariant variations. We organise the calculation by looking first at all terms varying into χ and then at terms varying into the current J. The sum of terms varying into χ give

$$\begin{aligned}
\Delta_\xi \Big[\mathcal{L}_{\text{top}}\big|_{\zeta=0} \Big] \Big|_{\chi \partial^2 \xi} &= \frac{1}{2} \Pi_{\tilde{\alpha}}{}^{MN} \Big[- 2 T_\alpha{}^R{}_P T^{\alpha\tilde{\alpha}}{}_{\tilde{\beta}} \partial_M \partial_R \xi^P \chi_N{}^{\tilde{\beta}} \\
&\quad + T^{\alpha\tilde{\alpha}}{}_{\tilde{\beta}} T_\alpha{}^R{}_P \partial_M \partial_R \xi^P \chi_N^{\tilde{\beta}} + T^{\alpha\tilde{\alpha}}{}_{\tilde{\beta}} T_\alpha{}^S{}_Q \mathcal{M}_{SP} \mathcal{M}^{QR} \partial_M \partial_R \xi^P \chi_N{}^{\tilde{\beta}} \Big] \\
&= - \Pi_{\tilde{\alpha}}{}^{U[M} T^{\alpha N]}{}_U T_\alpha{}^S{}_Q \mathcal{M}_{SP} \mathcal{M}^{QR} \partial_M \partial_R \xi^P \chi_N{}^{\tilde{\alpha}} \,,
\end{aligned} \tag{3.8}$$

where we used the identity on all terms and the fact that the first two vanish using the section constraint.

The terms whose non-covariant gauge variation contains a current J are

$$
\begin{aligned}
&\Delta_\xi\Big[\mathcal{L}_{\rm top}|_{\zeta=0}\Big]\Big|_{J\partial^2\xi}\\
&=\frac{1}{2}\Pi_{\tilde\alpha}{}^{MN}\Big[-2T^{\alpha R}{}_PK_\alpha{}^{\tilde\alpha}{}_\beta-2T_\beta{}^U{}_QT^{\tilde\alpha Q}{}_S\mathcal{M}_{UP}\mathcal{M}^{SR}+2T^{\tilde\alpha U}{}_QT_\beta{}^Q{}_S\mathcal{M}_{UP}\mathcal{M}^{SR}\\
&\qquad+2\Pi^{\tilde\alpha}{}_{QP}T_\beta{}^Q{}_S\mathcal{M}^{SR}-T_\beta{}^{\tilde\alpha}{}_{\tilde\beta}T^{\tilde\beta R}{}_P-T_\beta{}^{\tilde\alpha}{}_{\tilde\beta}T^{\tilde\beta S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}-T_\beta{}^{\tilde\alpha}{}_{\tilde\beta}\Pi^{\tilde\beta}{}_{QP}\mathcal{M}^{QR}\\
&\qquad+2K_{[\alpha}{}^{\tilde\alpha}{}_{\beta]}\Big(T^{\alpha R}{}_P+T^{\alpha S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\Big)\Big]\partial_M\partial_R\xi^PJ_N{}^\beta\\
&=\frac{1}{2}\Pi_{\tilde\alpha}{}^{MN}\Big[-2K_{(\alpha}{}^{\tilde\alpha}{}_{\beta)}T^{\alpha R}{}_P-T_\beta{}^{\tilde\alpha}{}_{\tilde\beta}T^{\tilde\beta R}{}_P+(2K_{(\alpha}{}^{\tilde\alpha}{}_{\beta)}T^{\alpha S}{}_Q+T_\beta{}^{\tilde\alpha}{}_{\tilde\beta}T^{\tilde\beta S}{}_Q)\mathcal{M}_{SP}\mathcal{M}^{QR}\\
&\qquad\qquad-2T_\beta{}^S{}_{(Q}\Pi^{\tilde\alpha}{}_{P)S}\mathcal{M}^{QR}\Big]\partial_M\partial_R\xi^PJ_N{}^\beta\,, \qquad (3.9)\\
&=-\frac{1}{2}\Omega_{IJ}C^{IM}{}_{\widehat\alpha}C^{JN}{}_\beta T^{\widehat\alpha S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta-\Pi_{\tilde\alpha}{}^{MN}T_\beta{}^S{}_{(Q}\Pi^{\tilde\alpha}{}_{P)S}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta\,,
\end{aligned}
$$

where in the first step we have used the commutation relation, in the last step we have used the identities to write the first line in terms of the C-tensors and combined the α and $\tilde\alpha$ components into an $\widehat\alpha$ index.

Combined non-covariant gauge variation at $\zeta = 0$

Collecting all the terms from above we therefore find

$$
\begin{aligned}
&\Delta_\xi\Big[\mathcal{L}|_{\zeta=0}\Big]+2\partial_{[M}\Big(\partial_{N]}\partial_P\xi^M\mathcal{M}^{NP}\Big) \qquad (3.10)\\
&=\frac{1}{2}\mathcal{M}_{IJ}C^{JN}{}_{\widehat\beta}\Big(C^{IM}{}_{\widehat\alpha}T^{\widehat\alpha S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}-C^{IM}{}_{\tilde\alpha}\Pi^{\tilde\alpha}{}_{QP}\mathcal{M}^{QR}\Big)\partial_M\partial_R\xi^PJ_N{}^{\widehat\beta}\\
&\quad-\Pi_{\tilde\alpha}{}^{U[M}T^{\alpha N]}{}_UT_\alpha{}^S{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\partial_M\partial_R\xi^P\chi_N{}^{\tilde\alpha}\\
&\quad-\frac{1}{2}\Omega_{IJ}C^{IM}{}_{\widehat\alpha}C^{JN}{}_\beta T^{\widehat\alpha S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta-\Pi_{\tilde\alpha}{}^{MN}T_\beta{}^S{}_{(Q}\Pi^{\tilde\alpha}{}_{P)S}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta
\end{aligned}
$$

where the first line combines (3.5) and (3.4) while the remaining lines come from the variation of the topological term given in (3.8) and (3.9).

So far we have avoided using any identity that mixes and $L(\Lambda_{10})\oplus L(\Lambda_4)$. The only equation that does this is the master identity and we shall apply it now to the first line above. Continuing from (3.10) we then obtain

$$
\begin{aligned}
&\Delta_\xi\Big[\mathcal{L}|_{\zeta=0}\Big]+2\partial_{[M}\Big(\partial_{N]}\partial_P\xi^M\mathcal{M}^{PN}\Big)\\
&=\frac{1}{2}\Omega_{IJ}C^{JN}{}_{\widehat\beta}\Big(C^{IM}{}_{\widehat\alpha}T^{\widehat\alpha S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}-C^{IM}{}_{\tilde\alpha}\Pi^{\tilde\alpha}{}_{QP}\mathcal{M}^{QR}\Big)\partial_M\partial_R\xi^PJ_N{}^{\widehat\beta}\\
&\quad+\frac{1}{2}(\mathcal{M}_{IJ}+\Omega_{IJ})C^{IN}{}_{\widehat\beta}C^{JM}\Pi_{QP}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^{\widehat\beta}\\
&\quad-\Pi_{\tilde\alpha}{}^{U[M}T^{\alpha N]}{}_UT_\alpha{}^S{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\partial_M\partial_R\xi^P\chi_N{}^{\tilde\alpha}\\
&\quad-\frac{1}{2}\Omega_{IJ}C^{IM}{}_{\widehat\alpha}C^{JN}{}_\beta T^{\widehat\alpha S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta-\Pi_{\tilde\alpha}{}^{MN}T_\beta{}^S{}_{(Q}\Pi^{\tilde\alpha}{}_{P)S}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta\\
&=\frac{1}{2}(\mathcal{M}_{IJ}+\Omega_{IJ})C^{IN}{}_{\widehat\beta}C^{JM}\Pi_{QP}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^{\widehat\beta}\\
&\quad-\frac{1}{2}\Pi_{\tilde\alpha}{}^{MN}T_\alpha{}^{\tilde\alpha}{}_{\tilde\beta}T^{\alpha S}{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\partial_M\partial_R\xi^P\chi_N{}^{\tilde\beta}+\frac{1}{2}\Pi_{\tilde\beta}{}^{MN}T_\beta{}^{\tilde\beta}{}_{\tilde\alpha}\Pi^{\tilde\alpha}{}_{QP}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta\\
&\quad-\Pi_{\tilde\alpha}{}^{U[M}T^{\alpha N]}{}_UT_\alpha{}^S{}_Q\mathcal{M}_{SP}\mathcal{M}^{QR}\partial_M\partial_R\xi^P\chi_N{}^{\tilde\alpha}-\Pi_{\tilde\alpha}{}^{MN}T_\beta{}^S{}_{(Q}\Pi^{\tilde\alpha}{}_{P)S}\mathcal{M}^{QR}\partial_M\partial_R\xi^PJ_N{}^\beta
\end{aligned}
$$

$$= \frac{1}{2}(\mathcal{M}_{IJ} + \Omega_{IJ}) C^{JM} (\Delta_\xi \zeta_M) C^{IN}{}_{\widehat{\beta}} J_N{}^{\widehat{\beta}} - \partial_N (\Pi_{\tilde{\alpha}}{}^{MN} \Pi^{\tilde{\alpha}}{}_{RP} \mathcal{M}^{RQ} \partial_M \partial_Q \xi^P) \tag{3.11}$$

where we have used the identities to remove most Ω_{IJ} terms when going to the second equality. The remaining term can be written as the non-covariant variation of ζ_M as shown. This result strongly suggests that one might be able to obtain a pseudo-Lagrangian invariant under generalised diffeomorphisms by adding the relevant ζ_M dependent terms. This is indeed what we will show next.

3.2 Gauge invariance

In order to demonstrate gauge-invariance of $\mathcal{L}$, we now consider the ζ_M dependent terms. These appear in the kinetic term and in the topological term. Their non-covariant gauge variation is given by

$$\begin{aligned} \Delta_\xi \Big[\mathcal{L} - \mathcal{L}|_{\zeta=0} \Big] &= -\frac{1}{2}(\mathcal{M}_{IJ} + \Omega_{IJ}) C^{IM}{}_{\widehat{\alpha}} C^{JN} J_M{}^{\widehat{\alpha}} \Delta_\xi \zeta_N \\ &\quad - \frac{1}{2}(\mathcal{M}_{IJ} + \Omega_{IJ}) \Big(C^{IM}{}_{\widehat{\alpha}} T^{\widehat{\alpha} S}{}_Q \mathcal{M}_{SP} + C^{IM}{}_{\tilde{\alpha}} \Pi^{\tilde{\alpha}}{}_{QP} + C^{IM} \Pi_{QP} \Big) \mathcal{M}^{QR} \partial_M \partial_R \xi^P C^{JN} \zeta_N \\ &= -\frac{1}{2}(\mathcal{M}_{IJ} + \Omega_{IJ}) C^{JM} (\Delta_\xi \zeta_M) C^{IN}{}_{\widehat{\beta}} J_N{}^{\widehat{\beta}} \end{aligned} \tag{3.12}$$

where in the first step we have written out the non-covariant variations of $J_M{}^{\widehat{\alpha}}$, cancelled one term using the identity to add one vanishing term and group terms together into the non-covariant variation of F^I. In the second step we have then applied the master identity twice to cancel the middle line.

Now we can collect all terms contributing to the variation of the pseudo-Lagrangian and obtain from (3.11) and (3.12)

$$\delta_\xi \mathcal{L} = \partial_M \Big(\xi^M \mathcal{L} \Big) \tag{3.13}$$

where we used moreover that the total derivative terms in (3.11) cancel. We have therefore proved that the pseudo-Lagrangian is gauge-invariant up to a total derivative as claimed. Note moreover that it transforms under generalised diffeomorphisms as a density, whereas the non-covariant variation usually only vanishes up to a total derivative.

4 Non-Riemannian Backgrounds in $\mathrm{O}(D,D)$ DFT

In this section we first revisit the possible parametrisations of $\mathrm{O}(D,D)$ generalised metrics from the perspective of the coset projector. We demonstrate how the classification of $\mathrm{O}(D,D)$ non-Riemannian parametrisations of Morand and Park [35] fits into this picture. Then, we will review the explicit details of these parametrisations and look at some examples which will inspire us in our later study of the SL(5) ExFT.

4.1 Generalised metric and coset projectors

Let us first recall that the generalised metric of DFT may be defined as a symmetric matrix $\mathcal{H}_{MN}$ obeying the compatibility condition $\mathcal{H}_{MK}\eta^{KL}\mathcal{H}_{LN} = \eta_{MN}$ with the $\mathrm{O}(D,D)$ structure. It transforms under $\mathrm{O}(D,D)$ generalised diffeomorphisms generated by a generalised vector $\Lambda^M = (\Lambda^i, \lambda_i)$ according to the generalised Lie derivative (2.2) with the Y-tensor $Y^{MN}{}_{PQ} = \eta^{MN}\eta_{PQ}$ and $\omega = 0$. The $\mathrm{O}(D,D)$ section condition $\eta^{MN}\partial_M \otimes \partial_N = 0$ may be solved by $\partial_i \neq 0, \tilde{\partial}^i = 0$, where the doubled coordinates are $Y^M = (Y^i, \tilde{Y}_i)$, After solving the section condition in this way, generalised diffeomorphisms produce D-dimensional diffeomorphisms generated by Λ^i and B-field gauge transformations with parameter λ_i.

This leads to the usual parametrisation in terms of the spacetime metric, g_{ij}, in string frame, and the B-field. The generalised dilaton may then be identified as $e^{-2\mathbf{d}} = e^{-2\Phi}\sqrt{|g|}$, where Φ is the spacetime dilaton. There is an implicit assumption that the $D \times D$ block $\mathcal{H}^{ij}$, which is identified with the inverse spacetime metric, is invertible.

The $\mathrm{O}(D, D)$ compatibility condition implies the existence of two projectors

$$P_M^N = \frac{1}{2}(\delta_M^N + \eta^{NP}\mathcal{H}_{PM}), \quad \bar{P}_M^N = \frac{1}{2}(\delta_M^N - \eta^{NP}\mathcal{H}_{PM}), \tag{4.1}$$

such the projector $P_{MN}{}^{KL}$, that appears in the generalised Lie derivative of the generalised metric (2.7), factorises as

$$P_{MN}{}^{KL} = 2P_M^{(K}\bar{P}_N{}^{L)}. \tag{4.2}$$

In the usual parametrisation, the trace $\eta^{MN}\mathcal{H}_{MN}$ is zero, and hence $P_{MN}{}^{MN} = D^2$, as expected for the $\mathrm{O}(D, D)/\mathrm{O}(D) \times \mathrm{O}(D)$ coset.

Let us suppose instead that the trace is not necessarily zero. Then, as P_M^N and $\bar{P}_M^N$ are still projectors, we can have $\eta^{MN}\mathcal{H}_{MN} = 2y$, for some integer y, with $-D \leq y \leq D$, such that $P_M^M = D + y$, $\bar{P}_M^M = D - y$.

We can define "square roots" of the projectors, namely matrices V_{MA} and $\bar{V}_{M\bar{A}}$, where $A = 1, \ldots, D+y$, $\bar{A} = 1, \ldots D - y$. These obey

$$V_{MA}h^{AB}V_{NB} = \frac{1}{2}(\mathcal{H}_{MN} + \eta_{MN}), \quad V_{MA}\eta^{MN}V_{NB} = h_{AB}, \quad \mathcal{H}^{MN}V_{NA} = \eta^{MN}V_{NA}, \tag{4.3}$$

$$\bar{V}_{M\bar{A}}\bar{V}_{N\bar{B}}\bar{h}^{\bar{A}\bar{B}} = \frac{1}{2}(\mathcal{H}_{MN} - \eta_{MN}), \quad \bar{V}_{M\bar{A}}\eta^{MN}\bar{V}_{N\bar{B}} = -\bar{h}^{\bar{A}\bar{B}}, \quad \mathcal{H}^{MN}\bar{V}_{N\bar{A}} = -\eta^{MN}\bar{V}_{N\bar{A}}, \tag{4.4}$$

where h_{AB} and $\bar{h}_{\bar{A}\bar{B}}$ are respectively $(D + y) \times (D + y)$ and $(D - y) \times (D - y)$ diagonal matrices of signatures (p, q) and $(\bar{p}, \bar{q})$. This is quite general; we will see how different choices of signature allow for different coset descriptions and constrains (p, q) and $(\bar{p}, \bar{q})$. Constructing a vielbein for the full generalised metric,

$$E_M{}^{\mathcal{A}} = (V_M{}^A, \bar{V}_M{}^{\bar{A}}), \quad \mathcal{H}_{MN} = E_M{}^{\mathcal{A}}E_N{}^{\mathcal{B}}\mathcal{H}_{\mathcal{A}\mathcal{B}}, \tag{4.5}$$

where the $2D \times 2D$ flat metric,

$$\mathcal{H}_{\mathcal{A}\mathcal{B}} \equiv \begin{pmatrix} h_{AB} & 0 \\ 0 & \bar{h}_{\bar{A}\bar{B}} \end{pmatrix}, \tag{4.6}$$

is of signature $(p + \bar{p}, q + \bar{q})$ we can check that

$$\eta^{\mathcal{A}\mathcal{B}} \equiv E_M{}^{\mathcal{A}}E_N{}^{\mathcal{B}}\eta^{MN} = \begin{pmatrix} h^{AB} & 0 \\ 0 & -\bar{h}^{\bar{A}\bar{B}} \end{pmatrix} \tag{4.7}$$

then has signature $(p + \bar{q}, q + \bar{p})$. Now, $E_M{}^{\mathcal{A}}$ must be an $\mathrm{O}(D, D)$ group element. This means that $\eta^{\mathcal{A}\mathcal{B}}$ should have signature (D, D) and so be equivalent (by a choice of basis for the flat indices) to η^{MN}. Hence the only possibilities obey $p + \bar{q} = D$, $q + \bar{p} = D$. This means that $p - \bar{p} = q - \bar{q} = y$ which is consistent with the trace being $\eta^{MN}\mathcal{H}_{MN} = \eta^{\mathcal{A}\mathcal{B}}\mathcal{H}_{\mathcal{A}\mathcal{B}} = p + q - \bar{p} - \bar{q} = 2y$. Note that the explicit parametrisation that will be used in the subsequent subsection does not make this component counting manifest, as it uses variables which are written in a D-dimensionally covariant manner. As a result, there are shift symmetries present (see (4.12) below) which complicate the choice of what should be regarded as the true independent variables. This suggests there ought to be an alternative formulation which exhibits the coset structure more clearly.

4.2 Review of Morand-Park classification

Dropping the assumption of the invertibility of the $D \times D$ block $\mathcal{H}^{ij}$ in the normal parametrisation led to the classification of $\mathrm{O}(D,D)$ generalised metrics in [35]. Taking the section condition solution, $\partial_i \neq 0$, $\tilde{\partial}^i = 0$, they found that the most general parametrisation of the generalised metric is given by

$$\mathcal{H}_{MN} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_{ij} & X_i^a Y_a^j - \bar{X}_i^{\bar{a}} \bar{Y}_{\bar{a}}^j \\ X_j^a Y_a^i - \bar{X}_j^{\bar{a}} \bar{Y}_{\bar{a}}^i & H^{ij} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} . \tag{4.8}$$

Here both H^{ij} and K_{ij} are symmetric $D \times D$ matrices which may be non-invertible, with $\{X, \bar{X}\}$ spanning the kernel of H^{ij} and $\{Y, \bar{Y}\}$ spanning the kernel of K_{ij}. Both kernels have dimensions $n+\bar{n}$, and we index the zero vectors by $a = 1, \dots, n$ and $\bar{a} = 1, \dots \bar{n}$. Explicitly,

$$H^{ij} X_j^a = 0 \,, \quad H^{ij} \bar{X}_j^{\bar{a}} = 0 \,, \quad K_{ij} Y_a^j = 0 \,, \quad K_{ij} \bar{Y}_{\bar{a}}^j = 0 \,. \tag{4.9}$$

We have some completeness relations which are necessary for the invertibility of $\mathcal{H}_{MN}$, namely

$$H^{ik} K_{kj} + Y_a^i X_j^a + \bar{Y}_{\bar{a}}^i \bar{X}_j^{\bar{a}} = \delta_j^i \,, \quad Y_a^i X_i^b = \delta_a^b \,, \quad \bar{Y}_{\bar{a}}^i \bar{X}_i^{\bar{b}} = \delta_{\bar{a}}^{\bar{b}} \,, \quad Y_a^i \bar{X}_i^{\bar{b}} = 0 = \bar{Y}_{\bar{a}}^i X_i^b \,, \tag{4.10}$$

which imply $H^{ik} K_{kl} H^{lj} = H^{ij}$, $K_{ik} H^{kl} K_{lj} = K_{ij}$. These objects are all tensors under diffeomorphisms and invariant under B-field gauge transformations. We see that the trace of the generalised metric is no longer zero, but given by $\mathcal{H}^M{}_M = 2(n - \bar{n})$, in agreement with the analysis of the previous subsection, with $0 \leq n + \bar{n} \leq D$. Note that $X, \bar{X}$ and $Y, \bar{Y}$ are a preferred basis for the zero vectors of H and K. Any other basis $X_i^{\prime u}$, $Y_u^{\prime i}$, where $u = 1, \dots n + \bar{n}$, would be such that

$$Z_i{}^j \equiv X_i^a Y_a^j - \bar{X}_i^{\bar{a}} \bar{Y}_{\bar{a}}^j = X_i^{\prime u} \sigma_u{}^v Y^{\prime j}{}_v \tag{4.11}$$

where $\sigma_u{}^v$ is conjugate to $\mathrm{diag}(\delta_b^a, -\delta_{\bar{b}}^{\bar{a}})$. Thus $X, \bar{X}$ and $Y, \bar{Y}$ diagonalise $\sigma_u{}^v$. Finally, note there is also a shift symmetry preserving the parametrisation (4.8), involving arbitrary parameters b_{ia}, $\bar{b}_{i\bar{a}}$:

$$\begin{aligned} Y_a^i &\to Y_a^i + H^{ij} b_{ja} \,, \\ \bar{Y}_{\bar{a}}^i &\to \bar{Y}_{\bar{a}}^i + H^{ij} \bar{b}_{j\bar{a}} \,, \\ K_{ij} &\to K_{ij} - 2 X_{(i}^a K_{j)k} H^{kl} b_{la} - 2 \bar{X}_{(i}^{\bar{a}} K_{j)k} H^{kl} \bar{b}_{l\bar{a}} + (X_i^a b_{ka} + \bar{X}_i^{\bar{a}} \bar{b}_{k\bar{a}}) H^{kl} (X_j^b b_{lb} + \bar{X}_j^{\bar{b}} \bar{b}_{l\bar{b}}) \,, \\ B_{ij} &\to B_{ij} - 2 X_{[i}^a b_{j]a} + 2 \bar{X}_{[i}^{\bar{a}} \bar{b}_{j]\bar{a}} + 2 X_{[i}^a \bar{X}_{j]}^{\bar{a}} \left(Y_a^k \bar{b}_{k\bar{a}} + \bar{Y}_{\bar{a}}^k b_{ka} + b_{ka} H^{kl} \bar{b}_{l\bar{a}} \right) \,, \end{aligned} \tag{4.12}$$

which we can view as eliminating some components of the B-field in the non-Riemannian geometry.

A variety of interesting example have been considered in [35]. For instance, $(n, \bar{n}) = (D, 0)$ corresponds to the maximally non-Riemannian case, $\mathcal{H}_{MN} = \eta_{MN}$. When $n = \bar{n}$ the parametrisations may be connected by $\mathrm{O}(D,D)$ transformations to Riemannian parametrisations. An example, which we will discuss below, is the $(1,1)$ non-Riemannian metric corresponding to the Gomis-Ooguri limit of string theory, or to the T-dual of a supergravity solution. The case $(n, \bar{n}) = (D-1, 0)$ gives an ultra-relativistic (Carroll) geometry, while $(n, \bar{n}) = (1, 0)$ or $(0, 1)$ provides a version of non-relativistic Newton-Cartan geometry. (In this case, the transformation (4.12) in fact reduces to known non-relativistic transformations termed Milne transformations or Galilean boosts [35].) In general, the non-Riemannian background (4.8) can be studied using the doubled sigma model, and it was shown in [35] that the zero vectors $X_i{}^a$ pick out n string target space coordinates which become chiral, while the $\bar{X}_i{}^{\bar{a}}$ lead to $\bar{n}$ antichiral directions.

5 Riemannian Backgrounds and Exotic Supergravities in SL(5) ExFT

We will now focus on the SL(5) ExFT, a good testing ground as it is simple enough to allow one to realise various constructions very explicitly, and simultaneously complex enough to be interesting. Already at the level of Riemannian parametrisations, the SL(5) ExFT describes not only the conventional 10- and 11-dimensional supergravities, but exotic variants, with all information about the nature of the spacetime theory encoded in the generalised metric via the choice of parametrisation. We should however note that though these exotic variants appear to give valid parametrisations of the ExFT variables, their role in the full quantum string and M-theory is less clear as they involve spacetimes of non-Minkowskian signatures, and they are not expected to exist as the low energy limits of fully fledged variants of string and M-theory, though they may still appear as complex saddle points in the path integral.

Spacetime decompositions

In general, in order to match exceptional field theory with standard supergravity, it is convenient to start with an intelligent decomposition of the fields of the latter. For instance, the 11- or 10-dimensional Einstein frame metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ can be decomposed in the following manner (corresponding to a partial fixing of Lorentz symmetry): splitting the 11- or 10-dimensional index $\hat{\mu} = (\mu, i)$, where μ is an n-dimensional index, let

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} |\phi|^{\omega} g_{\mu\nu} + A_{\mu}{}^{k} A_{\nu}{}^{l} \phi_{kl} & A_{\mu}{}^{k}\phi_{kj} \\ A_{\nu}{}^{k}\phi_{ki} & \phi_{ij} \end{pmatrix} , \tag{5.1}$$

where ω is the intrinsic weight appearing in the generalised Lie derivative. For SL(5), $\omega = -1/5$. The ExFT formalism will work regardless of the signatures of the blocks $g_{\mu\nu}$ and ϕ_{ij}. We will denote the signature of metrics by (t, s). Let ϕ_{ij} be a d-dimensional metric with signature (t, s), so that $\phi \equiv \det\phi = (-1)^t|\phi|$. Define $\epsilon_{i_1\dots i_d} = |\phi|^{1/2}\eta_{i_1\dots i_d}$, $\epsilon^{i_1\dots i_d} = |\phi|^{-1/2}\eta^{i_1\dots i_d}$ with both $\eta^{1\dots d} = \eta_{1\dots d} = +1$. Then we have $\epsilon^{i_1\dots i_d} = (-1)^t \phi^{i_1 i_1'} \dots \phi^{i_d i_d'} \epsilon_{i_1'\dots i_d'}$ and there are no extra signs in the contractions between ϵ with indices up and those with indices down.

As well as the metric, it can be convenient to redefine the components of the gauge fields which carry the external μ, ν indices, making use of the field $A_{\mu}{}^{i}$. The details are not important in the present paper.

The SL(5) ExFT

For SL(5), the representation R_1 is the antisymmetric 10-dimensional representation; we will write an R_1 index M as an antisymmetic pair of five-dimensional indices a, b, so that $V^M \equiv V^{ab} = -V^{ba}$. We will contract indices with a factor of 1/2, $V^M W_M \equiv \frac{1}{2} V^{ab} W_{ab}$, meaning that $\delta^M{}_N = 2\delta^{[ab]}_{cd} = \delta^a_c\delta^b_d - \delta^b_c\delta^a_d$. The generalised Lie derivative is defined by giving the Y-tensor, which is $Y^{MN}{}_{KL} = \eta^{aa'bb'e}\eta_{cc'dd'e}$, and the section condition is $\eta^{abcde}\partial_{bc}\partial_{de} = 0$.

The generalised metric, $\mathcal{M}_{MN}$, carries a pair of symmetric R_1 indices. We can also define a "little" generalised metric in the fundamental five-dimensional representation, such that

$$\mathcal{M}_{ab,cd} = \pm(m_{ac}m_{bd} - m_{ad}m_{bc}) , \tag{5.2}$$

where the overall sign is needed to describe exceptional field theory in the case where the Y^M coordinates include timelike directions. The little metric is constrained to have unit determinant, $\det m_{ab} = 1$. Note that it is immediate from this decomposition that $\epsilon^{abcde}\mathcal{M}_{ab,cd} = 0$ and hence $Y^{MN}{}_{PQ}\mathcal{M}_{MN} = 0$, so that referring to the projector trace $P_{MN}{}^{MN}$ in (2.10) we find that $\mathcal{M}_{ab,cd}$ has 14 components, corresponding to the coset SL(5)/SO(5) (or SL(5)/SO(2, 3)). The situation with the sign choice in (5.2), meanwhile, is a little subtle. We choose to fix the sign differently in different parametrisations, such that the "generalised line element"

$$g_{\mu\nu}dX^{\mu}dX^{\nu} + \mathcal{M}_{MN}(dY^M + \mathcal{A}_{\mu}dX^{\mu})(dY^N + \mathcal{A}_{\nu}dX^{\nu}) \tag{5.3}$$

when written out in terms of the spacetime metric, $\hat{g}_{\hat{\mu}\hat{\nu}}$ (as in (5.1)), and spacetime coordinates, $\hat{X}^{\hat{\mu}} = (X^\mu, Y^i)$, always equals

$$|\phi|^{-\omega}\hat{g}_{\hat{\mu}\hat{\nu}}dX^{\hat{\mu}}dX^{\hat{\nu}} + \dots \tag{5.4}$$

where the ellipsis denotes terms involving dual coordinates. Pullbacks of the expression (5.3) are used to construct particle and string actions with target space the extended geometry of ExFT, and the relative sign between the two terms is fixed by the appropriate notion of gauge covariance under the ExFT gauge symmetries. As it is $\mathcal{M}_{MN}$ that appears in (5.3), we stress that it is the parametrisation of this version of the generalised metric which must be considered fundamental, though we will almost always write down explicit expressions using the more compact notation of the little metric m_{ab}. (Note we can also express m_{ab} via $m_{ab} = \frac{1}{6}\eta_{aMN}\eta_{bPQ}\mathcal{M}^{MP}\mathcal{M}^{NQ}$.)

The gauge fields of the SL(5) ExFT appearing in the action are a one-form $\mathcal{A}_\mu{}^M$, two-form, $\mathcal{B}_{\mu\nu a}$ with field strength $\mathcal{H}_{\mu\nu\rho a}$, and three-form, $\mathcal{C}_{\mu\nu\rho}{}^a$, whose field strength $\mathcal{J}_{\mu\nu\rho\sigma}{}^a$ appears in the Chern-Simons term but does not have a kinetic term. The equation of motion for $\mathcal{C}_{\mu\nu\rho}{}^a$ accordingly amounts to a duality relation relating it to the degrees of freedom in the other gauge fields. The action is defined by

$$\begin{aligned} S = \int \mathrm{d}^7X\mathrm{d}^{10}Y\sqrt{|g|}\Bigg(&\hat{R}[g] + \frac{1}{12}g^{\mu\nu}D_\mu\mathcal{M}_{MN}D_\nu\mathcal{M}^{MN} - V(\mathcal{M},g) + \frac{1}{\sqrt{|g|}}\mathcal{L}_{CS} \\ &- \frac{1}{4}eg^{\mu\rho}g^{\nu\sigma}\mathcal{M}_{MN}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma}{}^N - \frac{1}{2}m^{ab}\mathcal{H}_{\mu\nu\rho a}\mathcal{H}^{\mu\nu\rho}{}_b\Bigg) \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} -V(\mathcal{M},g) &= \frac{1}{12}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} - \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_K\mathcal{M}_{LN} + \frac{1}{2}\partial_M\mathcal{M}^{MN}\partial_N\ln|g| \\ &\quad + \frac{1}{4}\mathcal{M}^{MN}\left(\partial_M g_{\mu\nu}\partial_N g^{\mu\nu} + \partial_M\ln|g|\partial_N\ln|g|\right) \\ &= \pm\Bigg(\frac{1}{8}m^{ac}m^{bd}\partial_{ab}m_{ef}\partial_{cd}m^{ef} + \frac{1}{2}m^{ac}m^{bd}\partial_{ab}m^{ef}\partial_{ec}m_{df} + \frac{1}{2}\partial_{ab}m^{ac}\partial_{cd}m^{bd} \\ &\quad + \frac{1}{2}m^{ac}\partial_{ab}m^{bd}\partial_{cd}\ln|g| + \frac{1}{8}m^{ac}m^{bd}(\partial_{ab}g^{\mu\nu}\partial_{cd}g_{\mu\nu} + \partial_{ab}\ln|g|\partial_{cd}\ln|g|)\Bigg) \end{aligned} \tag{5.6}$$

and the Chern-Simons term is described in [65].

5.1 Fixing the coefficients of the SL(5) ExFT

We have already seen a truncated form of this theory in the current literature, and described the tensor hierarchy fields in deep details. Recall we use $\mathcal{M},\mathcal{N} = 1,\dots 5$ to denote five-dimensional fundamental indices, while the R_1 representation of generalised vectors is the 10-dimensional antisymmetric representation, for which we write a 10-dimensional index $M = [\mathcal{M}\mathcal{N}]$ as an antisymmetric pair of five-dimensional indices.

The field content of the SL(5) exceptional field theory is

$$\left\{g_{\mu\nu}, \mathcal{M}_{\mathcal{M}\mathcal{N},\mathcal{P}\mathcal{Q}}, \mathcal{A}_\mu{}^{\mathcal{M}\mathcal{N}}, \mathcal{B}_{\mu\nu\mathcal{M}}, \mathcal{C}_{\mu\nu\rho}{}^{\mathcal{M}}, \dots\right\}. \tag{5.7}$$

Here we have the 7-dimensional metric $g_{\mu\nu}$, the generalised metric $\mathcal{M}_{\mathcal{M}\mathcal{N},\mathcal{P}\mathcal{Q}}$ parametrizing the coset SL(5)/SO(5), plus the tensor hierarchy fields: the one-form $\mathcal{A}_\mu{}^{\mathcal{M}\mathcal{N}}$, two-form $\mathcal{B}_{\mu\nu\mathcal{M}}$, and $\mathcal{C}_{\mu\nu\rho}{}^{\mathcal{M}}$. The corresponding field strengths of the tensor hierarchy fields are $\mathcal{F}_{\mu\nu}{}^{\mathcal{MN}}$, $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$ and $\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{M}}$. The four-form $\mathcal{D}_{\mu\nu\rho\sigma\mathcal{MN}}$ appears in the definition of $\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{M}}$, but drops out of the field equations. Hence this does not describe additional physical degrees of freedom.

All these fields are taken to depend on the 7-dimensional coordinates, x^μ, and the 10-dimensional extended coordinates, Y^M. The coordinate dependence of the fields on the latter is subject to the physical section condition which picks a subspace of the exceptional extended space. This section condition can be formulated in terms of the SL(5) invariant $\epsilon^{\mathcal{MNPQK}}$

$$\epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\partial_{\mathcal{PQ}}\Phi = 0\,, \quad \epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\Phi\partial_{\mathcal{PQ}}\Psi = 0\,, \tag{5.8}$$

where Φ and Ψ denote any field or gauge parameter.

It is convenient to decompose the generalised metric as

$$\mathcal{M}_{\mathcal{MN},\mathcal{PQ}} = m_{\mathcal{MP}}m_{\mathcal{QN}} - m_{\mathcal{MQ}}m_{\mathcal{PN}}\,, \tag{5.9}$$

where $m_{\mathcal{MN}}$ is symmetric and has unit determinant. We denote its inverse by $m^{\mathcal{MN}}$.

Then we can write the action **??** specialised to SL(5) as

$$\begin{aligned} S_{\mathrm{SL}(5)} = \int \mathrm{d}^7x\,\mathrm{d}Y\,\sqrt{|g|}\Big(& R_{\mathrm{ext}}(g) + \frac{1}{4}\mathcal{D}_\mu m^{\mathcal{MN}}\mathcal{D}^\mu m_{\mathcal{MN}} \\ & - \frac{1}{8}m_{\mathcal{MP}}m_{\mathcal{NQ}}\mathcal{F}_{\mu\nu}{}^{\mathcal{MN}}\mathcal{F}^{\mu\nu\mathcal{PQ}} - \frac{1}{12}m^{\mathcal{MN}}\mathcal{H}_{\mu\nu\rho\mathcal{M}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{N}} \\ & + \mathcal{L}_{\mathrm{int}}(m,g) + \sqrt{|g|}^{-1}\mathcal{L}_{\mathrm{top}}\Big)\,. \end{aligned} \tag{5.10}$$

In this case, the internal Lagrangian or potential can be expressed as

$$\begin{aligned} \mathcal{L}_{\mathrm{int}}(m,g) = & \frac{1}{8}m^{\mathcal{MP}}m^{\mathcal{NQ}}\partial_{\mathcal{MN}}m_{\mathcal{KL}}\partial_{\mathcal{PQ}}m^{\mathcal{KL}} + \frac{1}{2}m^{\mathcal{MP}}m^{\mathcal{NQ}}\partial_{\mathcal{MN}}m^{\mathcal{KL}}\partial_{\mathcal{KP}}m_{\mathcal{QL}} \\ & + \frac{1}{2}\partial_{\mathcal{MN}}m^{\mathcal{MP}}\partial_{\mathcal{PQ}}m^{\mathcal{NQ}} + \frac{1}{2}m^{\mathcal{MP}}\partial_{\mathcal{MN}}m^{\mathcal{NQ}}\partial_{\mathcal{PQ}}\ln|g| \\ & + \frac{1}{8}m^{\mathcal{MP}}m^{\mathcal{NQ}}(\partial_{\mathcal{MN}}g^{\mu\nu}\partial_{\mathcal{PQ}}g_{\mu\nu} + \partial_{\mathcal{MN}}\ln|g|\partial_{\mathcal{PQ}}\ln|g|)\,. \end{aligned} \tag{5.11}$$

It can be explicitly checked that this is a scalar under generalised diffeomorphisms. (This is the direct generalisation of the miniature SL(5) ExFT we wrote down in 2.14, with the scalar Δ there replaced by the full 7-dimensional metric $g_{\mu\nu}$ here.)

The topological term is best represented by writing it in terms of an integral over an auxiliary 8-dimensional spacetime:

$$S_{\mathrm{top}} = \kappa\int \mathrm{d}^8x\,\mathrm{d}Y\,\epsilon^{\mu_1\dots\mu_8}\left(\frac{1}{4}\hat{\partial}\mathcal{J}_{\mu_1\dots\mu_4}\bullet\mathcal{J}_{\mu_5\dots\mu_8} - 4\mathcal{F}_{\mu_1\mu_2}\bullet(\mathcal{H}_{\mu_3\mu_4\mu_5}\bullet\mathcal{H}_{\mu_6\mu_7\mu_8})\right)\,, \tag{5.12}$$

where the coefficients have been chosen so that its variation is a total derivative

$$\begin{aligned} \delta S_{\mathrm{top}} = 2\kappa\int \mathrm{d}^8x\,\mathrm{d}Y\,\epsilon^{\mu_1\dots\mu_8}\,\mathcal{D}_{\mu_1}\Big(& -4\delta\mathcal{A}_{\mu_2}\bullet(\mathcal{H}_{\mu_3\mu_4\mu_5}\bullet\mathcal{H}_{\mu_6\mu_7\mu_8}) \\ & - 12\mathcal{F}_{\mu_2\mu_3}\bullet(\Delta\mathcal{B}_{\mu_4\mu_5}\bullet\mathcal{H}_{\mu_6\mu_7\mu_8}) \\ & + (\hat{\partial}\Delta\mathcal{C}_{\mu_2\mu_3\mu_4})\bullet\mathcal{J}_{\mu_5\dots\mu_8}\Big)\,, \end{aligned} \tag{5.13}$$

and coefficient κ is determined to be

$$\kappa = \frac{1}{12\cdot 4!}\,. \tag{5.14}$$

In 5.1, we demonstrate this by requiring invariance under 7-dimensional diffeomorphisms.

Kinetic terms are included for the generalised metric and the gauge fields $\mathcal{A}_\mu$ and $\mathcal{B}_{\mu\nu}$. On the other hand, the field strength $\mathcal{J}_{\mu\nu\rho\sigma}$ of the gauge field $\mathcal{C}_{\mu\nu\rho}$ only appears in the topological term. This gauge field also appears in the field strength $\mathcal{H}_{\mu\nu\rho}$. We can find its equation of motion:

$$\partial_{\mathcal{N}\mathcal{M}}\left(\sqrt{|g|}m^{\mathcal{M}\mathcal{P}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{P}} - 12\kappa\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_4}\mathcal{J}_{\sigma_1\ldots\sigma_4}{}^{\mathcal{M}}\right) = 0\,. \tag{5.15}$$

This implies a duality relation between the gauge field $\mathcal{C}_{\mu\nu\rho}$ and the gauge field $\mathcal{B}_{\mu\nu}$.

Finally, note that although the four-form $\mathcal{D}_{\mu\nu\rho\sigma\mathcal{M}\mathcal{N}}$ appears in the definition of $\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{M}}$, it does so in the form $\hat{\partial}\mathcal{D}_{\mu\nu\rho\sigma}$ and consequently drops out of the variation 11.9, as $\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{M}}$ only appears accompanied by $\hat{\partial}$. Specifically, we can integrate by parts and use nilpotency of $\hat{\partial}$ to see this.

We will give a sense of the type of calculation involved in verifying the invariance of the ExFT invariance under n-dimensional diffeomorphisms, and how this fixes the coefficients in the action. We will work with the case of SL(5) that we described. We start with the topological term written as an integral over one dimension higher is:

$$S_{\text{top}} = \kappa\int \mathrm{d}^8x\,\mathrm{d}Y\,\epsilon^{\mu_1\ldots\mu_8}\left(\frac{1}{4}\hat{\partial}\mathcal{J}_{\mu_1\ldots\mu_4}\bullet\mathcal{J}_{\mu_5\ldots\mu_8} - 4\mathcal{F}_{\mu_1\mu_2}\bullet(\mathcal{H}_{\mu_3\mu_4\mu_5}\bullet\mathcal{H}_{\mu_6\mu_7\mu_8})\right) \tag{5.16}$$

where the coefficients have been chosen so that its variation is a total derivative:

$$\begin{aligned}\delta S_{\text{top}} = 2\kappa\int \mathrm{d}^8x\,\mathrm{d}Y\,\epsilon^{\mu_1\ldots\mu_8}\ \mathcal{D}_{\mu_1}\Big(&-4\delta\mathcal{A}_{\mu_2}\bullet(\mathcal{H}_{\mu_3\mu_4\mu_5}\bullet\mathcal{H}_{\mu_6\mu_7\mu_8})\\ &-12\mathcal{F}_{\mu_2\mu_3}\bullet(\Delta\mathcal{B}_{\mu_4\mu_5}\bullet\mathcal{H}_{\mu_6\mu_7\mu_8})\\ &+(\hat{\partial}\Delta\mathcal{C}_{\mu_2\mu_3\mu_4})\bullet\mathcal{J}_{\mu_5\ldots\mu_8}\Big)\,.\end{aligned} \tag{5.17}$$

Using the definition of $\bullet$, this is:

$$\begin{aligned}\delta S_{\text{top}} = 2\kappa\int \mathrm{d}^8x\,\mathrm{d}Y\,\epsilon^{\mu_1\ldots\mu_8}\ \mathcal{D}_{\mu_1}\Big(&+2\delta\mathcal{A}_{\mu_2}{}^{\mathcal{M}\mathcal{N}}\mathcal{H}_{\mu_3\mu_4\mu_5\mathcal{M}}\mathcal{H}_{\mu_6\mu_7\mu_8\mathcal{N}}\\ &+6\mathcal{F}_{\mu_2\mu_3}{}^{\mathcal{M}\mathcal{N}}\Delta\mathcal{B}_{\mu_4\mu_5\mathcal{M}}\mathcal{H}_{\mu_6\mu_7\mu_8\mathcal{N}}\\ &+\partial_{\mathcal{N}\mathcal{M}}\Delta\mathcal{C}_{\mu_2\mu_3\mu_4}{}^{\mathcal{N}}\mathcal{J}_{\mu_5\ldots\mu_8}{}^{\mathcal{M}}\Big)\,.\end{aligned} \tag{5.18}$$

The kinetic term for the one- and two-forms are

$$S_1 = -\frac{1}{8}\int \mathrm{d}^7x\,\mathrm{d}Y\,\sqrt{|g|}m_{\mathcal{M}\mathcal{P}}m_{\mathcal{N}\mathcal{Q}}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}\mathcal{F}^{\mu\nu\mathcal{P}\mathcal{Q}}\,, \tag{5.19}$$

$$S_2 = -\frac{1}{12}\int \mathrm{d}^7x\,\mathrm{d}Y\,\sqrt{|g|}m^{\mathcal{M}\mathcal{N}}\mathcal{H}_{\mu\nu\rho\mathcal{M}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{N}}\,. \tag{5.20}$$

Recall that

$$\begin{aligned}\delta\mathcal{F}_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} &= 2\mathcal{D}_{[\mu}\delta\mathcal{A}_{\nu]}{}^{\mathcal{M}\mathcal{N}} + \frac{1}{2}\epsilon^{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}}\partial_{\mathcal{P}\mathcal{Q}}\Delta\mathcal{B}_{\mu\nu\mathcal{K}}\,,\\ \delta\mathcal{H}_{\mu\nu\rho\mathcal{M}} &= 3\mathcal{D}_{[\mu}\Delta\mathcal{B}_{\nu\rho]\mathcal{M}} - \frac{3}{4}\epsilon_{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}}\delta\mathcal{A}_{[\mu}{}^{\mathcal{N}\mathcal{P}}\mathcal{F}_{\nu\rho]}{}^{\mathcal{Q}\mathcal{K}} + \partial_{\mathcal{N}\mathcal{M}}\Delta\mathcal{C}_{\mu\nu\rho}{}^{\mathcal{N}}\,,\end{aligned} \tag{5.21}$$

and hence the field equation for $\mathcal{C}_{\mu\nu\rho}$ is

$$\partial_{\mathcal{N}\mathcal{M}}\left(\frac{1}{6}\sqrt{|g|}m^{\mathcal{M}\mathcal{P}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{P}} - 2\kappa\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_4}\mathcal{J}_{\sigma_1\ldots\sigma_4}{}^{\mathcal{M}}\right) = 0\,. \tag{5.22}$$

Under external diffeomorphisms, we take

$$
\begin{aligned}
\delta_\xi \mathcal{A}_\mu{}^{\mathcal{M}\mathcal{N}} &= \xi^\sigma \mathcal{F}_{\sigma\mu}{}^{\mathcal{M}\mathcal{N}} + m^{\mathcal{M}\mathcal{P}} m^{\mathcal{N}\mathcal{Q}} g_{\mu\nu} \partial_{\mathcal{P}\mathcal{Q}} \xi^\nu \,, \\
\Delta_\xi \mathcal{B}_{\mu\nu\mathcal{M}} &= \xi^\sigma \mathcal{H}_{\sigma\mu\nu\mathcal{M}} \,, \\
\Delta_\xi \mathcal{C}_{\mu\nu\rho}{}^{\mathcal{M}} &= -\frac{1}{12\cdot 4!\cdot 3!\cdot\kappa} \sqrt{|g|} \xi^\sigma \epsilon_{\sigma\mu\nu\rho\lambda_1\lambda_2\lambda_3} m^{\mathcal{M}\mathcal{N}} \mathcal{H}^{\lambda_1\lambda_2\lambda_3}{}_{\mathcal{N}} \,.
\end{aligned}
\tag{5.23}
$$

The complete variation of the topological term is then:

$$
\begin{aligned}
\delta_\xi S_{\text{top}} = 2\kappa \int \mathrm{d}^8 x\, \mathrm{d}Y\, \epsilon^{\mu_1\ldots\mu_7} \Big(&+ 2\xi^\sigma \mathcal{F}_{\sigma\mu_1}{}^{\mathcal{M}\mathcal{N}} \mathcal{H}_{\mu_2\mu_3\mu_4\mathcal{M}} \mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}} \\
&+ 2 m^{\mathcal{M}\mathcal{P}} m^{\mathcal{N}\mathcal{Q}} g_{\mu_1\sigma} \partial_{\mathcal{P}\mathcal{Q}} \xi^\sigma \mathcal{H}_{\mu_2\mu_3\mu_4\mathcal{M}} \mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}} \\
&+ 6\xi^\sigma \mathcal{F}_{\mu_1\mu_2}{}^{\mathcal{M}\mathcal{N}} \mathcal{H}_{\sigma\mu_3\mu_4\mathcal{M}} \mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}} \Big) \\
+ \frac{1}{6} \int \mathrm{d}^8 x\, \mathrm{d}Y\, \partial_{\mathcal{N}\mathcal{M}} &\left(\sqrt{|g|} \xi^\mu m^{\mathcal{N}\mathcal{P}} \mathcal{H}^{\nu\rho\sigma}{}_{\mathcal{P}} \right) \mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{M}} \,.
\end{aligned}
\tag{5.24}
$$

The first and third lines here cancel via the Schouten identity. For the second line, we write

$$
\epsilon^{\mu_1\ldots\mu_7} g_{\mu_1\sigma} \partial_{\mathcal{P}\mathcal{Q}} \xi^\sigma \mathcal{H}_{\mu_2\mu_3\mu_4\mathcal{M}} \mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}} = -|g| \epsilon_{\sigma\lambda_1\ldots\lambda_6} \partial_{\mathcal{P}\mathcal{Q}} \xi^\sigma \mathcal{H}^{\lambda_1\lambda_2\lambda_3}{}_{\mathcal{M}} \mathcal{H}^{\lambda_4\lambda_5\lambda_6}{}_{\mathcal{N}} \,. \tag{5.25}
$$

Then we integrate by parts to obtain the final expression

$$
\begin{aligned}
\delta_\xi S_{\text{top}} = 8\kappa \int \mathrm{d}^8 x\, \mathrm{d}Y \sqrt{|g|} \xi^\sigma \epsilon_{\sigma\mu_1\ldots\mu_6} \partial_{\mathcal{P}\mathcal{Q}} &\left(\sqrt{|g|} m^{\mathcal{M}\mathcal{P}} \mathcal{H}^{\mu_1\mu_2\mu_3}{}_{\mathcal{M}} \right) m^{\mathcal{N}\mathcal{Q}} \mathcal{H}^{\mu_4\mu_5\mu_6}{}_{\mathcal{N}} \\
+ \frac{1}{6} \int \mathrm{d}^8 x\, \mathrm{d}Y\, \partial_{\mathcal{N}\mathcal{M}} &\left(\sqrt{|g|} \xi^\mu m^{\mathcal{N}\mathcal{P}} \mathcal{H}^{\nu\rho\sigma}{}_{\mathcal{P}} \right) \mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{M}} \,.
\end{aligned}
\tag{5.26}
$$

Next, we consider

$$
\begin{aligned}
\delta_\xi \mathcal{H}_{\mu\nu\rho\mathcal{M}} = L_\xi \mathcal{H}_{\mu\nu\rho\mathcal{M}} & \\
- \frac{3}{4} \epsilon_{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}} & m^{\mathcal{N}\mathcal{N}'} m^{\mathcal{P}\mathcal{P}'} g_{\sigma[\mu|} \partial_{\mathcal{N}'\mathcal{P}'} \xi^\sigma \mathcal{F}_{|\nu\rho]}{}^{\mathcal{Q}\mathcal{K}} \\
- \xi^\sigma \partial_{\mathcal{M}\mathcal{N}} & \mathcal{J}_{\sigma\mu\nu\rho}{}^{\mathcal{N}} - \frac{1}{12\cdot 4!\cdot 3!\cdot\kappa} \partial_{\mathcal{P}\mathcal{M}} \left(\sqrt{|g|} \xi^\sigma \epsilon_{\sigma\mu\nu\rho\lambda_1\lambda_2\lambda_3} m^{\mathcal{P}\mathcal{Q}} \mathcal{H}^{\lambda_1\lambda_2\lambda_3}{}_{\mathcal{Q}} \right)
\end{aligned}
\tag{5.27}
$$

after using the Bianchi identity for $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$ to simplify the expression. The final two lines can be viewed as the anomalous variation of this field strength under the 7-dimensional diffeomorphisms. The very final line gives:

$$
\begin{aligned}
\delta_\xi^{\text{anom}} S_2 \supset & \int \mathrm{d}^7 x\, \mathrm{d}Y \sqrt{|g|} \frac{1}{6} m^{\mathcal{M}\mathcal{N}} \mathcal{H}^{\mu\nu\rho}{}_{\mathcal{M}} \xi^\sigma \partial_{\mathcal{N}\mathcal{P}} \mathcal{J}_{\sigma\mu\nu\rho}{}^{\mathcal{P}} \\
+ \frac{1}{6} \frac{1}{12\cdot 4!\cdot 3!\cdot\kappa} & \int \mathrm{d}^7 x\, \mathrm{d}Y\, \partial_{\mathcal{P}\mathcal{M}} \left(\sqrt{|g|} m^{\mathcal{M}\mathcal{N}} \mathcal{H}^{\mu\nu\rho}{}_{\mathcal{N}} \right) \sqrt{|g|} \xi^\sigma \epsilon_{\sigma\mu\nu\rho\lambda_1\lambda_2\lambda_3} m^{\mathcal{P}\mathcal{Q}} \mathcal{H}^{\lambda_1\lambda_2\lambda_3}{}_{\mathcal{Q}}
\end{aligned}
\tag{5.28}
$$

after integrating by parts. The choice of sign is immaterial and we pick the plus sign. Next, we can consider

$$
\delta_\xi \mathcal{F}_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} = L_\xi \mathcal{F}_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} + \frac{1}{2} \epsilon^{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}} \partial_{\mathcal{P}\mathcal{Q}} \xi^\sigma \mathcal{H}_{\sigma\mu\nu\mathcal{K}} + 2\mathcal{D}_{[\mu|} (m^{\mathcal{M}\mathcal{P}} m^{\mathcal{N}\mathcal{Q}} \partial_{\mathcal{K}\mathcal{L}} \xi^\sigma g_{|\nu]\sigma}) \,. \tag{5.29}
$$

It is straightforward to check that the contribution to the variation of the kinetic term for this field strength arising from the second term here cancels against the remaining piece coming from the anomalous variation of $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$, i.e. the second line in 5.27. This fixes the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$ relative to each other. With further work, the third term here can be shown to cancel against a term coming from the variation of the Einstein-Hilbert term, $R_{\rm ext}(g)$, for the metric $g_{\mu\nu}$. This fixes the relative coefficients of $\mathcal{L}_1$ and $R_{\rm ext}(g)$. Finally, further anomalous variations from $R_{\rm ext}(g)$, the kinetic term for the generalised metric and the internal part of the Lagrangian all conspire to cancel against each other and fix all relative coefficients (up to the overall scale). We refer the diligent reader to the original literature to check the precise details.

5.2 M-theory parametrisations

The M-theory solution of the section condition is based on splitting $a = (i, 5)$, where i is a four-dimensional index, and choosing the physical coordinates to be $Y^i \equiv Y^{i5}$ and the dual coordinates to be Y^{ij}, with the section condition solution then provided by $\partial_i \neq 0$, $\partial_{ij} = 0$. Generalised diffeomorphisms are generated by $\Lambda^{ab} = (\Lambda^{i5}, \Lambda^{ij})$. The vector Λ^i is then found to generate four-dimensional diffeomorphisms, while $\Lambda^{ij} = \frac{1}{2}\eta^{ijkl}\lambda_{kl}$ produces gauge transformations of the three-form. This allows us to parametrise the generalised metric in terms of the internal spacetime metric, ϕ_{ij}, and the internal components of the three-form, C_{ijk}. It is convenient to turn C_{ijk} into a vector by defining $v^i \equiv \frac{1}{3!}\epsilon^{ijkl}C_{jkl}$. Then we have:

$$m_{ab} = \begin{pmatrix} \lambda|\phi|^{-2/5}\phi_{ij} & -\lambda|\phi|^{1/10}v_i \\ -\lambda|\phi|^{1/10}v_j & |\phi|^{3/5}((-1)^t + \lambda v^k v_k) \end{pmatrix} . \tag{5.30}$$

This parametrisation incorporates two sign factors. The first of these is $(-1)^t$, which depends on the number of timelike directions t in ϕ_{ij}. This appears in order that the generalised metric parametrise the correct coset SL(5)/SO(2, 3) rather than SL(5)/SO(5), and ensures that the determinant is +1. Such timelike variants of the classic G/H cosets were analysed. The second sign factor is denoted by λ, and controls the sign of the kinetic term of the three-form, providing an ExFT parametrisation for exotic variants of 11-dimensional supergravity related to timelike dualities. The parametrisation of the big generalised metric that we use corresponds to

$$\mathcal{M}_{ab,cd} = \lambda(-1)^t(m_{ac}m_{bd} - m_{ad}m_{bc}). \tag{5.31}$$

Studying the gauge transformations of the ExFT gauge fields in this solution of the section condition, we find that the obvious components of the 11-dimensional three-form can be identified with certain components of the ExFT gauge fields, schematically $\mathcal{A}_\mu{}^{ij} = \frac{1}{2}\eta^{ijkl}C_{\mu kl}$, $\mathcal{B}_{\mu\nu i} = C_{\mu\nu i}$, $\mathcal{C}_{\mu\nu\rho} = C_{\mu\nu\rho}$. Apart from the obvious identification $\mathcal{A}_\mu{}^i = A_\mu{}^i$, the other components of the gauge fields are related to the dual 11-dimensional six-form, and can be eliminated from the ExFT action using duality relations. As a result, one finds by explicit calculation that the ExFT action is equivalent to that of 11-dimensional supergravity:

$$S = \int d^{11}X\sqrt{|\hat{g}|}\left(R(\hat{g}) - \lambda\frac{1}{48}F^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \frac{1}{\sqrt{|\hat{g}|}}\mathcal{L}_{\rm CS}\right) . \tag{5.32}$$

In general we see that $\lambda = +1$ corresponds to the usual relative sign between the Ricci scalar and F^2 term, while $\lambda = -1$ flips the sign of the F^2 term. The latter variant of supergravity can be thought of as the low energy effective action of an exotic M-theory, called M^- theory, of signature $(2, 9)$ and containing M2 branes whose worldvolume has Euclidean signature.

We can summarise some of the sign choices appearing in the little generalised metric (5.30), with reference to figure 2:

- The signature of ϕ_{ij} is $(0, 4)$ and $\lambda = +1$ so that the signature of m_{ab} is $(0, 5)$, and if the external metric has signature $(1, 6)$ this describes the usual 11-dimensional SUGRA.

- The signature of ϕ_{ij} is $(1,3)$ and $\lambda = +1$ so that the signature of m_{ab} is $(2,3)$, and if the external metric has signature $(0,7)$ this describes the usual 11-dimensional SUGRA.
- The signature of ϕ_{ij} is $(2,2)$, and $\lambda = -1$ so that the signature of m_{ab} is $(2,3)$, and if the external metric has signature $(0,7)$ this describes the unusual 11-dimensional SUGRA with signature $(2,9)$ and wrong sign kinetic term, the low energy limit of the M^* theory (see diagram 2).
- Other choices can correspond to ExFT descriptions of other exotic variants of M-theory.

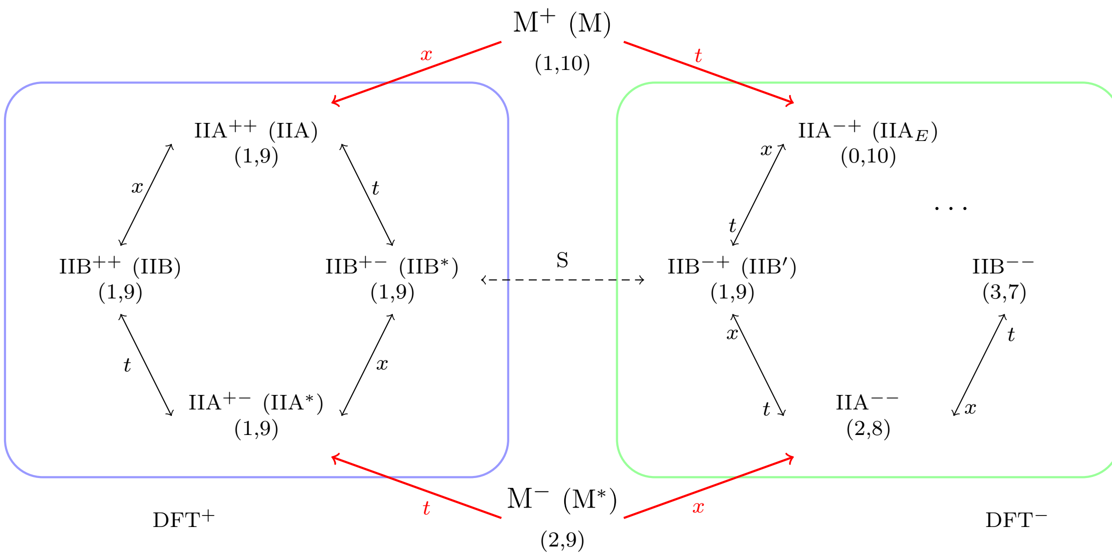


Figure 2: The exotic duality web. Red arrows denote timelike or spacelike reductions from 11 to 10 dimensions. Black arrows denote T-dualities. The dashed arrow in the centre denotes S-duality. All these theories are described by choosing different parametrisations of exceptional field theory. The superscript IIA/B$^{\pm\pm}$ denotes whether, firstly, fundamental strings and, secondly, D-branes have Lorentzian or Euclidean worldvolumes, and hence determines which gauge fields have wrong sign kinetic terms. Similarly M$^{\pm}$ denotes whether M2 branes have Lorentzian or Euclidean worldvolumes. There are additional versions of these theories with more exotic signatures.

5.3 IIB parametrisations

For the IIB solution of the section condition we split $a = (i, \alpha)$ where i a three-dimensional index, and α is a two-dimensional index associated to the unbroken SL(2) S-duality symmetry of IIB. The physical coordinates are then the three coordinates Y^{ij}. It can be convenient to view the i index as being naturally down, i.e. $Y^M = (Y_{ij}, Y_i{}^\alpha, Y^{\alpha\beta})$, such that the physical coordinates can be defined to have the usual index position via $Y^i = \eta^{ijk} Y_{jk}$.

The generalised diffeomorphism parameter $\Lambda^{ab} = (\eta_{ijk}\Lambda^k, \Lambda_i{}^\alpha, \Lambda^{\alpha\beta})$ now produces three-dimensional diffeomorphisms generated by Λ^i, gauge transformations $\Lambda_i{}^\alpha$ of the two-form doublet, and gauge transformations $\Lambda^{\alpha\beta} \equiv \varepsilon^{\alpha\beta}\frac{1}{3!}\eta^{ijk}\lambda_{ijk}$ of the four-form singlet.

The generalised metric can be parametrised in terms of the internal metric, ϕ_{ij}, the two two-forms $(C_{ij}, B_{ij}) = C_{ij}{}^\alpha$ (which we again write as vectors, $v^{i\alpha} \equiv \frac{1}{2}\epsilon^{ijk} C_{jk}{}^\alpha$), and a two-by-two matrix, $\mathcal{H}_{\alpha\beta}$, containing the dilaton Φ and RR zero-form C_0. We write

$$m_{ab} = \begin{pmatrix} |\phi|^{3/5}((-1)^t\sigma_F\sigma_D\phi^{ij} + \mathcal{H}_{\gamma\delta}v^{i\gamma}v^{j\delta}) & |\phi|^{1/10}\mathcal{H}_{\alpha\gamma}v^{i\gamma} \\ |\phi|^{1/10}\mathcal{H}_{\beta\gamma}v^{j\gamma} & |\phi|^{-2/5}\mathcal{H}_{\alpha\beta} \end{pmatrix}, \tag{5.33}$$

$$\mathcal{H}_{\alpha\beta} = \sigma_F e^{\Phi}\begin{pmatrix} 1 & C_0 \\ C_0 & \sigma_F\sigma_D e^{-2\Phi} + C_0^2 \end{pmatrix}. \tag{5.34}$$

Again, we allow for a general distribution of sign factors when the coset is SL(5)/SO(2, 3). Here the signs $\sigma_i = \pm$ dictate whether the parametrisation corresponds to a set of variants of type IIB, denoted IIB$^{\sigma_F\sigma_D}$, where IIB^{++} is the standard IIB, IIB^{+-} is obtained by a timelike T-dualisation of type IIA, IIB^{-+} is the S-dual of IIB^{+-} and is a theory where the fundamental strings have Euclidean worldsheet, and IIB^{--} is obtained by further T-dualities. The subscript on σ_F means that the sign corresponds to the F1 having Lorentzian/Euclidean worldvolume, while that on σ_D means that the sign corresponds to D-branes having Lorentzian/Euclidean worldsheets. In this case, the parametrisation of the big generalised metric that we use corresponds to

$$\mathcal{M}_{ab,cd} = (-1)^t(m_{ac}m_{bd} - m_{ad}m_{bc}). \tag{5.35}$$

We also identify the gauge fields such that (schematically) $\mathcal{A}_{\mu ij} = \eta_{ijk}A_\mu{}^k$, $\mathcal{A}_{\mu i}{}^\alpha = (C_{\mu i}, B_{\mu i})$, $\mathcal{A}_\mu{}^{\alpha\beta} = \varepsilon^{\alpha\beta}\frac{1}{3!}\eta^{ijk}C_{\mu ijk}$ and similarly for the higher rank fields. Then the SL(5) ExFT dynamics are equivalent to those following from the type pseudo-IIB action of the form

$$\begin{aligned} S = \int d^{10}X\,\sqrt{|\hat g|}\Big(& R(\hat g) + \frac{1}{4}\hat g^{\hat\mu\hat\nu}\partial_{\hat\mu}\mathcal{H}_{\alpha\beta}\partial_{\hat\nu}\mathcal{H}^{\alpha\beta} - \frac{1}{12}\sigma_D\sigma_F\mathcal{H}_{\alpha\beta}F_{\hat\mu\hat\nu\hat\rho}{}^\alpha F^{\hat\mu\hat\nu\hat\rho\beta} \\ & - \frac{1}{4\cdot 5!}\sigma_D\sigma_F F_{\hat\mu_1\ldots\hat\mu_5}F^{\hat\mu_1\ldots\hat\mu_5} + \frac{1}{\sqrt{|\hat g|}}\mathcal{L}_{\text{CS}}\Big), \end{aligned} \tag{5.36}$$

which matches the Einstein frame action exactly for the type IIB$^{\sigma_F\sigma_D}$ supergravities [56]. We see that the choice of signs σ_F, σ_D will determines which kinetic terms come with the wrong sign. When $\sigma_F = -1$, the NSNS B-field does, while when $\sigma_D = -1$ the RR two-form does.

6 Membrane Newton-Cartan Fundamental Limit and Exotic Eleven Dimensional Supergravity

6.1 Setting up the expansion

Metric We start by writing the 11-dimensional metric and its inverse as

$$\hat g_{\mu\nu} = c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu}\,, \quad \hat g^{\mu\nu} = cH^{\mu\nu} + c^{-2}\tau^{\mu\nu}\,. \tag{6.1}$$

We can view this simply as a field redefinition which introduces the 11-dimensional Newton-Cartan variables alongside the (dimensionless) parameter c. We will seek to send c to infinity and interpret the result as a non-relativistic limit. In principle, we can also think of this ansatz as containing the first terms in an infinite expansion in c^{-3}, and we will occasionally allow such a perspective to influence our presentation. However, we leave the development of the full non-relativistic expansion to future work. To see that the field redefinition (6.1) makes sense in Newton-Cartan terms we look at the condition $\delta_\mu^\nu = \hat g_{\mu\rho}\hat g^{\rho\nu}$, which gives at order c^3, c^0 and c^{-3} respectively the following three conditions:

$$\tau_{\mu\rho}H^{\rho\nu} = 0\,, \quad \tau_{\mu\rho}\tau^{\rho\nu} + H_{\mu\rho}H^{\rho\nu} = \delta_\mu^\nu\,, \quad H_{\mu\rho}\tau^{\rho\nu} = 0\,. \tag{6.2}$$

We view these as the defining conditions for $\tau_{\mu\nu}$, viewed as a longitudinal Newton-Cartan metric (of Lorentzian signature), and $H^{\mu\nu}$, viewed as the corresponding orthogonal transverse Newton-Cartan metric (of Euclidean signature). Letting $A = 0, 1, 2$ and $a = 1, \ldots 8$ denote longitudinal and transverse flat indices, respectively, we can introduce projective vielbeins such that

$$\tau_{\mu\nu} \equiv \tau_\mu{}^A \tau_\nu{}^B \eta_{AB} , \quad \tau^{\mu\nu} \equiv \tau^\mu{}_A \tau^\nu{}_B \eta^{AB} , \quad \tau^\mu{}_A \tau_\mu{}^B = \delta^B_A , \tag{6.3}$$

$$H^{\mu\nu} \equiv h^\mu{}_a h^\nu{}_b \delta^{ab} , \quad H_{\mu\nu} \equiv h^a{}_\mu h^b{}_\nu \delta_{ab} , \quad h^\mu{}_a h^b{}_\mu = \delta^b_a , \tag{6.4}$$

and hence obeying the Newton-Cartan completeness relations following from (6.2). Here η_{AB} is the flat three-dimensional Minkowski metric and δ_{ab} is the flat Euclidean 8-dimensional metric. We can then compute the determinant of the 11-dimensional metric:

$$\det \hat{g}_{\mu\nu} = -c^{-2}\Omega^2 , \quad \Omega^2 \equiv -\tfrac{1}{3!8!}\epsilon^{\mu_1\ldots\mu_{11}}\epsilon^{\nu_1\ldots\nu_{11}}\tau_{\mu_1\nu_1}\tau_{\mu_2\nu_2}\tau_{\mu_3\nu_3}H_{\mu_4\nu_4}\ldots H_{\mu_{11}\nu_{11}} , \tag{6.5}$$

where $\epsilon^{\mu_1\ldots\mu_{11}}$ denotes the 11-dimensional Levi-Civita symbol. Hence $\sqrt{-\hat{g}} = c^{-1}\Omega$ and it is Ω which will be used as the measure factor in the non-relativistic action. In terms of the vielbeins, we can write

$$\Omega = \left| \tfrac{1}{3!8!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_8}\epsilon_{ABC}\epsilon_{a_1\ldots a_8}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C h^{a_1}{}_{\sigma_1}\ldots h^{a_8}{}_{\sigma_8} \right| \tag{6.6}$$

and note that

$$\partial_\mu \ln \Omega = \tau^\nu{}_A \partial_\mu \tau_\nu{}^A + h^\nu{}_a \partial_\mu h^a{}_\nu . \tag{6.7}$$

We can obtain further useful identities by substituting the expressions (6.1) into contractions of the Levi-Civita symbol and the metric. One that we will use later is

$$n!\, H^{[\mu_1|\nu_1}\ldots H^{|\mu_n]\nu_n} = -\epsilon^{\mu_1\ldots\mu_n\lambda_1\ldots\lambda_{11-n}}\epsilon^{\nu_1\ldots\nu_n\sigma_1\ldots\Omega^2}\tau_{\lambda_1\sigma_1}\ldots\tau_{\lambda_3\sigma_3}H_{\lambda_4\sigma_4}\ldots H_{\lambda_{11-n}\sigma_{11-n}} . \tag{6.8}$$

Three-form For the three-form, let

$$\hat{C}_3 = C_3 - \tfrac{1}{6}c^3\epsilon_{ABC}\tau^A\wedge\tau^B\wedge\tau^C + c^{-3}\widetilde{C}_3 , \tag{6.9}$$

so that

$$\hat{F}_4 = F_4 - \tfrac{1}{2}c^3\epsilon_{ABC}d\tau^A\wedge\tau^B\wedge\tau^C + c^{-3}\widetilde{F}_4 , \tag{6.10}$$

where

$$F_4 \equiv dC_3 , \quad \widetilde{F}_4 \equiv d\widetilde{C}_3 . \tag{6.11}$$

Although $\widetilde{C}_3$ is subleading, it will explicitly appear in the action and dynamics of the non-relativistic limit. Its equation of motion will impose a self-duality constraint on F_4, and we will be able to identify a certain projection of its field strength with the totally longitudinal components of the dual seven-form field strength. We can therefore interpret the subleading part of $\hat{C}_3$ as being 'dual' to the finite part. This is clearly a general fact: the Hodge star itself has an expansion in powers of c and so inevitably mixes up the terms at different powers of c in any p-form it acts on. What is non-trivial is that the Chern-Simons term of the 11-dimensional theory will lead to both C_3 and $\widetilde{C}_3$ playing a role in the non-relativistic limit.

6.2 Expanding the action

The action for the eleven-dimensional metric and three-form is

$$S = \int \mathrm{d}^{11}x \left(\sqrt{|\hat{g}|}\left[\hat{R}(\hat{g}) - \tfrac{1}{48}\hat{F}^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu\rho\sigma}\right] + \tfrac{1}{144^2}\epsilon^{\mu_1\ldots\mu_{11}}\hat{F}_{\mu_1\ldots\mu_4}\hat{F}_{\mu_5\ldots\mu_8}\hat{C}_{\mu_9\mu_{10}\mu_{11}} \right) . \tag{6.12}$$

Here $\hat{F}_4 = d\hat{C}_3$. In form notation the Chern-Simons term is $\frac{1}{6}\hat{F}_4\wedge\hat{F}_4\wedge\hat{C}_3$, the equation of motion of the three-form is $d\hat{\star}\hat{F}_4 = \frac{1}{2}\hat{F}_4\wedge\hat{F}_4$ and its Bianchi identity is $d\hat{F}_4 = 0$. The Hodge dual field strength is $\hat{F}_7 = \hat{\star}\hat{F}_4$, which obeys the Bianchi identity $d\hat{F}_7 = \frac{1}{2}\hat{F}_4\wedge\hat{F}_4$ and the equation of motion $d\hat{\star}\hat{F}_7 = 0$.

Chern-Simons term We start with the expansion of the Chern-Simons term. Leaving wedge products implicit, we can simply compute

$$\begin{aligned}\tfrac{1}{6}\hat{F}_4\hat{F}_4\hat{C}_3 &= \tfrac{1}{6}F_4F_4C_3 - \tfrac{1}{6}(3c^3F_4F_4 + 6F_4\widetilde{F}_4)\tfrac{1}{6}\epsilon_{ABC}\tau^A\tau^B\tau^C \\ &\quad - \tfrac{1}{3}d\left(c^3F_4C_3\tfrac{1}{6}\epsilon_{ABC}\tau^A\tau^B\tau^C + \tfrac{1}{6}\epsilon_{ABC}\tau^A\tau^B\tau^C(F_4\widetilde{C}_3 + C_3\widetilde{F}_4)\right) + \mathcal{O}(c^{-3})\,.\end{aligned} \tag{6.13}$$

We drop the total derivative.

Kinetic term for three-form First, let's write the component expression

$$\hat{F}_{\mu_1\mu_2\mu_3\mu_4} = -6c^3T_{[\mu_1\mu_2}{}^A\tau_{\mu_3}{}^B\tau_{\mu_4]}{}^C\epsilon_{ABC} + F_{\mu_1\mu_2\mu_3\mu_4} + c^{-3}\widetilde{F}_{\mu_1\mu_2\mu_3\mu_4} \tag{6.14}$$

where we introduce the Newton-Cartan torsion

$$T_{\mu\nu}{}^A \equiv 2\partial_{[\mu}\tau_{\nu]}{}^A\,. \tag{6.15}$$

Any term involving three $H^{\mu\nu}$ contracting the first term in (6.14) vanishes as one $H^{\mu\nu}$ must necessarily contract a $\tau_\mu{}^A$. As a result,

$$\begin{aligned}&\sqrt{|\hat{g}|}\hat{g}^{\mu_1\mu_4}\dots\hat{g}^{\mu_4\nu_4}\hat{F}_{\mu_1\dots\mu_4}\hat{F}_{\nu_1\dots\nu_4} \\ &= \Omega\Big(c^3\left(H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4} - 12H^{\mu_1\nu_1}H^{\mu_2\nu_2}T_{\mu_1\mu_2}{}^AT_{\nu_1\nu_2}{}^B\eta_{AB}\right) \\ &\qquad - 24H^{\mu\nu}T_{\mu\rho}{}^AT_{\nu\sigma}{}^B\tau^\rho{}_A\tau^\sigma{}_B - 12H^{\mu_1\nu_1}H^{\mu_2\nu_2}F_{\mu_1\mu_2\mu_3\mu_4}T_{\nu_1\nu_2}{}^A\tau^{\mu_3B}\tau^{\mu_4C}\epsilon_{ABC} \\ &\qquad + 4H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}\tau^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4} \\ &\qquad\quad + 2H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4}\Big) + \mathcal{O}(c^{-3})\,.\end{aligned} \tag{6.16}$$

Kinetic term/Chern-Simons cancellations and self-duality We now examine the $\mathcal{O}(c^3)$ terms in (6.13) and (6.16) which involve a field strength F_4, as well as the $\mathcal{O}(c^0)$ terms involving the subleading $\widetilde{F}_4$. These cannot possibly be cancelled by a contribution from the expansion of the Ricci scalar. The relevant terms are:

$$\begin{aligned}&-\tfrac{1}{2\cdot 4!}\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}(c^3F_{\nu_1\nu_2\nu_3\nu_4} + 2\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4}) \\ &\qquad - \tfrac{1}{2\cdot 4!4!3!}\epsilon^{\mu_1\dots\dots\mu_{11}}F_{\mu_1\mu_2\mu_3\mu_4}(c^3F_{\mu_5\mu_6\mu_7\mu_8} + 2\widetilde{F}_{\mu_5\mu_6\mu_7\mu_8})\epsilon_{ABC}\tau_{\mu_9}{}^A\tau_{\mu_{10}}{}^B\tau_{\mu_{11}}{}^C\end{aligned} \tag{6.17}$$

To cancel the terms at order c^3, we are led to require the following constraint:

$$\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\nu_1\nu_2\nu_3\nu_4} = -\tfrac{1}{4!3!}\epsilon^{\mu_1\dots\dots\mu_{11}}F_{\mu_5\mu_6\mu_7\mu_8}\epsilon_{ABC}\tau_{\mu_9}{}^A\tau_{\mu_{10}}{}^B\tau_{\mu_{11}}{}^C\,. \tag{6.18}$$

This says that the totally transverse part of $F_{\mu\nu\rho\sigma}$ is self-dual (or anti-self-dual). This is self-consistent thanks to current solutions. We will refer to this as the self-duality constraint.

Three-form equation of motion As a sanity check that requiring the constraint (6.18) is sensible and necessary, let us at this point also take the limit at the level of the equation of motion of the three-form gauge field. We will revisit the equations of motion, including that of the metric, in more detail in section 7. For the three-form, we have originally:

$$\partial_\sigma(\sqrt{|\hat{g}|}\hat{g}^{\mu\lambda_1}\hat{g}^{\nu\lambda_2}\hat{g}^{\rho\lambda_3}\hat{g}^{\sigma\lambda_4}\hat{F}_{\lambda_1\dots\lambda_4}) = \tfrac{1}{2\cdot 4!4!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_8}\hat{F}_{\sigma_1\dots\sigma_4}\hat{F}_{\sigma_5\dots\sigma_8}\,. \tag{6.19}$$

Inserting the expansion, one has firstly at $\mathcal{O}(c^3)$ that

$$\partial_\sigma\left(\Omega H^{\mu\lambda_1}H^{\nu\lambda_2}H^{\rho\lambda_3}H^{\sigma\lambda_4}F_{\lambda_1\dots\lambda_4}\right) = -\tfrac{1}{3!4!}\epsilon^{\mu\nu\rho\sigma\sigma_1\dots\sigma_7}\partial_\sigma(F_{\sigma_1\dots\sigma_4}\epsilon_{ABC}\tau_{\sigma_5}{}^A\tau_{\sigma_6}{}^B\tau_{\sigma_7}{}^C)\,, \tag{6.20}$$

which is the duality relation (6.18) under a derivative.

At $\mathcal{O}(c^0)$ we have the finite equation of motion

$$\begin{aligned}\partial_\sigma\Big(\Omega(4H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}H^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}F_{\lambda_1\ldots\lambda_4} - 6H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}\tau^{|\rho|B}\tau^{|\sigma]C}T_{\lambda_1\lambda_2}{}^A\epsilon_{ABC}\\ + H^{\mu\lambda_1}H^{\nu\lambda_2}H^{\rho\lambda_3}H^{\sigma\lambda_4}\widetilde{F}_{\lambda_1\ldots\lambda_4})\Big)\\ = \tfrac{1}{2\cdot 4!4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_8}(F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8} - 12\epsilon_{ABC}T_{\sigma_1\sigma_2}{}^A\tau_{\sigma_3}{}^B\tau_{\sigma_4}{}^C\widetilde{F}_{\sigma_5\ldots\sigma_8})\,.\end{aligned} \tag{6.21}$$

This will be reproduced from the action that we find below.

Ricci scalar Now we come to the Ricci scalar. A very quick way to take the limit is to start with the explicit expression for the Ricci scalar in terms of the metric and its derivatives:

$$\begin{aligned}\hat{R} &= \frac{1}{4}\hat{g}^{\mu\nu}\partial_\mu\hat{g}_{\rho\sigma}\partial_\nu\hat{g}^{\rho\sigma} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\nu\hat{g}^{\rho\sigma}\partial_\rho\hat{g}_{\mu\sigma}\\ &\quad - \frac{1}{4}\hat{g}^{\mu\nu}\partial_\mu\ln\hat{g}\,\partial_\nu\ln\hat{g} - \hat{g}^{\mu\nu}\partial_\mu\partial_\nu\ln\hat{g} - \partial_\mu\ln\hat{g}\,\partial_\nu\hat{g}^{\mu\nu} - \partial_\mu\partial_\nu\hat{g}^{\mu\nu}\,.\end{aligned} \tag{6.22}$$

Calculating the expansion is trivial. One has $\hat{R} = c^4R^{(4)} + cR^{(0)} + \mathcal{O}(c^{-2})$ with

$$\begin{aligned}R^{(4)} &= \frac{1}{4}H^{\mu\nu}\partial_\nu H^{\rho\sigma}\partial_\mu\tau_{\rho\sigma} - \frac{1}{2}H^{\mu\nu}\partial_\nu H^{\rho\sigma}\partial_\rho\tau_{\mu\sigma}\,,\\ R^{(0)} &= \frac{1}{4}H^{\mu\nu}(\partial_\mu\tau_{\rho\sigma}\partial_\nu\tau^{\rho\sigma} + \partial_\mu H_{\rho\sigma}\partial_\nu H^{\rho\sigma}) + \frac{1}{4}\tau^{\mu\nu}\partial_\mu\tau_{\rho\sigma}\partial_\nu H^{\rho\sigma}\\ &\quad - \frac{1}{2}H^{\mu\nu}\partial_\nu\tau^{\rho\sigma}\partial_\rho\tau_{\mu\sigma} - \frac{1}{2}H^{\mu\nu}\partial_\nu H^{\rho\sigma}\partial_\rho H_{\mu\sigma} - \frac{1}{2}\tau^{\mu\nu}\partial_\nu H^{\rho\sigma}\partial_\rho\tau_{\mu\sigma}\\ &\quad - H^{\mu\nu}\partial_\mu\ln\Omega\,\partial_\nu\ln\Omega - 2H^{\mu\nu}\partial_\mu\partial_\nu\ln\Omega - 2\partial_\mu\ln\Omega\,\partial_\nu H^{\mu\nu} - \partial_\mu\partial_\nu H^{\mu\nu}\,.\end{aligned} \tag{6.23}$$

Recall that the measure $\sqrt{-\hat{g}}$ introduces a further power of c^{-1}. The singular piece can be easily rewritten as

$$R^{(4)} = -\tfrac{1}{2}H^{\mu\nu}H^{\rho\sigma}(\partial_\mu\tau_\rho{}^A\partial_\nu\tau_\sigma{}^B - \partial_\rho\tau_\mu{}^A\partial_\nu\tau_\sigma{}^B)\eta_{AB} = -\tfrac{1}{4}H^{\mu\nu}H^{\rho\sigma}T_{\mu\rho}{}^AT_{\nu\sigma}{}^B\eta_{AB}\,. \tag{6.24}$$

This cancels exactly the remaining singular term appearing in the expansion (6.16) of the kinetic term for the three-form. An entirely similar cancellation appeared in the NSNS sector expansion of [27], and as noted there is reminiscent of what happens when taking the Gomis-Ooguri limit on the string worldsheet. In the conclusions in section 15 we discuss the comparison with this limit in more detail.

Action and constraint Combining (6.13), (6.16) and (6.23) we obtain the expansion of the 11-dimensional SUGRA action in the form $S = c^3S^{(3)} + c^0S^{(0)} + \ldots$. The singular part is:

$$S^{(3)} = -\int d^{11}x\,\tfrac{1}{2\cdot 4!}F_{\mu_1\ldots\mu_4}\left(\Omega H^{\mu_1\nu_1}\ldots H^{\mu_4\nu_4} + \tfrac{1}{4!3!}\epsilon^{\mu_1\ldots\mu_4\nu_1\ldots\nu_7}\epsilon_{ABC}\tau_{\nu_5}{}^A\tau_{\nu_6}{}^B\tau_{\nu_7}{}^C\right)F_{\nu_1\ldots\nu_4}\,, \tag{6.25}$$

and in order to have a good $c\to\infty$ limit, we impose the constraint

$$\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\nu_1\nu_2\nu_3\nu_4} = -\tfrac{1}{4!3!}\epsilon^{\mu_1\ldots\ldots\mu_{11}}F_{\mu_5\mu_6\mu_7\mu_8}\epsilon_{ABC}\tau_{\mu_9}{}^A\tau_{\mu_{10}}{}^B\tau_{\mu_{11}}{}^C\,, \tag{6.26}$$

to ensure that $S^{(3)}$ vanishes. The finite part of the supergravity action is:

$$
\begin{aligned}
S^{(0)} = \int \mathrm{d}^{11}x\,\Omega\Big(& R^{(0)} + \tfrac{1}{2}H^{\mu\nu}T_{\mu\rho}{}^{A}T_{\nu\sigma}{}^{B}\tau^{\rho}{}_{A}\tau^{\sigma}{}_{B} \\
&- \tfrac{1}{12}H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}\tau^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4} \\
&+ \tfrac{1}{4}H^{\mu_1\nu_1}H^{\mu_2\nu_2}F_{\mu_1\ldots\mu_4}\epsilon_{ABC}T_{\nu_1\nu_2}{}^{A}\tau^{\mu_3B}\tau^{\mu_4C} \\
&- \tfrac{1}{\cdot 4!}\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4}\big(H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4} \\
&\qquad + \tfrac{1}{4!3!\Omega}\epsilon^{\nu_1\nu_2\nu_3\nu_4\mu_1\ldots\mu_7}F_{\mu_1\mu_2\mu_3\mu_4}\epsilon_{ABC}\tau_{\mu_5}{}^{A}\tau_{\mu_6}{}^{B}\tau_{\mu_7}{}^{C}\big)\Big) \\
&+ \tfrac{1}{6}F_4\wedge F_4\wedge C_3\,,
\end{aligned}
\tag{6.27}
$$

where $R^{(0)}$ is as defined in (6.23). The equation of motion of $C_{\mu\nu\rho}$ gives exactly (6.21), and we will discuss the equations of motion of the Newton-Cartan fields in detail in section 7. The equation of motion of $\widetilde{C}_{\mu\nu\rho}$ is (6.20), giving the constraint under a derivative. Alternatively if we were just to take the action (6.27) at face value, forgetting about its origin via an expansion of the three-form, we could make the choice to view $\widetilde{F}_{\mu\nu\rho\sigma}$ as an independent field, serving as a Lagrange multiplier imposing the constraint in its form (6.26).

Symmetries The action is diffeomorphism invariant (as follows from the covariant rewriting we carry out below), as well as gauge invariant under $\delta C_3 = d\lambda_2$, $\delta\widetilde{C}_3 = d\widetilde{\lambda}_2$. The vielbeins $h^a{}_\mu$ and $\tau^A{}_\mu$ transform under SO(8) and SO(1, 2) rotational symmetries respectively, which are also symmetries of the action. The non-relativistic theory is also invariant under Galilean boosts and a dilatation symmetry.

The Galilean boosts mix the longitudinal and transverse degrees of freedom. The parameter for such a boost is denoted $\Lambda_a{}^A$. Letting $\Lambda_\mu{}^A \equiv h^a{}_\mu\Lambda_a{}^A$ such that $\tau^\mu{}_A\Lambda_\mu{}^B = 0$, we can give the (infinitesimal) action of these symmetries as

$$
\delta_\Lambda H_{\mu\nu} = 2\Lambda_{(\mu}{}^{A}\tau_{\nu)A}\,,\quad \delta_\Lambda\tau^\mu{}_A = -H^{\mu\nu}\Lambda_{\nu A}\,,\quad \delta_\Lambda C_{\mu\nu\rho} = -3\epsilon_{ABC}\Lambda_{[\mu}{}^{A}\tau_\nu{}^{B}\tau_{\rho]}{}^{C}\,. \tag{6.28}
$$

The action $S^{(0)}$ is invariant under these transformations on using the self-duality constraint. One way for the action to be exactly invariant would be to treat $\widetilde{F}_{\mu\nu\rho\sigma}$ as an independent field transforming as $\delta_\Lambda\widetilde{F}_{\mu\nu\rho\sigma} = -4\Lambda_{[\mu}{}^{A}F_{\nu\rho\sigma]\lambda}\tau^\lambda{}_A$, or to have $\widetilde{C}_{\mu\nu\rho}$ transform in a way leading to this transformation.

The dilatations are meanwhile induced by the expansion in powers of c, with the dilatation weight of each field equal to the power of c which accompanies it in the expansion. The (infinitesimal) action of dilatations is hence:

$$
\delta_\lambda H^{\mu\nu} = +\lambda H^{\mu\nu}\,,\quad \delta_\lambda H_{\mu\nu} = -\lambda H_{\mu\nu}\,,\quad \delta_\lambda\tau^\mu{}_A = -\lambda\tau^\mu_A\,,\quad \delta_\lambda\tau_\mu{}^A = +\lambda\tau_\mu{}^A\,,\quad \delta_\lambda C_{\mu\nu\rho} = 0\,. \tag{6.29}
$$

Note $\delta\Omega = -\lambda\Omega$. For λ coordinate dependent this is a symmetry of the action $S^{(0)}$ on using the self-duality constraint (6.26). If we treat $\widetilde{F}_{\mu\nu\rho\sigma}$ as an independent field transforming as $\delta_\lambda\widetilde{F}_{\mu\nu\rho\sigma} = -3\lambda\widetilde{F}_{\mu\nu\rho\sigma}$, then the action $S^{(0)}$ is exactly invariant. We will explicitly verify the invariance of the action and study these symmetries in more detail in section 7.

Newton-Cartan connections and covariant rewriting The way we obtained the action (6.27) was by a straightforward computation at the level of the metric and three-form. To better understand the result, we rewrite the action in a covariant way by introducing the following connection

$$
\Gamma^\rho_{\mu\nu} = \tau^\rho{}_A\partial_\mu\tau_\nu{}^A + \frac{1}{2}H^{\rho\sigma}\left(\partial_\mu H_{\sigma\nu} + \partial_\nu H_{\mu\sigma} - \partial_\sigma H_{\mu\nu}\right)\,, \tag{6.30}
$$

whose covariant derivative we denote by ∇_μ. This satisfies the following metric compatibility conditions:

$$
\nabla_\rho H^{\mu\nu} = 0 = \nabla_\rho\tau_\mu{}^A\,, \tag{6.31}
$$

though it is not the unique solution. The antisymmetric component of (6.30) is the torsion (6.15):

$$\Gamma^{\rho}_{[\mu\nu]} = \frac{1}{2}\tau^{\rho}{}_{A}T_{\mu\nu}{}^{A} . \tag{6.32}$$

It is also useful to define the 'acceleration' and its trace

$$a_{\mu}{}^{AB} \equiv -\tau^{\rho A}T^{B}_{\rho\mu} , \qquad a_{\mu} \equiv a_{\mu}{}^{AB}\eta_{AB} , \tag{6.33}$$

as well as its symmetric traceless component

$$a_{\mu}{}^{\{AB\}} \equiv a_{\mu}{}^{(AB)} - \frac{1}{d_L}\eta^{AB}a_{\mu} , \tag{6.34}$$

where d_L is the dimension of the longitudinal space (which is $d_L = 3$ here, but we will also use this notation in the reduction to the $d_L = 2$ case of SNC in section 8.1). The final tensor that will appear is the extrinsic curvature defined by

$$\mathcal{K}_{\mu\nu A} = \frac{1}{2}\mathcal{L}_{\tau^{\rho A}}H_{\mu\nu} , \qquad \mathcal{K}_{A} \equiv H^{\mu\nu}\mathcal{K}_{\mu\nu A} , \tag{6.35}$$

and obeying the following useful identities

$$\tau^{\mu(A}\mathcal{K}_{\mu\nu}{}^{B)} = 0, \qquad \nabla_{\mu}\tau^{\nu A} = H^{\nu\rho}\mathcal{K}_{\mu\rho}{}^{A} . \tag{6.36}$$

Finally, let's introduce some notation to make the expressions more compact. Given an arbitrary tensor $M_{\mu\nu}$ carrying lower indices, we will employ for convenience the following short-hand notation:

$$M^{\mu\nu} \equiv H^{\mu\rho}H^{\nu\sigma}M_{\rho\sigma} , \quad M_{AB} \equiv \tau^{\mu}{}_{A}\tau^{\nu}{}_{B}M_{\mu\nu} , \quad \nabla_{\rho}M_{AB} \equiv \nabla_{\rho}\left(\tau^{\mu}{}_{A}\tau^{\nu}{}_{B}M_{\mu\nu}\right) , \tag{6.37}$$

and similarly for tensors of arbitrary rank. The meaning of raised indices should then hopefully clear from context – note that e.g. the field strengths, Newton-Cartan torsion and covariant derivative are all naturally defined with lower curved indices so when they appear instead with raised curved or longitudinal flat indices this uses the above notation.

The action can then be written in terms of these manifestly covariant quantities as

$$S = \int \mathrm{d}^{11}x\, \Omega\, \left(\mathcal{L} + \mathcal{L}_{\widetilde{F}} + \Omega^{-1}\mathcal{L}_{\mathrm{top}}\right) , \tag{6.38}$$

with

$$\begin{aligned} \mathcal{L} &= \mathcal{R} - a^{\mu AB}a_{\mu(AB)} + \frac{3}{2}a^{\mu}a_{\mu} - \frac{1}{12}F^{\mu\nu\rho A}F_{\mu\nu\rho A} + \frac{1}{4}\epsilon_{ABC}F^{AB\mu\nu}T_{\mu\nu}{}^{C} , \\ \mathcal{L}_{\widetilde{F}} &= -\frac{1}{4!}\widetilde{F}_{\nu_1\ldots\nu_4}\left(F^{\nu_1\ldots\nu_4} + \frac{1}{4!\,3!\,\Omega}\epsilon^{\nu_1\ldots\nu_4\mu_1\ldots\mu_7}F_{\mu_1\ldots\mu_4}\epsilon_{ABC}\tau_{\mu_5}{}^{A}\tau_{\mu_6}{}^{B}\tau_{\mu_7}{}^{C}\right) , \\ \mathcal{L}_{\mathrm{top}} &= \frac{1}{6}F_4 \wedge F_4 \wedge C_3 = \frac{1}{6}\frac{1}{3!\,4!^2}\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}C_{\mu_9\ldots\mu_{11}} , \end{aligned} \tag{6.39}$$

where the Ricci scalar $\mathcal{R}$ is defined in terms of the usual Riemann curvature tensor of the connection (6.30) via

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} , \quad \mathcal{R} = \mathcal{R}^{\rho}{}_{\mu\rho\nu}H^{\mu\nu} . \tag{6.40}$$

Dual field strength

The appearance of the two field strengths F_4 and $\widetilde{F}_4$ in the finite action (6.27) may seem rather exotic. In fact, we can relate the latter to components of the dual seven-form field strength, revealing that the non-relativistic action involves a partially democratic treatment of what are originally dual degrees of freedom. In 11-dimensional SUGRA, we have

$$\hat{F}_7 = d\hat{C}_6 + \tfrac{1}{2}\hat{C}_3 \wedge \hat{F}_4 \,, \qquad \hat{F}_7 = \hat{\star}\hat{F}_4 \,. \tag{6.41}$$

With our expansion, we can compute $\hat{\star}\hat{F}_4$ in components:

$$\begin{aligned}(\hat{\star}\hat{F}_4)_{\mu_1\dots\mu_7} = \Omega\epsilon_{\mu_1\dots\mu_7\nu_1\dots\nu_4}(& c^3 H^{\nu_1\lambda_1}\dots H^{\nu_4\lambda_4}F_{\rho_1\dots\rho_4} + H^{\nu_1\lambda_1}\dots H^{\nu_4\lambda_4}\widetilde{F}_{\lambda_1\dots\lambda_4} \\ &+ 4H^{\nu_1\lambda_1}\dots H^{\nu_3\lambda_3}\tau^{\nu_4\lambda_4}F_{\lambda_1\dots\lambda_4} \\ &- 6H^{\nu_1\lambda_1}H^{\nu_2\lambda_2}T_{\lambda_1\lambda_2}{}^{A}\tau^{\nu_3 B}\tau^{\nu_4 C}\epsilon_{ABC}) + \mathcal{O}(c^{-3})\,.\end{aligned} \tag{6.42}$$

We then search for an expansion of $\hat{C}_6$ that can reproduce the singular term and lead to a sensible definition of the dual six-form in the non-relativistic theory. This is provided by

$$\hat{C}_6 = -\tfrac{1}{2}c^3 C_3 \wedge \tfrac{1}{6}\epsilon_{ABC}\tau^A \wedge \tau^B \wedge \tau^C + C_6 - \tfrac{1}{2}\widetilde{C}_3 \wedge \tfrac{1}{6}\epsilon_{ABC}\tau^A \wedge \tau^B \wedge \tau^C + \mathcal{O}(c^{-3})\,, \tag{6.43}$$

leading to

$$\hat{F}_7 = -\tfrac{1}{6}c^3\epsilon_{ABC}\tau^A \wedge \tau^B \wedge \tau^C \wedge F_4 + dC_6 + \tfrac{1}{2}C_3 \wedge F_4 - \tfrac{1}{6}\epsilon_{ABC}\tau^A \wedge \tau^B \wedge \tau^C \wedge \widetilde{F}_4 + \mathcal{O}(c^{-3})\,. \tag{6.44}$$

The singular piece in (6.44) agrees with that in (6.42) on using the self-duality constraint (6.26) obeyed by F_4. From the finite terms, we can define in the non-relativistic limit the quantity $F_7 \equiv dC_6 + \frac{1}{2}C_3 \wedge F_4$ which obeys again $dF_7 = \frac{1}{2}F_4 \wedge F_4$. We could also define this quantity directly in the non-relativistic theory after taking the limit by starting with the equation of motion (6.21) of the gauge field. In that case, we would define the dual seven-form field strength to be the quantity appearing under the exterior derivative, including all terms on the left-hand side of (6.21) as well as that involving $d\tau$ on the right-hand side. In components, this means

$$\begin{aligned}F_{\mu_1\dots\mu_7} = \tfrac{1}{4!}\Omega\epsilon_{\mu_1\dots\mu_7\nu_1\dots\nu_4}(& H^{\nu_1\lambda_1}\dots H^{\nu_4\lambda_4}\widetilde{F}_{\lambda_1\dots\lambda_4} + 4H^{\nu_1\lambda_1}\dots H^{\nu_3\lambda_3}\tau^{\nu_4\lambda_4}F_{\lambda_1\dots\lambda_4} \\ &- 6H^{\nu_1\lambda_1}H^{\nu_2\lambda_2}T_{\lambda_1\lambda_2}{}^{A}\tau^{\nu_3 B}\tau^{\nu_4 C}\epsilon_{ABC} \\ &+ \tfrac{1}{4!3!}\Omega^{-1}\epsilon^{\nu_1\dots\nu_4\lambda_1\dots\lambda_7}\epsilon_{ABC}\tau_{\lambda_1}{}^{A}\tau_{\lambda_2}{}^{B}\tau_{\lambda_3}{}^{C}\widetilde{F}_{\lambda_4\dots\lambda_7})\,.\end{aligned} \tag{6.45}$$

Now, we can take the totally longitudinal contraction

$$\begin{aligned}F_{\mu_1\dots\mu_4\sigma_1\sigma_2\sigma_3}\tau^{\sigma_1}{}_{A}\tau^{\sigma_2}{}_{B}\tau^{\sigma_3}{}_{C} = {}& \tfrac{1}{4!}\Omega\epsilon_{\mu_1\dots\mu_4\nu_1\dots\nu_4\sigma_1\sigma_2\sigma_3}\tau^{\sigma_1}{}_{A}\tau^{\sigma_2}{}_{B}\tau^{\sigma_3}{}_{C}H^{\nu_1\lambda_1}\dots H^{\nu_4\lambda_4}\widetilde{F}_{\lambda_1\dots\lambda_4} \\ &+ \epsilon_{ABC}\widetilde{F}_{\mu_1\dots\mu_4}\,.\end{aligned} \tag{6.46}$$

Using (**??**), it can be shown that whereas the transverse part of $F_{\mu\nu\rho\sigma}$ obeys a self-duality constraint, the longitudinal part of $F_{\mu_1\dots\mu_7}$ obeys an anti-self-duality constraint:

$$\begin{aligned}&\Omega H^{\mu_1\nu_1}\dots H^{\mu_4\nu_4}F_{\mu_1\dots\mu_4\sigma_1\sigma_2\sigma_3}\tau^{\sigma_1}{}_{A}\tau^{\sigma_2}{}_{B}\tau^{\sigma_3}{}_{C} \\ &\qquad = +\tfrac{1}{4!3!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\mu_4\lambda_1\dots\lambda_3}\epsilon_{DEF}\tau_{\lambda_1}{}^{D}\tau_{\lambda_2}{}^{E}\tau_{\lambda_3}{}^{F}F_{\mu_1\dots\mu_4\sigma_1\sigma_2\sigma_3}\tau^{\sigma_1}{}_{A}\tau^{\sigma_2}{}_{B}\tau^{\sigma_3}{}_{C}\,.\end{aligned} \tag{6.47}$$

The conclusion is that (6.46) shows that the totally longitudinal part of $F_{\mu_1\dots\mu_7}$ can be identified with the anti-self-dual transverse part of $\widetilde{F}_{\mu\nu\rho\sigma}$. Notice that the longitudinal part of the latter projects trivially out of the action, and in fact it is exactly the projection as on the right-hand side of (6.46) which appears in (6.27). Hence we can re-express the terms in the Lagrangian involving $\widetilde{F}_{\mu\nu\rho\sigma}$ as

$$\begin{aligned}\mathcal{L}_{\widetilde{F}} = -\tfrac{1}{2}\tfrac{1}{4!}&F_{\mu_1\dots\mu_4\lambda_1\dots\lambda_3}\tfrac{1}{6}\epsilon^{ABC}\tau^{\lambda_1}{}_{A}\tau^{\lambda_2}{}_{B}\tau^{\lambda_3}{}_{C}\times \\ &\times(H^{\mu_1\nu_1}\dots H^{\mu_4\nu_4} + \tfrac{1}{4!3!\Omega}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7}\epsilon_{DEF}\tau_{\nu_5}{}^{D}\tau_{\nu_6}{}^{E}\tau_{\nu_7}{}^{F})F_{\nu_1\nu_2\nu_3\nu_4}\,.\end{aligned} \tag{6.48}$$

This appearance of (components of) both the four-form and its dual together in the action is again reminiscent of exceptional field theory.

7 Equations of Motion and Symmetries

We have expanded the action, and now we turn our attention to the equations of motion, and the role played by the non-relativistic dilatation and boost symmetries.

7.1 Equations of motion from expansion

To keep track of the equations of motion at each order, we will consider the result of expanding the variation of the action. We will explicitly find that this gives the same results as varying the expansion of the action we considered previously. The reason we take this approach is that it will provide a useful way to keep track of which parts of the expansion of the eleven-dimensional equations of motion appear at which order. Recall that we view our non-relativistic limit as arising from a field redefinition, and we do not consider possible subleading terms which would occur in a true non-relativistic expansion. That said, we set up the expansion below in a way that would be reminiscent of such an expansion.

The relativistic equations of motion are obtained from the variation of the action (6.12):

$$\delta S = \int \mathrm{d}^{11}x(\sqrt{|\hat{g}|}\delta\hat{g}^{\mu\nu}\mathcal{G}_{\mu\nu} + \delta\hat{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho})\,, \tag{7.1}$$

where

$$\begin{aligned}\mathcal{G}_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{12}\hat{F}_{\mu\rho_1\ldots\rho_3}\hat{F}_\nu{}^{\rho_1\ldots\rho_3} - \tfrac{1}{2}\hat{g}_{\mu\nu}(R - \tfrac{1}{48}\hat{F}^{\rho_1\ldots\rho_4}\hat{F}_{\rho_1\ldots\rho_4})\,,\\ \mathcal{E}^{\mu\nu\rho} &= -\tfrac{1}{6}\left(\partial_\sigma(\sqrt{|\hat{g}|}\hat{F}^{\mu\nu\rho\sigma}) - \tfrac{1}{2\cdot 4!\cdot 4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_8}\hat{F}_{\sigma_1\ldots\sigma_4}\hat{F}_{\sigma_5\ldots\sigma_8}\right)\,.\end{aligned} \tag{7.2}$$

We consider the non-relativistic expansion of the fields, in the form

$$\hat{g}^{\mu\nu} = cH^{\mu\nu} + c^{-2}\tau^{\mu\nu}\,,\quad \hat{g}_{\mu\nu} = c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu}\,,\quad \hat{C}_{\mu\nu\rho} = c^3\omega_{\mu\nu\rho} + C_{\mu\nu\rho} + c^{-3}\widetilde{C}_{\mu\nu\rho}\,, \tag{7.3}$$

where $\omega_{\mu\nu\rho} = -\epsilon_{ABC}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C$. Both $\mathcal{G}$ and $\mathcal{E}$ admit an expansion in powers of c^3, with

$$\mathcal{G} = c^6\mathcal{G}^{(6)} + c^3\mathcal{G}^{(3)} + c^0\mathcal{G}^{(0)} + c^{-3}\mathcal{G}^{(-3)} + \ldots\,,\qquad \mathcal{E} = c^3\mathcal{E}_{(3)} + c^0\mathcal{E}_{(0)} + c^{-3}\mathcal{E}_{(-3)} + \ldots\,. \tag{7.4}$$

We now re-organise the variation of the action that results from (7.3), by inserting the expressions (7.3) for the metric and three-form. We choose to consider the variations of $\tau^\mu{}_A$ and $H^{\mu\nu}$ as independent, in terms of which

$$\delta\omega_{\mu\nu\rho} = -\omega_{\mu\nu\rho}\tau_\lambda^D\delta\tau^\lambda{}_D - 3\omega_{\lambda[\mu\nu}H_{\rho]\kappa}\delta H^{\lambda\kappa}\,. \tag{7.5}$$

The general result at order c^{3n} following from (7.1) is that

$$\begin{aligned}\delta S^{(3n)} = \int \mathrm{d}^{11}x\Big[&\delta H^{\mu\nu}(\Omega\mathcal{G}^{(3n)}_{\mu\nu} - 3\omega_{\mu\rho\sigma}H_{\lambda\nu}\mathcal{E}^{\rho\sigma\lambda}_{(3n-3)})\\ &+ \delta\tau^\mu{}_A(2\tau^{\nu A}\Omega\mathcal{G}^{(3n+3)}_{\mu\nu} - \tau^A{}_\mu\omega_{\rho\sigma\lambda}\mathcal{E}^{\rho\sigma\lambda}_{(3n-3)})\\ &+ \delta C_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(3n)} + \delta\widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(3n+3)}\Big]\,,\end{aligned} \tag{7.6}$$

using $\sqrt{|\hat{g}|} = \Omega c^{-1}$. Hence, in general, if we expand the theory up to order $3k$, for $k \le n \le 2$, the equations of motion will be

$$\mathcal{G}^{(3n)}_{\langle\mu\nu\rangle} = 3H_{\lambda\langle\mu}\omega_{\nu\rangle\rho\sigma}\Omega^{-1}\mathcal{E}^{\lambda\rho\sigma}_{(3n-3)}\,,\quad 2\mathcal{G}^{(3n+3)}_{\mu A} = \tau_{\mu A}\omega_{\rho\sigma\lambda}\Omega^{-1}\mathcal{E}^{\rho\sigma\lambda}_{(3n-3)}\,,\quad \mathcal{E}^{\mu\nu\rho}_{(3n)} = 0\,, \tag{7.7}$$

with the understanding that $\mathcal{G}^{(9)} = \mathcal{E}^{(6)} = 0$. The angle bracket notation takes into account that the variation of $H^{\mu\nu}$ is constrained by $\delta H^{\mu\nu}\tau_\mu{}^A\tau_\nu{}^B = 0$. We can solve this constraint by letting $\delta H^{\mu\nu} = H^{\rho(\mu}H_{\rho\sigma}M^{\nu)\sigma}$ such that the naive variation $\delta H^{\mu\nu}T_{\mu\nu}$ implies instead the equation of motion

$$T_{\langle\mu\nu\rangle} = \tfrac{1}{2}(H_{\mu\rho}H^{\rho\sigma}T_{(\sigma\nu)} + H_{\nu\rho}H^{\rho\sigma}T_{(\mu\sigma)}) \tag{7.8}$$

which is symmetric and obeys $\tau^\mu{}_A\tau^\nu{}_B T_{\langle\mu\nu\rangle}=0$. Note that the equation of motion for $\widetilde{C}$ at each order is exactly that of C at the previous order.

We should contrast the equations of motion (7.7) with the result of independently expanding $\mathcal{G}$ and $\mathcal{E}$. If we naively set each other of the expansion of the latter to zero, we would find the equations $\mathcal{G}^{(3n)}=0=\mathcal{E}^{(3n)}$ at any given order. However, in the non-relativistic expansion, treating $\tau^\mu{}_A$ and $H^{\mu\nu}$ as independent fields, then equation (7.7) says that we cannot simply expand the relativistic equations and set each order independently to zero unless we consider the full expansion (potentially infinite if treating subleading terms). A similar subtlety is the question of which equations of motion we are meant to expand. For instance, in the relativistic theory both $\mathcal{E}^{\mu\nu\rho}=0$ and $g_{\mu\sigma}g_{\rho\kappa}g_{\sigma\lambda}\mathcal{E}^{\sigma\kappa\lambda}=0$ are equivalent, but lead to different truncations to finite order in the $1/c$ expansion. Here we have made the choice to expand the equations of motion that appear conjugate to the variations $\delta g^{\mu\nu}$ and $\delta C_{\mu\nu\rho}$.

Let us look for example at the first two orders, c^6 and c^3. If we simply wanted to expand the theory up to order c^6 we would find the equation $\big(\mathcal{G}^{(6)}-3\omega\mathcal{E}^{(3)}H\big)_{\langle\mu\nu\rangle}=0$, however if we proceed with expanding up to order c^3 we find that the equation for the 3-form tells us that $\mathcal{E}^{(3)}=0$, so that we can safely impose the two equations $\mathcal{G}^{(6)}_{\langle\mu\nu\rangle}=\mathcal{E}^{(3)}=0$ *independently*.

Matters are further complicated by a number of 'off-shell' identities obeyed by the terms appearing in the expansion of $\mathcal{G}$ and $\mathcal{E}$. These identities will feature heavily below, and in fact are crucial for the consistency and symmetries of the non-relativistic truncation.

To put all these ideas together, we now look in detail at the first orders of the expansion of (7.1).

Terms at $\mathcal{O}(c^6)$ Here we encounter the leading terms in the expansions of $\mathcal{G}$ and $\mathcal{E}$. First of all, we have

$$\mathcal{G}^{(6)}_{\mu\nu}=\tfrac{1}{2}\tau_{\mu\nu}\left(\tfrac{1}{2}T^A_{\rho_1\sigma_1}T^B_{\rho_2\sigma_2}\eta_{AB}H^{\rho_1\sigma_1}H^{\rho_2\sigma_2}+\tfrac{1}{48}H^{\rho_1\sigma_1}\dots H^{\rho_4\sigma_4}F_{\rho_1\dots\rho_4}F_{\sigma_1\dots\sigma_4}\right) \tag{7.9}$$

which obeys $\mathcal{G}_{\langle\mu\nu\rangle}=0$ automatically. Hence the $\delta H^{\mu\nu}$ variation at order c^6 does not imply an actual equation of motion. One also has

$$\mathcal{E}^{\mu\nu\rho}_{(3)}=-\tfrac{1}{6}\partial_\sigma\left(\Omega H^{\mu\lambda_1}H^{\nu\lambda_2}H^{\rho\lambda_3}H^{\sigma\lambda_4}F_{\lambda_1\dots\lambda_4}+\tfrac{1}{3!4!}\epsilon^{\mu\nu\rho\sigma\sigma_1\dots\sigma_7}F_{\sigma_1\dots\sigma_4}\epsilon_{ABC}\tau_{\sigma_5}{}^A\tau_{\sigma_6}{}^B\tau_{\sigma_7}{}^C\right). \tag{7.10}$$

This is the self-duality constraint under a derivative. It obeys $\tau_\mu{}^A\tau_\nu{}^B\mathcal{E}^{\mu\nu\rho}_{(3)}=0$, and so also the $\delta\tau$ variation at order c^6 vanishes identically. This is however necessary for consistency: the expansion of the action itself started only at order c^3, i.e. $S^{(6)}\equiv 0$. Hence at this order we do not obtain any equations of motion.

Terms at $\mathcal{O}(c^3)$ At this order, there was a non-zero $S^{(3)}$ given by (6.25), for which we required the self-duality constraint (6.26) to set to zero. Let us see how this information is reproduced. First of all, the variation of C_3 coming from (7.6) at this order implies $\mathcal{E}_{(3)}=0$. The variation of $\tau^\mu{}_A$ involves a contribution from $\mathcal{E}_{(0)}$, which can be read off from the finite part of the expansion of the three-form equation of motion, which was (6.21). For convenience, we repeat this here:

$$\begin{aligned}\mathcal{E}^{\mu\nu\rho}_{(0)}=-\tfrac{1}{6}\partial_\sigma\Big(\Omega\big(&4H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}H^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}F_{\lambda_1\dots\lambda_4}-6H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}\tau^{|\rho|B}\tau^{|\sigma]C}T_{\lambda_1\lambda_2}{}^A\epsilon_{ABC}\\&+H^{\mu\lambda_1}H^{\nu\lambda_2}H^{\rho\lambda_3}H^{\sigma\lambda_4}\widetilde{F}_{\lambda_1\dots\lambda_4}\big)\Big)\\&+\tfrac{1}{2\cdot 3!4!4!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_8}\big(F_{\sigma_1\dots\sigma_4}F_{\sigma_5\dots\sigma_8}-12\epsilon_{ABC}T_{\sigma_1\sigma_2}{}^A\tau_{\sigma_3}{}^B\tau_{\sigma_4}{}^C\widetilde{F}_{\sigma_5\dots\sigma_8}\big).\end{aligned} \tag{7.11}$$

What one finds then is that

$$\begin{aligned}&2\tau^{\nu A}\Omega\mathcal{G}^{(6)}_{\mu\nu}-\tau_\mu{}^A\omega_{\rho\sigma\lambda}\mathcal{E}^{(0)\rho\sigma\lambda}\\&\quad=\tfrac{1}{2\cdot 4!}\tau_\mu{}^A\Omega F_{\nu_1\dots\nu_4}\left(H^{\nu_1\rho_1}\dots H^{\nu_4\rho_4}F_{\rho_1\dots\rho_4}+\tfrac{1}{\Omega 3!4!}\epsilon^{\nu_1\dots\nu_4\rho_1\dots\rho_7}F_{\rho_1\dots\rho_4}\epsilon_{ABC}\tau_{\rho_5}{}^A\tau_{\rho_6}{}^B\tau_{\rho_7}{}^C\right),\end{aligned} \tag{7.12}$$

which is proportional to the self-duality constraint. For the terms accompanying the $\delta H^{\mu\nu}$ variation one finds

$$\begin{aligned}&\delta H^{\mu\nu}(\Omega\mathcal{G}^{(3)}_{\mu\nu} - 3\omega_{(\mu|\rho\sigma}H_{\lambda|\nu)}\mathcal{E}^{\rho\sigma\lambda}_{(0)})\\ &\quad= \delta H^{\mu\nu}(\tfrac{1}{4\cdot 4!^2}\epsilon_{ABC}H_{\lambda_1(\mu}\tau_{\nu)}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8}\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}\\ &\qquad - \tfrac{\Omega}{12}F_{\mu\rho_1\ldots\rho_3}F_\nu{}^{\rho_1\ldots\rho_3} + \tfrac{\Omega}{96}H_{\mu\nu}F^2)\end{aligned} \tag{7.13}$$

such that after projecting using (7.8)

$$\begin{aligned}\Omega\mathcal{G}^{(3)}_{\langle\mu\nu\rangle} - 3\omega_{\langle\mu|\rho\sigma}H_{\lambda|\nu\rangle}\mathcal{E}^{\rho\sigma\lambda}_{(0)} &= \tfrac{1}{8\cdot 4!^2}\epsilon_{ABC}H_{\lambda_1(\mu}\tau_{\nu)}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8}\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}\\ &\quad + \tfrac{\Omega}{96}H_{\mu\nu}F^2 - \tfrac{\Omega}{12}H_{\kappa(\mu}F_{\nu)\rho\sigma\lambda}F^{\kappa\rho\sigma\lambda}\,,\end{aligned} \tag{7.14}$$

using the obvious shorthand for raised indices and F^2 instead of writing $H^{\mu\nu}$ multiple times. This exactly reproduces the variation $\delta S^{(3)}$ of the leading part of the expansion of the action (6.25). Then, after projecting and using the Schouten identity (7.13) or (7.14) can be shown to again be proportional to the self-duality constraint (specifically: the time-space projection of the first term combines with the time-space projection of the third term, and the space-space projection of the second term combines with the space-space projection of the third term).

Hence the sole equation of motion we obtain at this order is the self-duality constraint. This is consistent with what we required from the expansion of the action.

Terms at $\mathcal{O}(c^0)$ We next consider (7.6) with $n = 0$. First of all, the equation of motion of C indeed gives $\mathcal{E}_{(0)}$, as in (7.11), while that of $\widetilde{C}$ gives the constraint in the form $\mathcal{E}_{(3)}$. This is exactly what we obtain from varying the finite action $S^{(0)}$ directly. Note that the longitudinal projection of $\mathcal{E}_{(0)}$ in conjunction with the self-duality constraint implies the equation

$$\tfrac{1}{2}\eta_{AB}H^{\mu\rho}H^{\nu\sigma}T_{\mu\nu}{}^AT_{\rho\sigma}{}^B = -\tfrac{1}{48}H^{\mu_1\nu_1}\ldots H^{\mu_4\nu_4}F_{\mu_1\ldots\mu_4}F_{\nu_1\ldots\nu_4}\,, \tag{7.15}$$

thereby reproducing the equation we would get by setting $\mathcal{G}^{(6)} = 0$ (compare (7.9)). Hence although we could not set $\mathcal{G}^{(6)} = 0$ previously, the non-relativistic theory is not missing this equation. Note that for generic non-vanishing F_4, equation (7.15) is incompatible with imposing foliation-type constraints on the MNC torsion such that the left-hand side vanishes, however if F_4 is also restricted to vanish (for example) one could require such constraints (as is always possible in the NSNS sector case [27]).

Now we turn to the equations of motion following from the variations of τ and H. For simplicity, we present here the independent equations of motion after projecting onto longitudinal (time) and transverse (space) components. The temporal and spatial projectors are defined as

$$(\Delta_T)^\mu{}_\nu = \tau^\mu{}_A\tau_\nu{}^A, \qquad (\Delta_S)^\mu{}_\nu = H^{\mu\rho}H_{\rho\nu}, \qquad (\Delta_T)^\mu{}_\nu + (\Delta_S)^\mu{}_\nu = \delta^\mu_\nu. \tag{7.16}$$

We start with the equations of motion of τ. The trace of the time projection gives an equation involving the Ricci scalar:

$$\begin{aligned}\mathcal{R} &= \tfrac{7}{3}\nabla^\mu a_\mu + a^{\mu\{AB\}}a_{\mu AB} + \tfrac{7}{6}a^2 + \tfrac{1}{36}F_{A\nu\rho\sigma}F^{A\nu\rho\sigma} - \tfrac{1}{6}\epsilon_{ABC}F^{AB\rho\sigma}T_{\rho\sigma}{}^C\\ &\quad + \tfrac{1}{4!}\widetilde{F}_{\mu\nu\rho\sigma}\left(F^{\mu\nu\rho\sigma} + \tfrac{1}{\Omega 4!3!}\epsilon_{ABC}\epsilon^{\mu\nu\rho\sigma\lambda_1\ldots\lambda_7}F_{\lambda_1\ldots\lambda_4}\tau_{\lambda_5}{}^A\tau_{\lambda_6}{}^B\tau_{\lambda_7}{}^C\right)\,.\end{aligned} \tag{7.17}$$

The traceless part of the time-time projection is:

$$\begin{aligned}&\nabla^\mu a_{\mu\{AB\}} + a^\mu a_{\mu\{AB\}} + a_{\mu[C(A]}a^\mu{}_{\{B)D\}}\eta^{CD}\\ &\quad= -\tfrac{1}{12}F_A{}^{\mu\nu\rho}F_{B\mu\nu\rho} + \epsilon_{(A|CD}F_{|B)}{}^{C\mu\nu}T_{\mu\nu}{}^D - \tfrac{\eta_{AB}}{3}\left(-\tfrac{1}{12}F^{C\mu\nu\rho}F_{C\mu\nu\rho} + \epsilon_{CDE}F^{\mu\nu CD}T_{\mu\nu}{}^E\right)\,.\end{aligned} \tag{7.18}$$

The space projection is

$$\nabla_\rho T^{\mu\rho}{}_A + a_{\rho AC}T^{\mu\rho C} = \tfrac{1}{6}F^{\mu\nu\rho\sigma}F_{A\nu\rho\sigma} - \tfrac{1}{2}\epsilon_{ABC}F^{\mu\rho\sigma B}T_{\rho\sigma}{}^C \tag{7.19}$$

Finally, consider the equations of motion of H. The space-space projection is:

$$\begin{aligned}\mathcal{R}^{(\mu\nu)} - a^{\mu AB}a^\nu{}_{\{AB\}} + \tfrac{1}{6}\left(a^\mu a^\nu - a^2 H^{\mu\nu}\right)\\ = \tau^{\rho A}\nabla^{(\mu}T^{\nu)}{}_{\rho A} + \tfrac{1}{6}H^{\mu\nu}\nabla^\rho a_\rho + \tfrac{1}{4}F^{\mu\rho\sigma A}F^\nu{}_{\rho\sigma A} - \tfrac{1}{36}H^{\mu\nu}F^{A\rho\sigma\lambda}F_{A\rho\sigma\lambda}\\ - \tfrac{1}{2}\epsilon_{ABC}F^{(\mu|\rho AB}T^{|\nu)}{}_\rho{}^C + \tfrac{1}{24}H^{\mu\nu}\epsilon_{ABC}F^{\rho\sigma AB}T_{\rho\sigma}{}^C\\ + \tfrac{1}{6}F^{(\mu|\rho\sigma\lambda}\widetilde{F}^{|\nu)}{}_{\rho\sigma\lambda} - \tfrac{1}{48}H^{\mu\nu}F^{\rho\sigma\lambda\kappa}\widetilde{F}_{\rho\sigma\lambda\kappa}\\ + \tfrac{1}{2}H^{\mu\nu}\big(- \mathcal{R} + \tfrac{7}{3}\nabla^\mu a_\mu + a^{\mu\{AB\}}a_{\mu AB} + \tfrac{7}{6}a^2 + \tfrac{1}{36}F_{A\nu\rho\sigma}F^{A\nu\rho\sigma} - \tfrac{1}{6}\epsilon_{ABC}F^{AB\rho\sigma}T_{\rho\sigma}{}^C\big)\,.\end{aligned} \tag{7.20}$$

Combining the trace of (7.20) with (7.17) we find that the self-duality constraint (6.26) appears (contracted with $\widetilde{F}_{\mu\nu\rho\sigma}$).

The time-space projection is (with $\epsilon^A{}_{BC} \equiv \eta^{AD}\epsilon_{DBC}$)

$$\begin{aligned}\mathcal{R}^{(\mu A)} - a^\mu{}_{BC}a^{A(BC)} + \tfrac{1}{2}a_B a^{\mu BA}\\ = \tfrac{1}{4}\epsilon^A{}_{BC}\nabla^\rho F^\mu{}_\rho{}^{BC} + \tfrac{1}{4}\epsilon^A{}_{BC}a_\rho F^{\mu\rho BC} + \tfrac{1}{4}\epsilon_{BCD}a_\rho{}^{AB}F^{\mu\rho CD}\\ + \tfrac{1}{4}F^{AB\rho\sigma}F^\mu{}_{B\rho\sigma} + \tfrac{1}{4}\epsilon_{BCD}F^{ABC\rho}T_\rho{}^{\mu D}\\ + \tfrac{1}{2}a^{\rho BA}\nabla_\rho \tau^\mu{}_B - \tfrac{1}{2}\nabla^2\tau^{\mu A} - a^\rho\nabla_\rho\tau^{\mu A} - \tfrac{1}{2}a^{\mu BA}\mathcal{K}_B + \tfrac{1}{2}a^\mu\mathcal{K}^A\\ - \tfrac{1}{2}\nabla_B a^{\mu BA} + \nabla^A a^\mu + \tfrac{1}{2}T^\mu{}_{\sigma B}\nabla^B\tau^{\sigma A} + \tfrac{1}{2}\nabla_\rho\nabla^\mu\tau^{\rho A} - \tfrac{1}{2}\tau^\rho{}_B\nabla^\mu a_\rho{}^{AB}\\ + \tfrac{1}{6}F^{(\mu}{}_{\nu\rho\sigma}\widetilde{F}^{A)\nu\rho\sigma} - \tfrac{1}{4\cdot 4!^2\Omega}\epsilon^A{}_{BC}\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C H^{\mu\kappa}H_{\kappa\lambda_1}F_{\sigma_1\ldots\sigma_4}\widetilde{F}_{\sigma_5\ldots\sigma_8}\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}\,.\end{aligned} \tag{7.21}$$

We have verified that these are indeed exactly the equations of motions that one gets by varying the finite part of the action, $S^{(0)}$, given in (6.27).

7.2 Dilatations and a 'missing' equation of motion

We already mentioned the existence of a dilatation transformation given by (6.29), whose origin lay in the expansion in powers of c. There is evidently a freedom to rescale c by some constant while simultaneously rescaling the component fields such that the eleven-dimensional fields are unchanged. This *rigid dilatation* leaves the full action invariant. Hence for an infinitesimal dilatation, with $\delta_\lambda c = -\lambda c$, we have the transformations (6.29), and clearly order-by-order for the action we should have

$$\delta_\lambda S^{(6)} = 6\lambda S^{(6)}\,, \quad \delta_\lambda S^{(3)} = 3\lambda S^{(3)}\,, \quad \delta_\lambda S^{(0)} = 0\cdot\lambda S^{(0)}\,, \quad \delta_\lambda S^{(-3)} = -3\lambda S^{(-3)}\,,\ldots \tag{7.22}$$

Recall that $S^{(6)}$ and $\delta S^{(6)}$ vanish identically, so the first of these is just $0 = 0$.

A powerful consequence of the rigid dilatations is that if we know the equations of motion for the action $S^{(3k)}$ at a given order $k \neq 0$ we can immediately write down an action that produces them (which will agree up to total derivatives with that arising from the expansion). This works by applying the formula (7.6) for the variation and specialising to the dilatation variation. This is guaranteed to produce $3kS^{(3k)}$. This singles out the finite order action as being special, as here knowing the equations of motion and dilatation symmetry is not enough to determine its form. Furthermore, for this case we can promote the dilatation parameter to be coordinate dependent, and obtain a *local dilatation* symmetry.

Let's verify these statements. Under a rigid dilatation with parameter λ, the variation of the c^3 part of the action is

$$\delta_\lambda S^{(3)} = \int \mathrm{d}^{11}x\,\Omega\left(\lambda\mathcal{G}^{(3)}_{\mu\nu}H^{\mu\nu} - \lambda\left(2(\mathcal{G}^{(6)})^A{}_A + 3\epsilon_{ABC}\Omega^{-1}\mathcal{E}^{ABC}_{(0)}\right)\right)\,, \tag{7.23}$$

where $\mathcal{E}^{ABC} \equiv \tau_\mu{}^A \tau_\nu{}^B \tau_\rho{}^C \mathcal{E}^{\mu\nu\rho}$. It can be checked that $\mathcal{G}^{(3)}_{\mu\nu} H^{\mu\nu} = 0$. Then, if we denote the self-duality constraint by

$$\Theta^{\mu_1 \dots \mu_4} \equiv H^{\mu_1\rho_1} \dots H^{\mu_4\rho_4} F_{\rho_1\dots\rho_4} + \tfrac{1}{\Omega 3!4!} \epsilon^{\mu_1\dots\mu_4\rho_1\dots\rho_7} F_{\rho_1\dots\rho_4} \epsilon_{ABC} \tau_{\rho_5}{}^A \tau_{\rho_6}{}^B \tau_{\rho_7}{}^C \tag{7.24}$$

we have

$$2(\mathcal{G}^{(6)})^A{}_A + 3\epsilon_{ABC} \Omega^{-1} \mathcal{E}^{ABC}_{(0)} = 3 \tfrac{1}{2\cdot 4!} F_{\mu_1\dots\mu_4} \Theta^{\mu_1\dots\mu_4} \,, \tag{7.25}$$

hence indeed referring to (6.25) for $S^{(3)}$ we indeed have

$$\delta_\lambda S^{(3)} = 3\lambda S^{(3)} \,. \tag{7.26}$$

Next consider the finite part of the action, with:

$$\delta_\lambda S^{(0)} = \int \mathrm{d}^{11}x\, \Omega \left(\lambda \mathcal{G}^{(0)}_{\mu\nu} H^{\mu\nu} - \lambda \left(2(\mathcal{G}^{(3)})^A{}_A + 3\epsilon_{ABC} \Omega^{-1} \mathcal{E}^{ABC}_{(-3)} \right) + \Omega^{-1} \mathcal{E}^{(3)\mu\nu\rho} \delta_\lambda \widetilde{C}_{\mu\nu\rho} \right) \,. \tag{7.27}$$

Now we can show that

$$\mathcal{G}^{(0)}_{\mu\nu} H^{\mu\nu} - \left(2(\mathcal{G}^{(3)})^A{}_A + 3\epsilon_{ABC} \Omega^{-1} \mathcal{E}^{ABC}_{(-3)} \right) = -\tfrac{1}{8} \widetilde{F}_{\mu_1\dots\mu_4} \Theta^{\mu_1\dots\mu_4} \,, \tag{7.28}$$

such that using $\mathcal{E}^{\mu\nu\rho}_{(3)} = -\frac{1}{6} \partial_\sigma \Theta^{\mu\nu\rho\sigma}$ we have

$$\begin{aligned} \delta_\lambda S^{(0)} &= \int \mathrm{d}^{11}x \, (-\tfrac{1}{8} \lambda \widetilde{F}_{\mu\nu\rho\sigma} \Theta^{\mu\nu\rho\sigma} - \tfrac{1}{6} \partial_\sigma \Theta^{\mu\nu\rho\sigma} \delta_\lambda \widetilde{C}_{\mu\nu\rho}) \,, \\ &= \int \mathrm{d}^{11}x \, (-\tfrac{1}{8} \lambda \widetilde{F}_{\mu\nu\rho\sigma} \Theta^{\mu\nu\rho\sigma} - \tfrac{1}{24} \Theta^{\mu\nu\rho\sigma} \delta_\lambda \widetilde{F}_{\mu\nu\rho\sigma}) \,, \end{aligned} \tag{7.29}$$

after integrating by parts. For arbitrary local λ, we therefore have $\delta_\lambda S^{(0)} = 0$ *on imposing the self-duality constraint*, irrespective of the transformation of $\widetilde{C}_{\mu\nu\rho}$. Alternatively, if we require that

$$\delta_\lambda \widetilde{F}_{\mu\nu\rho\sigma} = -3\lambda \widetilde{F}_{\mu\nu\rho\sigma} \,, \tag{7.30}$$

then (7.29) vanishes identically without use of the constraint. This would mean accepting a non-local transformation for $\widetilde{C}_{\mu\nu\rho}$ itself, which is not completely outlandish given the discussion in section 6.2 suggests we may think of it as being a dual degree of freedom to C_3.

What this means in practice is that the action $S^{(0)}$ is invariant under variations of $H^{\mu\nu}$ and $\tau^\mu{}_A$ of the form (6.29). This implies that there is a 'direction' in the space of variations which leaves the action $S^{(0)}$ unchanged (or at best produces the self-duality constraint, which is not an independent equation of motion). Hence if we vary $S^{(0)}$ to obtain the equations of motion of $H^{\mu\nu}$ and $\tau^\mu{}_A$, we will find that we are 'missing' an equation of motion. This is exactly as in the NSNS sector case [26, 27] and reflects a known difficulty, even in the purely gravitational context, of obtaining the Poisson equation from an action principle for non-relativistic theories [52, 53], at least at first order.

Thus, in order to obtain an equation of motion for this missing variation, we go one step further in the expansion. The variation of $S^{(-3)}$, from (7.6), is:

$$\begin{aligned} \delta S^{(-3)} = \int \mathrm{d}^{11}x \Big[&\delta H^{\mu\nu} (\Omega \mathcal{G}^{(-3)}_{\mu\nu} - 3\omega_{\mu\rho\sigma} H_{\lambda\nu} \mathcal{E}^{\rho\sigma\lambda}_{(-6)}) + \delta\tau^\mu{}_A (2\tau^{\nu A} \Omega \mathcal{G}^{(0)}_{\mu\nu} - \tau^A{}_\mu \omega_{\rho\sigma\lambda} \mathcal{E}^{\rho\sigma\lambda}_{(-6)}) \\ &+ \delta C_{\mu\nu\rho} \mathcal{E}^{\mu\nu\rho}_{(-3)} + \delta \widetilde{C}_{\mu\nu\rho} \mathcal{E}^{\mu\nu\rho}_{(0)} \Big] \,, \end{aligned} \tag{7.31}$$

For dilatations we have

$$\delta_\lambda S^{(-3)} = \int \mathrm{d}^{11}x \Big[\lambda \left(H^{\mu\nu} \Omega \mathcal{G}^{(-3)}_{\mu\nu} - 2\Omega (\mathcal{G}^{(0)})_A{}^A - 3\epsilon_{ABC} \mathcal{E}^{ABC}_{(-6)} \right) + \delta_\lambda \widetilde{C}_{\mu\nu\rho} \mathcal{E}^{\mu\nu\rho}_{(0)} \Big] \,. \tag{7.32}$$

With constant λ, equation (7.22) implies that

$$S^{(-3)} = \int \mathrm{d}^{11}x\, (\Omega\, \mathcal{N} + \widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(0)})\,, \tag{7.33}$$

where we defined the combination

$$\mathcal{N} \equiv \tfrac{1}{3}(-H^{\mu\nu}\mathcal{G}^{(-3)}_{\mu\nu} + 2(\mathcal{G}^{(0)})_A{}^A) + \epsilon_{ABC}\Omega^{-1}\mathcal{E}^{ABC}_{(-6)}\,. \tag{7.34}$$

Crucially, (7.34) does not vanish on applying the self-duality constraint, unlike the combination of terms (7.25) and (7.28) which appeared at the previous orders, and nor is it a combination of any other equations of motion resulting from the finite action. It can therefore be used as the equation of motion of the 'dilatation mode'. (We are not really interested in the $\widetilde{C}$ variation appearing in (7.32), which multiplies something we have already taken into account as an equation of motion.) It involves the fully longitudinal part of $\mathcal{G}^{(0)}$, which has not yet appeared in the equations of motion. Hence, we identify it with the 'Poisson equation', in which the longitudinal part of $C_{\mu\nu\rho}$ plays the role of the Newton potential (as did the longitudinal part of the B-field in the Stueckelberg gauge-fixed NSNS sector). This is because $\mathcal{E}_{(-6)}$ is the first equation of motion which contains two derivatives acting on the former. Explicitly,

$$\begin{aligned}\mathcal{E}^{\mu\nu\rho}_{(-6)} = -\tfrac{1}{6}\partial_\sigma\Big(&\Omega(4H^{[\mu|\lambda_1}\tau^{|\nu|\lambda_2}\tau^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}F_{\lambda_1\dots\lambda_4} + 6H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}\tau^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}\widetilde{F}_{\lambda_1\dots\lambda_4}\Big)\\ &+ \tfrac{1}{2\cdot 4!4!3!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_8}\widetilde{F}_{\sigma_1\dots\sigma_4}\widetilde{F}_{\sigma_5\dots\sigma_8}\,.\end{aligned} \tag{7.35}$$

Intriguingly, the combination of $\mathcal{G}^{(-3)}$ and $\mathcal{G}^{(0)}$ appearing in (7.34) has a somewhat murky relationship to the 'trace-reversed' version of the metric equation of motion. The equation $\mathcal{G}_{\mu\nu} = 0$ in the original 11-dimensional theory can be simplified somewhat by taking its trace and solving that for the Ricci scalar. This trace is

$$\hat{g}^{\mu\nu}\mathcal{G}_{\mu\nu} = -\tfrac{9}{2}R + \tfrac{1}{32}\hat{F}^2 \tag{7.36}$$

and the equation of motion without the Ricci scalar is

$$\bar{\mathcal{G}}_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} - \tfrac{1}{9}\hat{g}_{\mu\nu}\hat{g}^{\rho\sigma}\mathcal{G}_{\rho\sigma} = R_{\mu\nu} - \tfrac{1}{12}\hat{F}_\mu{}^{\rho\sigma\lambda}\hat{F}_{\nu\rho\sigma\lambda} + \tfrac{1}{144}\hat{g}_{\mu\nu}\hat{F}^2\,, \tag{7.37}$$

for which

$$\tau^{\mu\nu}\bar{\mathcal{G}}^{(0)}_{\mu\nu} = \tfrac{1}{3}(2\tau^{\mu\nu}\mathcal{G}^{(0)}_{\mu\nu} - H^{\mu\nu}\mathcal{G}^{(-3)}_{\mu\nu})\,, \tag{7.38}$$

which is exactly the combination appearing in (7.34). Note the relative numerical factors here are the same as the relative numerical factors in the powers of c in the expansion.

Now, what exactly is the equation (7.34)? Expanding the metric equation contributions and covariantising everything, one arrives at

$$\begin{aligned}\tau^{\mu\nu}\bar{\mathcal{G}}^{(0)}_{\mu\nu} = {}& 2\tau^{\mu A}\nabla^\rho\mathcal{K}_{\mu\rho A} - \nabla^A\mathcal{K}_A - \tfrac{1}{4}a^{ABC}a_{ABC} - \tfrac{1}{2}a^{ABC}a_{ACB} - a^A a_A\\ &- \epsilon_{ABC}F^{DAB\rho}a_{\rho D}{}^C - \tfrac{1}{8}F^{AB\mu\nu}F_{AB\mu\nu} + \tfrac{1}{48}\widetilde{F}^{\mu\nu\rho\sigma}\widetilde{F}_{\mu\nu\rho\sigma} + \tfrac{1}{4}\epsilon_{ABC}\widetilde{F}^{\mu\nu AB}T_{\mu\nu}{}^C\\ &- a^A\mathcal{K}_A + \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} - 2\tau^{\mu A}\tau^{\nu B}\nabla_\nu a_{\mu[AB]} - \tau^{\mu\nu}\nabla_\mu a_\nu\,,\end{aligned} \tag{7.39}$$

$$\begin{aligned}\epsilon_{ABC}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C\Omega^{-1}\mathcal{E}^{\mu\nu\rho}_{(-6)} = {}& -\tfrac{1}{6}\epsilon_{ABC}\nabla^\mu F^{ABC}{}_\mu - \tfrac{1}{4}\epsilon_{ABC}\widetilde{F}^{AB\mu\nu}T_{\mu\nu}{}^C\\ &+ \tfrac{1}{2\cdot 4!^2\Omega}\tfrac{1}{6}\epsilon^{\lambda_1\dots\sigma_1\dots\sigma_8}\widetilde{F}_{\sigma_1\dots\sigma_4}\widetilde{F}_{\sigma_5\dots\sigma_8}\epsilon_{ABC}\tau_{\lambda_1}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C\,,\end{aligned} \tag{7.40}$$

hence the covariant Poisson equation is

$$\begin{aligned}\mathcal{N} = {}& -\tfrac{1}{6}\epsilon_{ABC}(\nabla^\mu F^{ABC}{}_\mu + a_\mu F^{ABC\mu} + 3a_{\mu D}{}^A F^{BCD\mu}) - \tfrac{1}{8}F^{AB\mu\nu}F_{AB\mu\nu}\\ &+ \tfrac{1}{48}\widetilde{F}^{\mu\nu\rho\sigma}\widetilde{F}_{\mu\nu\rho\sigma} + \tfrac{\Omega^{-1}}{2\cdot 4!^2 3!}\epsilon^{\lambda_1\dots\lambda_3\sigma_1\dots\sigma_8}\widetilde{F}_{\sigma_1\dots\sigma_4}\widetilde{F}_{\sigma_5\dots\sigma_8}\epsilon_{ABC}\tau_{\lambda_1}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C\\ &- \nabla^A\mathcal{K}_A - a^A\mathcal{K}_A - \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} - 2a^{\mu[AB]}\mathcal{K}_{\mu AB} - 2\tau^{\mu\nu}\nabla_\mu a_\nu\\ &- a^{ABC}(\tfrac{1}{4}a_{ABC} + \tfrac{1}{2}a_{ACB} + \eta_{BC}a_A)\\ = {}& 0\,.\end{aligned} \tag{7.41}$$

Note that this expression could equivalently be rewritten in terms of the Ricci tensor, using the following identity:

$$\mathcal{R}^A{}_A = \tau^{\mu\nu}\mathcal{R}_{\mu\nu} = -\nabla^A\mathcal{K}_A - \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} - a^{\mu AB}\mathcal{K}_{\mu AB}\,. \tag{7.42}$$

Remarkably, equation (7.41) transforms covariantly under *local* dilatations. Exactly this equation will also be selected by the exceptional field theory description as an 'extra' equation of motion that one can not find from the variation of the finite part of the action. Furthermore, under Galilean boosts (discussed in next subsection), it transforms into the other equations of motions. All this is in keeping with the properties of the missing Poisson equation in the NSNS sector [26,27] and supports including equation (7.41) as an equation of motion of the non-relativistic theory.

If we think in terms of the expansion it might seem strange to find the rest of the equations of motion from the expansion at order c^0 and this extra equation from order c^{-3}. Clearly, if we would vary the action $S^{(-3)}$ we would find additional $\mathcal{O}(c^{-3})$ contributions to the finite equations of motion, and if we would vary the action $S^{(-6)}$ we would find additional $\mathcal{O}(c^{-3})$ contributions to the equation of motion (7.41), i.e. it would become $\mathcal{N} = \mathcal{O}(c^{-3})$. The guiding philosophy is to find the lowest order non-zero equation of motion resulting from the variations of the action. For the Poisson equation associated to the degree of freedom that disappears into dilatations at the level of $S^{(0)}$, this happens to arise at lower order than the other equations of motion.

As a final remark, just as in the NSNS sector case [27], it is also possible to define a covariant derivative that is covariant with respect to dilatations. Letting b_μ denote this dilatation connection, and simultaneously introducing $\omega_\mu{}^{AB}$ as the longitudinal spin connection, we this new affine connection is defined by the following metric compatibility conditions

$$\widetilde{\nabla}_\mu\tau_\nu{}^A = \partial_\mu\tau_\nu{}^A - \omega_\mu{}^{AB}\tau_{\nu B} - b_\mu\tau_\nu{}^A - \widetilde{\Gamma}^\rho_{\mu\nu}\tau_\rho{}^A = 0\,, \tag{7.43}$$

$$\widetilde{\nabla}_\mu H^{\rho\sigma} = \partial_\mu H^{\rho\sigma} - b_\mu H^{\rho\sigma} + \widetilde{\Gamma}^\rho_{\mu\lambda}H^{\lambda\sigma} + \widetilde{\Gamma}^\sigma_{\mu\lambda}H^{\rho\lambda} = 0\,. \tag{7.44}$$

The solution to these equations is

$$\widetilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \tau^\rho{}_A\left(b_\mu\tau_\nu{}^A + \omega_\mu{}^{AB}\tau_{\nu B}\right) - \frac{1}{2}H^{\rho\sigma}\left(b_\mu H_{\nu\rho} + b_\nu H_{\mu\rho} - b_\rho H_{\mu\nu}\right) \tag{7.45}$$

where the dilatation and spin connections are explicitly given by

$$b_\mu = \frac{1}{3}a_\mu + \frac{1}{6}\tau_\mu{}^A a_A\,, \quad \omega_\mu{}^{AB} = -a_\mu{}^{[AB]} + \frac{1}{2}\tau_\mu{}^C a^{AB}{}_C + \tau_\mu{}^{[A}a^{B]}\,. \tag{7.46}$$

7.3 Boost invariance

Now let's consider the boost transformations defined in (6.28). The calculations are very similar to those in the previous subsection. The variation of $S^{(3)}$ under (6.28) vanishes identically. The variation of the finite action gives

$$\delta S^{(0)} = \int d^{11}x\Big[-\Lambda_\rho{}^A\left(2H^{\mu\rho}\tau^\nu{}_A\Omega\mathcal{G}^{(3)}_{\mu\nu} + 3\epsilon_{ABC}\tau_\mu{}^B\tau_\nu{}^C\mathcal{E}^{\mu\nu\rho}_{(0)}\right) + \delta_\Lambda\widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(3)}\Big]\,, \tag{7.47}$$

and the combination of $\mathcal{G}$ and $\mathcal{E}$ terms appearing here is

$$-2\Omega\mathcal{G}^{(3)}_{A\mu}\Lambda^{\mu A} - 3\epsilon_{ABC}\mathcal{E}^{\mu AB}_{(0)}\Lambda_\mu{}^C = \tfrac{1}{6}F^{A\mu\nu\rho}\Lambda^\sigma{}_A F_{\sigma\mu\nu\rho} - \tfrac{\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}}{4\cdot 4!^2\Omega}F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8}\Lambda_{\lambda_1}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C\epsilon_{ABC}\,. \tag{7.48}$$

Using $\Lambda_{\mu A}\tau^\mu{}_B = 0$ and the Schouten identity this can be shown to be proportional to the self-duality constraint. Hence the finite action $S^{(0)}$ is invariant under boosts up to a total derivative and the self-duality constraint. To make the action boost-invariant off-shell we must improve the transformations (**??**) by requiring $\widetilde{F}$ to transform as well, similarly to (7.30). The improved boost transformations are

$$\begin{aligned} \delta_\Lambda H_{\mu\nu} &= 2\Lambda_{(\mu}{}^A\tau_{\nu)A}\,, & \delta_\Lambda\tau^\mu{}_A &= -H^{\mu\nu}\Lambda_{\nu A}\,, \\ \delta_\Lambda C_{\mu\nu\rho} &= -3\epsilon_{ABC}\Lambda_{[\mu}{}^A\tau_\nu{}^B\tau_{\rho]}{}^C\,, & \delta_\Lambda\widetilde{F}_{\mu\nu\rho\sigma} &= -4\tau^\lambda{}_A F_{\lambda[\mu\nu\rho}\Lambda_{\sigma]}{}^A\,. \end{aligned} \tag{7.49}$$

Furthermore, one can then check that the set of equations of motion presented in the previous sections is boost-invariant (i.e. closed under boosts) as expected. This includes the extra equation of motion (7.41), which under boosts transforms into the time-space projection of the equation of motion of $H^{\mu\nu}$, equation (7.21), as well as the self-duality constraint. This further implies that it is consistent to include it on the same footing as the remaining equations of motion that can be derived by varying $S^{(0)}$. Indeed, one can obtain the boost variation directly from that of $S^{(-3)}$, which is:

$$\delta S^{(-3)} = \int \mathrm{d}^{11}x \Big[-\Lambda_\rho{}^A \left(2H^{\mu\rho}\tau^\nu{}_A \Omega \mathcal{G}^{(0)}_{\mu\nu} + 3\epsilon_{ABC}\tau_\mu{}^B \tau_\nu{}^C \mathcal{E}^{\mu\nu\rho}_{(-3)} \right) + \delta_\Lambda \widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(0)} \Big] . \tag{7.50}$$

The quantity in round brackets is exactly the time-space projection of the $H^{\mu\nu}$ equation of motion. (As a side-remark, note that this means that the boost variation of $S^{(-3)}$ is not identically zero, although it is zero on using the equations of motion following from the finite action.)

8 Dimensional Reductions and Type IIA Newton-Cartan

In this section we will propose reductions from the 11-dimensional Newton-Cartan theory to ten-dimensional type IIA Newton-Cartan theories. We have a choice of whether to reduce on a longitudinal or a transverse direction. Reducing on a longitudinal direction will lead to type IIA stringy Newton-Cartan with RR fields. Reducing on a transverse direction will lead to a novel type IIA Newton-Cartan geometry which can be thought of as arising from a non-relativistic limit associated to D2 branes rather than strings. Similar reductions have been carried out in [37, 48] from the M2 worldvolume theory.

For comparison with the reduction ansatzes below, let us record here the usual decomposition of the eleven-dimensional metric and three-form into ten-dimensional fields:

$$d\hat{s}^2_{11} = e^{4\hat{\Phi}/3}(dy + \hat{A}_1)^2 + e^{-2\hat{\Phi}/3} d\hat{s}^2_{10} , \quad \hat{C}_3 = \hat{A}_3 + \hat{B}_2 \wedge dy , \tag{8.1}$$

where y denotes the direction on which we reduce.

Index book-keeping In this section, we denote the 11-dimensional Newton-Cartan fields and curved spacetime indices with hats, thus $\hat{h}^a{}_{\hat{\mu}}$, $\hat{\tau}_{\hat{\mu}}{}^A$, $\hat{\Omega}$, and so on such that the 11-dimensional coordinates are $x^{\hat{\mu}} = (x^\mu, y)$, with $\mu = 0, \dots, 9$. We assume that we have an isometry in the y direction. The 11-dimensional three-forms are denoted $C_{\hat{\mu}\hat{\nu}\hat{\rho}}$, $\widetilde{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$.

8.1 Type IIA SNC

Here we present a reduction ansatz which produces the known Stueckelberg gauge-fixed form of the SNC NSNS sector action, supplemented with RR fields.

Reduction ansatz We want to reduce on a longitudinal direction. We therefore split the longitudinal index $A = (\mathsf{A}, 2)$ with $\mathsf{A} = 0, 1$. Then we single out

$$\hat{\tau}^2 \equiv e^{2\Phi/3}(dy + A_\mu dx^\mu) , \tag{8.2}$$

thereby defining the dilaton Φ and RR one-form A_μ that will appear in the reduced theory. If we take $\hat{\tau}_2 = e^{-2\Phi/3}\partial_y$ then the remaining pair of Newton-Cartan clock forms and vectors must have the form

$$\hat{\tau}^{\mathsf{A}} = e^{-\Phi/3}\tau_\mu{}^{\mathsf{A}} dx^\mu , \quad \hat{\tau}_{\mathsf{A}} = e^{+\Phi/3}(\tau^\mu{}_{\mathsf{A}}\partial_\mu, -\tau^\nu{}_{\mathsf{A}} A_\nu \partial_y) . \tag{8.3}$$

A compatible ansatz for the transverse vielbein is

$$\hat{h}^a{}_{\hat{\mu}} = (e^{-\Phi/3}h^a{}_\mu, 0) , \quad \hat{h}^{\hat{\mu}}{}_a = (e^{\Phi/3}h^\mu{}_a, -e^{\Phi/3}h^\nu{}_a A_\nu) . \tag{8.4}$$

These are such that ${\tau_\mu}^{\mathsf{A}}$, ${\tau^\mu}_{\mathsf{A}}$ and ${h^\mu}_a$, ${h^a}_\mu$ are ten-dimensional fields obeying the usual stringy Newton-Cartan completeness identities. We can define $\tau_{\mu\nu} \equiv {\tau_\mu}^{\mathsf{A}}{\tau_\nu}^{\mathsf{B}}\eta_{\mathsf{AB}}$, $H_{\mu\nu} \equiv {h^a}_\mu {h^b}_\nu \delta_{ab}$, and similarly for the projective inverses. We also have

$$\hat{\Omega} = e^{-8\Phi/3}\Omega\,, \quad \Omega \equiv \tfrac{1}{2!8!}\epsilon^{\mu\nu\sigma_1\ldots\sigma_8}\epsilon_{\mathsf{AB}}\epsilon_{a_1\ldots a_8}{\tau_\mu}^{\mathsf{A}}{\tau_\nu}^{\mathsf{B}}{h^{a_1}}_{\sigma_1}\ldots {h^{a_8}}_{\sigma_8}\,. \tag{8.5}$$

Finally, we make the traditional decomposition of the three-form and its field strength:

$$C_3 = A_3 + B_2\wedge dy\,, \quad F_4 = G_4 + H_3\wedge(dy + A_1)\,, \quad G_4 = dA_3 - A_1\wedge\mathcal{H}_3\,, \quad \mathcal{H}_3 = dB_2\,, \tag{8.6}$$

where $A_1 \equiv A_\mu dx^\mu$, along with

$$\widetilde{C}_3 = \widetilde{A}_3 + \widetilde{B}_2\wedge dy\,, \quad \widetilde{F}_4 = \widetilde{G}_4 + \widetilde{\mathcal{H}}_3\wedge(dy + A_1)\,, \quad \widetilde{G}_4 = d\widetilde{A}_3 - A_1\wedge\widetilde{\mathcal{H}}_3\,, \quad \widetilde{\mathcal{H}}_3 = d\widetilde{B}_2\,. \tag{8.7}$$

Interpretation as an expansion Inserting the above ansatz into the original limit (**??**) gives

$$\begin{aligned} d\hat{s}^2_{11} &= c^2 e^{4\Phi/3}(dy + A_1)^2 + e^{-2\Phi/3}(c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu})\,,\\ \hat{C}_3 &= -c^3\tfrac{1}{2}\epsilon_{\mathsf{AB}}\tau^{\mathsf{A}}\wedge\tau^{\mathsf{B}}\wedge dy + A_3 + B_2\wedge dy + c^{-3}(\widetilde{A}_3 + \widetilde{B}_2\wedge dy)\,. \end{aligned} \tag{8.8}$$

Hence according to (8.1) this translates into the following expansion of the ten-dimensional type IIA string frame metric $\hat{g}_{\mu\nu}$, NSNS two-form, $\hat{B}_2$, and dilaton $\hat{\Phi}$:

$$\begin{aligned} \hat{g}_{\mu\nu} &= c_s^2\tau_{\mu\nu} + H_{\mu\nu}\,,\\ \hat{B}_2 &= -c_s^2\epsilon_{\mathsf{AB}}\tau^{\mathsf{A}}\wedge\tau^{\mathsf{B}} + B_2 + c_s^{-2}\widetilde{B}_2\,,\\ e^{\hat{\Phi}} &= c_s e^{\Phi}\,, \end{aligned} \tag{8.9}$$

where $c_s \equiv c^{3/2}$. This is nothing but the limit leading to stringy Newton-Cartan. In addition, we have an expansion of the RR fields:

$$\hat{A}_3 = A_3 + c_s^{-2}\widetilde{A}_3\,, \quad \hat{A}_1 = A_1\,, \tag{8.10}$$

It is clear from these expressions that we can equivalently view this reduction as the result of the usual M-theory to type IIA reduction using (8.1) followed by the SNC field redefinitions of (8.9) and (8.10). At first glance, this is not completely general, given that the ansatz for the RR 1-form A_1 does not involve a subleading term while the other gauge fields do. A justification for the above ansatz is that it correctly produces the NSNS sector dynamics of SNC. Modifications to the ansatz would involve relaxing the implicit Stueckelberg gauge-fixing in 11-dimensions and comparing this to the possible 10-dimensional expansions. We do not consider this in this paper.

Constraint The constraint (6.26) becomes

$$\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}G_{\nu_1\nu_2\nu_3\nu_4} = -\tfrac{1}{4!2!}\epsilon^{\mu_1\ldots\mu_{10}}G_{\mu_5\mu_6\mu_7\mu_8}\epsilon_{\mathsf{AB}}{\tau_{\mu_9}}^{\mathsf{A}}{\tau_{\mu_{10}}}^{\mathsf{B}} \tag{8.11}$$

and so only involves the RR 4-form field strength. The field strength of the NSNS 2-form is not constrained. This is to be expected, as the limit of the NSNS sector alone makes sense without any constraint, and in the eleven-dimensional case the constraint arose as a consequence of the Chern-Simons term, which is not present in the truncation to the NSNS sector.

Type IIA SNC with RR fields The action obtained from the reduction ansatz (8.3) and (8.4) is

$$S_{\text{IIA SNC}} = \int \mathrm{d}^{10}x\, \Omega \left(e^{-2\Phi}\mathcal{L} + \mathcal{L}_{\widetilde{G}} + \Omega^{-1}\mathcal{L}_{\text{top}}\right) \tag{8.12}$$

with

$$\begin{aligned} \mathcal{L} =& \mathcal{R} - a^{\mu\mathsf{AB}}a_{\mu\{\mathsf{AB}\}} + (a^\mu - 2D^\mu\Phi)(a_\mu - 2D_\mu\Phi) - \tfrac{1}{12}\mathcal{H}^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho} - \tfrac{1}{2}\epsilon_{\mathsf{AB}}\tau^{\rho\mathsf{A}}\mathcal{H}_{\rho\mu\nu}T^{\mu\nu\mathsf{B}} \\ &- \tfrac{1}{2}e^{2\Phi}G^{\mu\mathsf{A}}G_{\mu\mathsf{A}} - \tfrac{1}{12}e^{2\Phi}G^{\mu\nu\rho\mathsf{A}}G_{\mu\nu\rho\mathsf{A}} + \tfrac{1}{4}e^{2\Phi}\epsilon^{\mathsf{AB}}G_{\mathsf{AB}\rho\sigma}G^{\rho\sigma}, \\ \mathcal{L}_{\widetilde{G}} =& -\tfrac{1}{4!}\widetilde{G}_{\nu_1\ldots\nu_4}\left(G^{\nu_1\ldots\nu_4} + \tfrac{1}{4!2!\Omega}\epsilon^{\nu_1\ldots\nu_4\mu_1\ldots\mu_6}G_{\mu_1\ldots\mu_4}\epsilon_{\mathsf{AB}}\tau_{\mu_5}{}^{\mathsf{A}}\tau_{\mu_6}{}^{\mathsf{B}}\right), \\ \mathcal{L}_{\text{top}} =& \tfrac{1}{2}dA_3\wedge dA_3\wedge B_2\,, \end{aligned} \tag{8.13}$$

using the field strengths defined in (8.6) and (8.7) along with $G_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}$. As before, we write for convenience $G^{\mu\nu} \equiv H^{\mu\rho}H^{\nu\sigma}G_{\rho\sigma}$. The Ricci scalar and connection, torsion, acceleration and so on are defined in the same way as before but for the SNC geometry. If we ignore the RR fields, this is exactly the Stueckelberg gauge fixed action for NSNS SNC (note that the subleading component $\widetilde{B}_2$ only appears in the definition of $\widetilde{G}_4$). Furthermore, one can check that the reduction of the Poisson equation agrees with the Poisson equation for SNC, with of course additional contributions from the RR sector. The reduced Poisson equation is found to be

$$\begin{aligned} &-\tfrac{1}{2}\epsilon_{\mathsf{AB}}\nabla_\mu\mathcal{H}^{\mathsf{AB}\mu} + \nabla^{\mathsf{A}}\mathcal{K}_{\mathsf{A}} - 2\tau^{\mu\nu}\nabla_\mu\nabla_\nu\Phi + 2\tau^{\mu\nu}\nabla_\mu a_\nu + \epsilon_{\mathsf{AB}}\mathcal{H}^{\mathsf{AB}\mu}\nabla_\mu\Phi - 2a^{\mathsf{A}}\nabla_{\mathsf{A}}\Phi \\ &\quad + \mathcal{K}^{\mu\nu\mathsf{A}}\mathcal{K}_{\mu\nu\mathsf{A}} + a^{\mathsf{A}}\mathcal{K}_{\mathsf{A}} + 2a^{\mu[\mathsf{AB}]}\mathcal{K}_{\mu\mathsf{AB}} + a^{\mathsf{ABC}}\left(\tfrac{1}{4}a_{\mathsf{ABC}} + \tfrac{1}{2}a_{\mathsf{ACB}} + \eta_{\mathsf{BC}}a_{\mathsf{A}}\right) \\ &\quad + \tfrac{1}{4}\mathcal{H}^{\mathsf{A}\mu\nu}\mathcal{H}_{\mathsf{A}\mu\nu} - \epsilon_{\mathsf{AB}}\mathcal{H}^{\mathsf{CB}\mu}\left(a_{\mu\mathsf{C}}{}^{\mathsf{A}} + \tfrac{1}{2}a_\mu\delta^{\mathsf{A}}_{\mathsf{C}}\right) + \tfrac{1}{4}e^{2\Phi}\left(G^{\mathsf{AB}}G_{\mathsf{AB}} + \tfrac{1}{2}G^{\mathsf{AB}\mu\nu}G_{\mathsf{AB}\mu\nu}\right) \\ &\quad - e^{2\Phi}\tfrac{1}{48}\left(\widetilde{G}^{\mu\nu\rho\sigma}\widetilde{G}_{\mu\nu\rho\sigma} + \tfrac{1}{48\Omega}\epsilon^{\lambda_1\lambda_2\mu_1\ldots\mu_8}\widetilde{G}_{\mu_1\ldots\mu_4}\widetilde{G}_{\mu_5\ldots\mu_8}\epsilon_{\mathsf{AB}}\tau_{\lambda_1}{}^{\mathsf{A}}\tau_{\lambda_2}{}^{\mathsf{B}}\right) \\ &\quad = 0\,. \end{aligned} \tag{8.14}$$

In this case [27], it is the longitudinal components of the NSNS 2-form playing the role of the Newton potential. It is also interesting to look at the reduction of the equation (7.15), which was the equation of motion of the longitudinal components of the three-form. This reduces to

$$\tfrac{1}{2}\eta_{\mathsf{AB}}H^{\mu\rho}H^{\nu\sigma}T_{\mu\nu}{}^{\mathsf{A}}T_{\rho\sigma}{}^{\mathsf{B}} = -\tfrac{1}{48}e^{2\Phi}H^{\mu_1\nu_1}\ldots H^{\mu_4\nu_4}G_{\mu_1\ldots\mu_4}G_{\nu_1\ldots\nu_4}\,, \tag{8.15}$$

and in particular in the truncation to the NSNS sector the right-hand side is zero. This allows imposing foliation constraints on the NSNS sector SNC torsion $T_{\mu\nu}{}^{\mathsf{A}}$, such as those discussed in [27].

8.2 Type IIA D2NC

General decompositions breaking local rotational invariance The next reduction we do involves reducing on a transverse reduction. This breaks part of the local SO(8) rotational invariance. Accordingly, write the flat index $a = (\mathsf{a}, \bar{\imath})$, with $\mathsf{a} = 1, \ldots, 8-q$ and $\bar{\imath} = 1\ldots q$. Simultaneously we can consider a *different* decomposition of the spacetime coordinate index $\hat{\mu} = (\mu, i)$ where μ is n-dimensional and i is $(11-n)$-dimensional. We then pick a lower triangular form for the vielbein $\hat{h}^a{}_{\hat{\mu}}$ such that

$$\hat{h}^a{}_{\hat{\mu}} = \begin{pmatrix} h^{\mathsf{a}}{}_\mu & 0 \\ A_\mu{}^k h^{\bar{\imath}}{}_k & h^{\bar{\imath}}{}_i \end{pmatrix}. \tag{8.16}$$

The condition $\hat{h}^a{}_{\hat{\mu}}\hat{\tau}^{\hat{\mu}}{}_A = 0$ implies

$$h^{\mathsf{a}}{}_\mu\hat{\tau}^\mu{}_A = 0\,, \quad h^{\bar{\imath}}{}_i(\hat{\tau}^i{}_A + A_\mu{}^i\hat{\tau}^\mu{}_A) = 0\,. \tag{8.17}$$

The diagonal blocks in (8.16) will in general not be square. Two interesting examples however are to take these blocks to be square and invertible. In this subsection, we will take the lower right block to be a non-zero 1×1 matrix, and perform a reduction to a novel type of type IIA Newton-Cartan geometry associated to D2 branes. In section 9, we will take the upper left block to be an invertible $(11-d) \times (11-d)$ matrix, and offer a description of the M-theory Newton-Cartan theory in terms of exceptional field theory.

Transverse reduction to type IIA The dimensional reduction to type IIA corresponds to taking $n = 10$, and $q = 1$ above. We again label the coordinates again as $x^{\hat{\mu}} = (x^\mu, y)$. In this case $h^{\bar{y}}{}_y$ is a scalar and we can identify it with the dilaton as $h^{\bar{y}}{}_y \equiv e^{2\Phi/3}$. Using the conditions (8.17), the full Kaluza-Klein ansatz is:

$$\hat{h}^a{}_{\hat{\mu}} = \begin{pmatrix} e^{-\Phi/3} h^{\mathsf{a}}{}_\mu & 0 \\ e^{2\Phi/3} A_\mu & e^{2\Phi/3} \end{pmatrix}, \quad \hat{h}^{\hat{\mu}}{}_a = \begin{pmatrix} e^{\Phi/3} h^\mu{}_{\mathsf{a}} & 0 \\ -e^{\Phi/3} A_\nu h^\nu{}_{\mathsf{a}} & e^{-2\Phi/3} \end{pmatrix}, \tag{8.18}$$

$$\hat{\tau}_{\hat{\mu}}{}^A = e^{-\Phi/3} (\tau_\mu{}^A, 0), \quad \hat{\tau}^{\hat{\mu}}{}_A = e^{+\Phi/3} (\tau^\mu{}_A, -A_\nu \tau^\nu{}_A), \tag{8.19}$$

plus the same definitions (8.6) and (8.7) for the three-forms and field strengths. We also have

$$\hat{\Omega} = e^{-8\Phi/3} \Omega, \quad \Omega \equiv \tfrac{1}{3!7!} \epsilon^{\mu\nu\rho\sigma_1\dots\sigma_7} \epsilon_{ABC} \epsilon_{\mathsf{a}_1\dots\mathsf{a}_7} \tau_\mu{}^A \tau_\nu{}^B \tau_\rho{}^C h^{\mathsf{a}_1}{}_{\sigma_1} \dots h^{\mathsf{a}_7}{}_{\sigma_1}. \tag{8.20}$$

Interpretation as an expansion Inserting the above ansatz into the original limit (**??**) gives

$$\begin{aligned} d\hat{s}^2_{11} &= c^{-1} e^{4\Phi/3} (dy + A_1)^2 + e^{-2\Phi/3} (c^2 \tau_{\mu\nu} + c^{-1} H_{\mu\nu}), \\ \hat{C}_3 &= -c^3 e^{-\Phi} \tfrac{1}{3!} \epsilon_{ABC} \tau^A \wedge \tau^B \wedge \tau^C + A_3 + B_2 \wedge dy + c^{-3} (\widetilde{A}_3 + \widetilde{B}_2 \wedge dy). \end{aligned} \tag{8.21}$$

Hence according to (8.1) this translates into the following expansion of the ten-dimensional type IIA string frame metric $\hat{g}_{\mu\nu}$, RR three-form, $\hat{C}_2$, and dilaton $\hat{\Phi}$:

$$\begin{aligned} \hat{g}_{\mu\nu} &= c_D^2 \tau_{\mu\nu} + c_D^{-2} H_{\mu\nu}, \\ \hat{C}_3 &= -c_D^4 \epsilon_{ABC} e^{-\Phi} \tau^A \wedge \tau^B \wedge \tau^C + C_3 + c_D^{-4} \widetilde{C}_3, \\ e^{\hat{\Phi}} &= c_D^{-1} e^{\Phi}, \end{aligned} \tag{8.22}$$

along with expansions for the NSNS two-form, $\hat{B}_2$, and RR one-form, $\hat{A}_1$:

$$\hat{B}_2 = B_2 + c_D^{-4} \widetilde{B}_2, , \quad \hat{A}_1 = A_1, \tag{8.23}$$

where $c_D \equiv c^{3/4}$. This is an expansion and non-relativistic limit associated to the D2 brane (the powers of c_D appear in the same way as those of the harmonic function in the D2 brane SUGRA solution). We can refer to it as D2 Newton-Cartan (D2NC).

Constraint The constraint (6.26) becomes

$$\begin{aligned} \Omega H^{\mu_1\nu_1} H^{\mu_2\nu_2} H^{\mu_3\nu_3} H^{\mu_4\nu_4} G_{\nu_1\nu_2\nu_3\nu_4} &= +\tfrac{1}{3!3!} e^{-\Phi} \epsilon^{\mu_1\dots\mu_{10}} H_{\mu_5\mu_6\mu_7} \epsilon_{ABC} \tau_{\mu_8}{}^A \tau_{\mu_9}{}^B \tau_{\mu_{10}}{}^C, \\ \Omega e^{-\Phi} H^{\mu_1\nu_1} H^{\mu_2\nu_2} H^{\mu_3\nu_3} \mathcal{H}_{\nu_1\nu_2\nu_3} &= +\tfrac{1}{4!3!} \epsilon^{\mu_1\dots\mu_{10}} G_{\mu_4\mu_5\mu_6\mu_7} \epsilon_{ABC} \tau_{\mu_8}{}^A \tau_{\mu_9}{}^B \tau_{\mu_{10}}{}^C, \end{aligned} \tag{8.24}$$

which are equivalent. So now we have a duality relation between the RR 3-form gauge field and the NSNS 2-form.

Type IIA D2 Newton-Cartan theory The action obtained from the reduction ansatz (8.18) and (8.2) is

$$S_{\text{D2NC}} = \int \mathrm{d}^{10}x\,\Omega\,\left(e^{-2\Phi}\mathcal{L} + \mathcal{L}_{\widetilde{G}} + \Omega^{-1}\mathcal{L}_{\text{top}}\right) \tag{8.25}$$

with

$$\begin{aligned}
\mathcal{L} =& \mathcal{R} - a^{\mu AB}a_{\mu(AB)} + \tfrac{3}{2}a^\mu a_\mu - 5a^\mu D_\mu\Phi + \tfrac{9}{2}D^\mu\Phi D_\mu\Phi - \tfrac{1}{4}\mathcal{H}^{\mu\nu A}\mathcal{H}_{\mu\nu A}\\
&- \tfrac{1}{4}e^{2\Phi}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{12}e^{2\Phi}G^{\mu\nu\rho A}G_{\mu\nu\rho A} + \tfrac{1}{4}e^{\Phi}\epsilon^{ABC}G_{AB\rho\sigma}T^{\rho\sigma}{}_C\,,\\
\mathcal{L}_{\widetilde{G}} =& -\tfrac{1}{4!}\widetilde{G}_{\nu_1\dots\nu_4}\left(G^{\nu_1\dots\nu_4} - \tfrac{1}{3!^2\Omega}e^{-\Phi}\epsilon^{\nu_1\dots\nu_4\mu_1\dots\mu_6}\mathcal{H}_{\mu_1\dots\mu_3}\epsilon_{ABC}\tau_{\mu_4}{}^A\tau_{\mu_5}{}^B\tau_{\mu_6}{}^C\right)\\
&- \tfrac{1}{3!}e^{-2\Phi}\widetilde{\mathcal{H}}_{\nu_1\dots\nu_3}\left(\mathcal{H}^{\nu_1\dots\nu_3} - \tfrac{1}{4!3!\Omega}e^{+\Phi}\epsilon^{\nu_1\dots\nu_3\mu_1\dots\mu_7}G_{\mu_1\dots\mu_4}\epsilon_{ABC}\tau_{\mu_5}{}^A\tau_{\mu_6}{}^B\tau_{\mu_7}{}^C\right),\\
=& -\tfrac{1}{4!}\left(\widetilde{G}_{\nu_1\dots\nu_4} - \tfrac{1}{3!}e^{-\Phi}\widetilde{\mathcal{H}}_{\rho_1\dots\rho_3}\epsilon^{\rho_1\dots\rho_3\sigma_1\dots\sigma_7}\tfrac{1}{3!\Omega}\epsilon_{ABC}H_{\nu_1\sigma_1}\dots H_{\nu_4\sigma_4}\tau_{\sigma_5}{}^A\tau_{\sigma_6}{}^B\tau_{\sigma_7}{}^C\right)\times\\
&\times\left(G^{\nu_1\dots\nu_4} - \tfrac{1}{3!^2\Omega}e^{-\Phi}\epsilon^{\nu_1\dots\nu_4\mu_1\dots\mu_6}\mathcal{H}_{\mu_1\dots\mu_3}\epsilon_{ABC}\tau_{\mu_4}{}^A\tau_{\mu_5}{}^B\tau_{\mu_6}{}^C\right),\\
\mathcal{L}_{\text{top}} =& \tfrac{1}{2}dA_3\wedge dA_3\wedge B_2\,,
\end{aligned} \tag{8.26}$$

where the field strengths are defined as in (8.6) and (8.7) with again $G_2 \equiv dA_1$. Note that we obtain what appears to be an extra contribution to the dilaton kinetic term due to the $e^{-\Phi}$ factor that in the expansion of $\hat{C}_3$ in (8.22). We could alter this by redefining the RR fields in the reduced theory. In addition, the reduction of the Poisson equation (7.41) gives

$$\begin{aligned}
&\tfrac{1}{6}e^{\Phi}\epsilon_{ABC}\left(\nabla_\mu G^{ABC\mu} + a_\mu G^{ABC\mu} + 3a_{\mu D}{}^A G^{DBC\mu}\right) - \tfrac{1}{3}e^{\Phi}\epsilon_{ABC}G^{ABC\mu}\nabla_\mu\Phi\\
&+ \nabla^A\mathcal{K}_A - 3\tau^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - 3a^A\nabla_A\Phi + 2\nabla^A\Phi\nabla_A\Phi - \mathcal{K}^A\nabla_A\Phi + 2\tau^{\mu\nu}\nabla_\mu a_\nu\\
&+ \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} + a^A\mathcal{K}_A + 2a^{\mu[AB]}\mathcal{K}_{\mu AB} + a^{ABC}\left(\tfrac{1}{4}a_{ABC} + \tfrac{1}{2}a_{ACB} + \eta_{BC}a_A\right)\\
&+ \tfrac{1}{4}\mathcal{H}^{AB\mu}\mathcal{H}_{AC\mu} + \tfrac{1}{8}e^{2\Phi}\left(G^{AB\mu\nu}G_{AB\mu\nu} + 4G^{A\mu}G_{A\mu}\right) - e^{2\Phi}\tfrac{1}{48}\widetilde{G}^{\mu\nu\rho\sigma}\widetilde{G}_{\mu\nu\rho\sigma} - \tfrac{1}{12}\widetilde{\mathcal{H}}^{\mu\nu\rho}\widetilde{\mathcal{H}}_{\mu\nu\rho}\\
&+ e^{-\Phi}\tfrac{1}{4!3!3!\Omega}\epsilon^{\lambda_1\lambda_2\lambda_3\mu_1\dots\mu_7}\epsilon_{ABC}\tau_{\lambda_1}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C\widetilde{G}_{\mu_1\dots\mu_4}\widetilde{\mathcal{H}}_{\mu_5\dots\mu_7} = 0\,.
\end{aligned} \tag{8.27}$$

As in the MNC case, the longitudinal components of the three-form gauge field play the role of the Newton potential.

9 Dimensional Decompositions and Exceptional Field Theory Description

9.1 Exceptional field theory

We will now discuss the exceptional field theory description of the 11-dimensional MNC theory. ExFT automatically has a number of features in common with the non-relativistic theory: breaking of 11-dimensional Lorentz symmetry, a geometry arising from mixing metric and form-field components, and the inclusion of dual degrees of freedom. We will see how it provides a unified framework treating the relativistic and non-relativistic theory on an equal footing, which demonstrates that the same exceptional Lie algebraic structures that underlie the relativistic theory are present in the non-relativistic one. In addition, the ExFT equations of motion include the additional missing Poisson equation.

We will focus particularly on the relatively unexceptional case of the SL(3) × SL(2) ExFT [54]. This makes use of an (8 + 3)-dimensional split of the 11-dimensional spacetime. As such, it is a very natural fit for the (8 + 3)-dimensional split into transverse and longitudinal directions present in the MNC expansion. The SL(3) × SL(2) ExFT includes a Riemannian metric for the 8-dimensional part of the spacetime, but the 3-dimensional part is described by an 'extended geometry' involving an SL(3)×SL(2) symmetric generalised metric. By decomposing the 11-dimensional Newton-Cartan theory appropriately, we will replace the transverse Newton-Cartan metric with an invertible 8-dimensional metric, $\hat{H}^{\hat{\mu}\hat{\nu}} \to g^{\mu\nu}$, and the longitudinal metric with an invertible 3-dimensional metric, $\hat{\tau}_{\hat{\mu}\nu} \to \tau_{ij}$, which will be embedded into the generalised metric description. This drastic simplification of the geometry is nonetheless sufficient to highlight the key features of the theory.

It would also be interesting to consider for example the opposite (3+8)-dimensional split corresponding to the $E_{8(8)}$ ExFT, embedding the transverse metric into the $E_{8(8)}$ generalised metric. However as the known formulation of ExFT makes use of a Riemannian metric for the unextended part of the spacetime, this is not immediately available for our purposes. Evidently, for any given $E_{d(d)}$ ExFT, one can construct or imagine multiple other 'hybrid' formulations depending on how one chooses to separate or mix the longitudinal and transverse directions. More ambitiously, one could choose to work with the recently fully constructed 'master' E_{11} ExFT [55], for which no coordinate decomposition is necessary. Evidently this would eschew the technical difficulties of the latter in favour of the technicalities associated to working with an infinite-dimensional algebra. In this paper, although many features that we will see are quite general, we describe the explicit details mainly for the $d \leq 4$ cases.

ExFT ingredients The basic idea behind ExFT is to replace d-dimensional vectors with *generalised vectors* V^M transforming in a specified representation of $E_{d(d)}$. This representation is such that we can decompose V^M under GL(d) as $V^M = (V^i, V_{ij}, V_{ijklm}, \dots)$ where V^i is a d-dimensional vector, V_{ij} and V_{ijklm} a two- and five-form, and the ellipsis corresponds to higher rank mixed symmetry tensors that appear for $d \geq 7$. Generalised vectors are used to provide an $E_{d(d)}$-compatible local symmetry of *generalised diffeomorphisms*. These are defined in terms of a *generalised Lie derivative* which acts on a generalised vector V^M of weight λ_V as

$$\delta_U V^M = \mathcal{L}_U V^M \equiv U^N \partial_N V^M - V^N \partial_N U^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q + (\lambda_V - \tfrac{1}{9-d}) \partial_N U^N V^M \,. \tag{9.1}$$

Here $Y^{MN}{}_{PQ}$ is constructed from invariant tensors of $E_{d(d)}$. This together with the weight term with coefficient $-1/(9-d)$ appear such that this generalised Lie derivative involves an infinitesimal $E_{d(d)}$, rather than GL(N) transformation. The partial derivatives written here formally involve an extended set of coordinates y^M. However, consistency requires the imposition of a constraint $Y^{MN}{}_{PQ}\partial_M \partial_N = 0$ where the derivatives can act on a single field or a product of fields. One solution to this constraint is to view the d-dimensional partial derivatives as being embedded such that $\partial_M = (\partial_i, 0, \dots, 0)$. We always assume we have made this choice below. (An alternative solution leads to a ten-dimensional type IIB description.)

Given this choice, for the $d \leq 4$ cases we will look at in detail, the action of $U^M = (u^i, \lambda_{ij})$ on $V^M = (V^i, V_{ij})$ (both having generalised diffeomorphism weight $1/(9-d)$) is $\mathcal{L}_U V^M = (L_u V^i, L_u V^{ij} - 3V^k \partial_{[k}\lambda_{ij]})$, where L_u denotes the usual d-dimensional Lie derivative. Identifying the two-form components λ_{ij} with the gauge transformation parameter of a three-form $\hat{C}_{ijk}$, this means we can write $V^M = (V^i, \widetilde{V}_{ij} - \hat{C}_{ijk}V^k)$, with $\widetilde{V}_{ij}$ gauge invariant. We use this to give explicit parametrisations for the ExFT fields.

The field content of ExFT is as follows. We now let $\mu, \nu, \dots$ be $(11-d)$-dimensional indices. We then have an $(11-d)$-dimensional metric, $g_{\mu\nu}$, which is a scalar of weight $-2/(9-d)$ under generalised diffeomorphisms. The $E_{d(d)}$ extended geometry is equipped with a generalised metric, $\mathcal{M}_{MN}$, transforming as a rank two symmetric tensor of weight zero under generalised diffeomorphisms. In addition, there is a 'tensor hierarchy' of gauge fields, starting with an $(11-d)$-dimensional one-form $\mathcal{A}_\mu{}^M$, and continuing with p-forms $\mathcal{B}_{\mu\nu}$, $\mathcal{C}_{\mu\nu\rho}$, ... in particular representations of $E_{d(d)}$. This set of fields mimics and extends what appears in a dimensional decomposition (or reduction) of the bosonic fields of supergravity.

Dimensional decomposition and field redefinitions We describe now the dimensional decomposition used to embed 11-dimensional SUGRA in the ExFT framework. We split the 11-dimensional coordinates $x^{\hat{\mu}} = (x^\mu, y^i)$, making an $(11-d)+d$ split. The supergravity degrees of freedom are then similarly decomposed under this split, classified according to their nature from the point of view of $(11-d)$-dimensional spacetime, and then rearranged into multiplets of the exceptional groups $E_{d(d)}$. We assume no restriction on the coordinate dependence. This can be viewed as a partial fixing of the local Lorentz symmetry in which we choose the 11-dimensional vielbein $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$ and hence metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ to be

$$\hat{e}^{\hat{a}}{}_{\hat{\mu}} = \begin{pmatrix} |\phi|^{-\frac{1}{2(9-d)}} e^{\mathsf{a}}{}_\mu & 0 \\ A_\mu{}^k \phi^{\bar{\imath}}{}_k & \phi^{\bar{\imath}}{}_i \end{pmatrix}, \quad \hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} |\phi|^{-\frac{1}{9-d}} g_{\mu\nu} + \phi_{kl} A_\mu{}^k A_\nu{}^l & \phi_{ik} A_\nu{}^l \\ \phi_{jk} A_\nu{}^k & \phi_{ij} \end{pmatrix}, \tag{9.2}$$

where $e^{\mathsf{a}}{}_\mu$ is a vielbein for an $(11-d)$-dimensional (Einstein frame) metric $g_{\mu\nu}$ and $\phi^{\bar{\imath}}{}_i$ is a vielbein for a d-dimensional metric ϕ_{ij}, with $|\phi| \equiv |\det(\phi_{ij})|$. Normally one takes $g_{\mu\nu}$ to be Lorentzian, such that this corresponds to fixing the Lorentz symmetry as $\mathrm{SO}(1,10) \to \mathrm{SO}(1,10-d) \times \mathrm{SO}(d)$, however we can also take it to be Euclidean, such that $\mathrm{SO}(1,10) \to \mathrm{SO}(11-d) \times \mathrm{SO}(1,d-1)$. The latter choice is relevant for the version of ExFT applicable to the non-relativistic theory.

The 'Kaluza-Klein vector' $A_\mu{}^i$ has a field strength defined by

$$F_{\mu\nu}{}^i = 2\partial_{[\mu} A_{\nu]}{}^i - 2A_{[\mu|}{}^j \partial_j A_{|\nu]}{}^i\,. \tag{9.3}$$

Letting L denote the d-dimensional Lie derivative, the Kaluza-Klein vector also appears as the connection in the derivative $D_\mu = \partial_\mu - L_{A_\mu}$ which is covariant with respect to d-dimensional diffeomorphisms, using the transformation $\delta_\Lambda A_\mu{}^i = D_\mu \Lambda^i$ induced by the action of 11-dimensional diffeomorphisms on (9.2).

For the three-form and its field strength, we define a succession of gauge field components (denoted by bold font) via

$$\hat{C}_3 = \hat{\mathbf{C}}_3 + \hat{\mathbf{C}}_{2i} Dy^i + \tfrac{1}{2}\hat{\mathbf{C}}_{1ij} Dy^i Dy^j + \tfrac{1}{3!}\hat{\mathbf{C}}_{ijk} Dy^i Dy^j Dy^k \tag{9.4}$$

where $Dy^i \equiv dy^i + A_\mu{}^i dx^\mu$, the subscripts on the right-hand side denote the form degree in $(11-d)$ dimensions, and we omit the implicit wedge products. Similarly, for $\hat{F}_4 = d\hat{C}_3$ we let

$$\hat{F}_4 = \hat{\mathbf{F}}_4 + \hat{\mathbf{F}}_{3i} Dy^i + \tfrac{1}{2}\hat{\mathbf{F}}_{2ij} Dy^i Dy^j + \tfrac{1}{3!}\hat{\mathbf{F}}_{1ijk} Dy^i Dy^j Dy^k + \tfrac{1}{4!}\hat{\mathbf{F}}_{ijkl} Dy^i Dy^j Dy^k Dy^l\,, \tag{9.5}$$

The persistence of hats reflects the fact that we still want to take the non-relativistic limit of all these quantities. Explicit component expressions can be found in appendix 10. We can make similar redefinitions for the dual six-form and its field strength.

Metric and generalised metrics The metric $g_{\mu\nu}$ appearing in (9.2) is directly used as the $(11-d)$-dimensional ExFT metric (the generalised diffeomorphism weight $-2/(9-d)$ follows from the conformal factor in (9.2)).

The generalised metric $\mathcal{M}_{MN}$, or its generalised vielbein, may be defined as an $E_{d(d)}$ element valued in a coset $E_{d(d)}/H_d$ where H_d is the maximal compact subgroup (in the Euclidean case) or a non-compact version thereof (in the Lorentzian case). Under generalised diffeomorphisms it transforms as a rank two symmetric tensor of weight zero. It is normally parametrised in terms of the wholly d-dimensional components of the eleven-dimensional fields, ϕ_{ij} and $\hat{\mathbf{C}}_{ijk}$, in a manner consistent with its transformation under generalised diffeomorphisms. For $d \geq 6$, this parametrisation also includes internal components of the dual-six form. For simplicity, we will restrict to $d \leq 4$, in which case the conventional parametrisation of the generalised metric is given by

$$\mathcal{M}_{MN} = |\phi|^{1/(9-d)} \begin{pmatrix} \phi_{ij} + \frac{1}{2}\hat{\mathbf{C}}_i{}^{pq}\hat{\mathbf{C}}_{jpq} & \hat{\mathbf{C}}_i{}^{kl} \\ \hat{\mathbf{C}}_k{}^{ij} & 2\phi^{i[k}\phi^{l]j} \end{pmatrix}. \tag{9.6}$$

The conformal factor here ensures that $|\det \mathcal{M}| = 1$.

In specific cases, we can find factorisations of the generalised metric leading to simpler expressions. This includes the SL(3) × SL(2) ExFT. Here, generalised vectors $V^M = (V^i, V_{ij})$ transform in the $(\mathbf{3},\mathbf{2})$ of SL(3)×SL(2), with $i, j, \dots$ three-dimensional. We can dualise V_{ij} using the three-dimensional epsilon symbol, and define $\widetilde{V}^i \equiv \frac{1}{2}\epsilon^{ijk}\widetilde{V}_{jk}$. Introduce an SL(2) fundamental index, $\alpha = 1, 2$, and let $V^M \equiv V^{i\alpha}$ with $V^{i1} \equiv V^i$ and $V^{i2} \equiv \widetilde{V}^i$. In terms of this basis we have a factorisation

$$\mathcal{M}_{MN} = \mathcal{M}_{i\alpha,j\beta} = \mathcal{M}_{ij}\mathcal{M}_{\alpha\beta}\,, \tag{9.7}$$

where $\mathcal{M}_{ij} = \mathcal{M}_{ji}$ with $|\det \mathcal{M}_{ij}| = 1$, and $\mathcal{M}_{\alpha\beta} = \mathcal{M}_{\beta\alpha}$ with $|\det \mathcal{M}_{\alpha\beta}| = 1$. When ϕ_{ij} has Lorentzian signature, the expressions which reproduce (9.6) are

$$\mathcal{M}_{ij} = |\phi|^{-1/3}\phi_{ij}\,,\quad \mathcal{M}_{\alpha\beta} = \begin{pmatrix} |\phi|^{1/2} - |\phi|^{-1/2}\hat{C}^2 & -|\phi|^{-1/2}\hat{C} \\ -|\phi|^{-1/2}\hat{C} & -|\phi|^{-1/2}\end{pmatrix}\,,\quad \hat{C} \equiv \tfrac{1}{3!}\epsilon^{ijk}\hat{C}_{ijk}\,, \tag{9.8}$$

Gauge fields and dual degrees of freedom Along with the Kaluza-Klein vector, $A_\mu{}^i$, coming from the metric decomposition (9.2), the p-forms obtained from the decomposition (9.4) of the three-form fit into $E_{d(d)}$-valued multiplets denoted $\mathcal{A}_\mu$, $\mathcal{B}_{\mu\nu}$, $\mathcal{C}_{\mu\nu\rho}$, ... Their field strengths are denoted $\mathcal{F}_{\mu\nu}$, $\mathcal{H}_{\mu\nu\rho}$, $\mathcal{J}_{\mu\nu\rho\sigma}$, ... To obtain full $E_{d(d)}$ representations, we have to include here the set of p-forms obtained by decomposing the dual six-form. This is unsurprising from the point of $E_{d(d)}$ U-duality transformations, which mix electric and magnetic degrees of freedom (e.g. M2 and M5 branes) coupling respectively to p-forms and their duals.

For $d = 3$, this works as follows [54]. The ExFT gauge fields $\mathcal{A}_\mu{}^{i\alpha}$, $\mathcal{B}_{\mu\nu i}$, $\mathcal{C}_{\mu\nu\rho}{}^\alpha$, $\mathcal{D}_{\mu\nu\rho\sigma}{}^i$ have weights $1/6, 2/6, 3/6, 4/6$ respectively, and their field strengths are denoted $\mathcal{F}_{\mu\nu}{}^{i\alpha}$, $\mathcal{H}_{\mu\nu\rho i}$, $\mathcal{J}_{\mu\nu\rho\sigma}{}^\alpha$ and $\mathcal{K}_{\mu\nu\rho\sigma\lambda}{}^i$ (the latter does not appear in the action). Under generalised diffeomorphisms, $\mathcal{F}^{i\alpha}$ transforms as a generalised vector of weight 1/6, while $\mathcal{H}$ and $\mathcal{J}$ transform via the generalised Lie derivative acting as

$$\mathcal{L}_\Lambda \mathcal{H}_i = \Lambda^{j\beta}\partial_{j\beta}\mathcal{H}_i + \partial_{i\beta}\Lambda^{j\beta}\mathcal{H}_j\,,\quad \mathcal{L}_\Lambda \mathcal{J}^\alpha = \Lambda^{j\beta}\partial_{j\beta}\mathcal{J}^\alpha - \partial_{j\beta}\Lambda^{j\alpha}\mathcal{J}^\beta + \partial_{j\beta}\Lambda^{j\beta}\mathcal{J}^\alpha\,. \tag{9.9}$$

These field strengths obey Bianchi identities:

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{i\alpha} = \epsilon^{ijk}\epsilon^{\alpha\beta}\partial_{j\beta}\mathcal{H}_{\mu\nu\rho k}\,, \tag{9.10}$$

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]i} + 3\epsilon_{ijk}\epsilon_{\alpha\beta}\mathcal{F}_{[\mu\nu}{}^{j\alpha}\mathcal{F}_{\rho\sigma]}{}^{k\beta} = \partial_{i\alpha}\mathcal{J}_{\mu\nu\rho\sigma}{}^\alpha\,, \tag{9.11}$$

$$5\mathcal{D}_{[\mu}\mathcal{J}_{\nu\rho\sigma\lambda]}{}^\alpha + 10\mathcal{F}_{[\mu\nu}{}^{i\alpha}\mathcal{H}_{\rho\sigma\lambda]i} = \epsilon^{\alpha\beta}\partial_{i\beta}\mathcal{K}_{\mu\nu\rho\sigma\lambda}{}^i\,, \tag{9.12}$$

where $\mathcal{D}_\mu \equiv \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu}$. The ExFT one-form can be simply identified as $\mathcal{A}_\mu{}^M = (A_\mu{}^i, \frac{1}{2}\epsilon^{ijk}C_{\mu jk})$. The two-form $\mathcal{B}_{\mu\nu i}$ transforms in the $(\bar{\mathbf{3}},\mathbf{1})$ of SL(3) × SL(2) and is identified (up to a further field redefinition) with $\hat{C}_{\mu\nu i}$. However, rather than give the precise field redefinitions for the potentials, it is simpler to work at the level of the field strengths. These are all tensors under generalised diffeomorphisms, meaining in particular that they transform in a particular way under d-dimensional three-form gauge transformations. This allows us to decompose in terms of manifestly gauge invariant combinations

$$\mathcal{F}_{\mu\nu}{}^{i1} \equiv F_{\mu\nu}{}^i\,,\quad \mathcal{F}_{\mu\nu}{}^{i2} \equiv \frac{1}{2}\epsilon^{ijk}(\hat{F}_{\mu\nu jk} - \hat{C}_{jkl}\hat{F}_{\mu\nu}{}^l)\,,\quad \mathcal{H}_{\mu\nu\rho i} \equiv -\hat{F}_{\mu\nu\rho i}\,, \tag{9.13}$$

where $F_{\mu\nu}{}^i$, $\hat{F}_{\mu\nu\rho i}$ and $\hat{F}_{\mu\nu jk}$ are gauge invariant and can be exactly identified with the quantities defined in (9.5) with $F_{\mu\nu}{}^i$ as in (9.3).

The three-form situation is then where it gets interesting. There is a single 8-dimensional three-form $\hat{C}_{\mu\nu\rho}$ obtained from the 11-dimensional one. There is also a single three-form $\hat{C}_{\mu\nu\rho ijk}$ coming from the 11-dimensional six-form. Together these form an SL(3) singlet and SL(2) doublet, for which the field strength obeys a self-duality constraint reproducing (in the relativistic case!) the correct duality relationship between the field strengths $\hat{F}_{\mu\nu\rho\sigma}$ and $\hat{F}_{\mu\nu\rho\sigma ijk}$. This duality constraint, which has to be imposed by hand, involves the eight-dimensional Hodge star acting on the 8-dimensional indices and the SL(2) generalised metric acting on the SL(2) indices:

$$\sqrt{|g|}\mathcal{M}_{\alpha\beta}\mathcal{J}^{\mu\nu\rho\sigma\beta} = -48\kappa\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma\lambda_1\dots\lambda_4}\mathcal{J}_{\lambda_1\dots\lambda_4}{}^{\beta}\,. \tag{9.14}$$

The coefficient κ is fixed via the self-consistency of (9.14) (in both the cases where $g_{\mu\nu}$ has Lorentzian or Euclidean signature, with $\mathcal{M}_{\alpha\beta}$ having the opposite) to be $\kappa = \pm\frac{1}{2\cdot(24)^2}$, with the choice of sign being a matter of convention (equivalent to changing the sign of the three-form in eleven dimensions). This is consistent with decomposing the SL(2) doublet of four-form field strengths as

$$\mathcal{J}_{\mu\nu\rho\sigma}{}^1 \equiv \hat{F}_{\mu\nu\rho\sigma}\,, \quad \mathcal{J}_{\mu\nu\rho\sigma}{}^2 \equiv \tfrac{1}{6}\epsilon^{ijk}(\hat{F}_{\mu\nu\rho\sigma ijk} - \hat{C}_{ijk}\hat{F}_{\mu\nu\rho\sigma})\,. \tag{9.15}$$

Thus in general, ExFT treats simultaneously degrees of freedom coming from the three-form with dual degrees of freedom coming from the six-form, encoding the duality relations between them in its dynamics.

Dynamics: SL(3) $\times$ SL(2) **ExFT pseudo-action** The ExFT Lagrangian can be uniquely fixed by the requirement of invariance under the local symmetries (generalised diffeomorphisms, gauge transformations of the tensor hierarchy, and finally $(11-d)$-dimensional diffeomorphisms). When $11-d$ is even, this gives a pseudo-action which must be accompanied by a self-duality constraint such as (9.14). This includes the case $d=3$. The pseudo-action in this case can be written as $S = \int \mathrm{d}^8x\, \mathrm{d}^6y\sqrt{|g|}\mathcal{L}_{\text{ExFT}}$ where the Lagrangian has the (quite general) expression

$$\mathcal{L}_{\text{ExFT}} = R_{\text{ext}}(g) + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} + \sqrt{|g|}^{-1}\mathcal{L}_{\text{top}}\,, \tag{9.16}$$

Here, with $\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu}$, we have

$$R_{\text{ext}}(g) = \frac{1}{4}g^{\mu\nu}\mathcal{D}_\mu g_{\rho\sigma}\mathcal{D}_\nu g^{\rho\sigma} - \frac{1}{2}g^{\mu\nu}\mathcal{D}_\mu g^{\rho\sigma}\mathcal{D}_\rho g_{\nu\sigma} + \frac{1}{4}g^{\mu\nu}\mathcal{D}_\mu \ln g\mathcal{D}_\nu \ln g + \frac{1}{2}\mathcal{D}_\mu \ln g\mathcal{D}_\nu g^{\mu\nu}\,, \tag{9.17}$$

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= \frac{1}{4}\mathcal{D}_\mu\mathcal{M}^{ij}\mathcal{D}^\mu\mathcal{M}_{ij} + \frac{1}{4}\mathcal{D}_\mu\mathcal{M}_{\alpha\beta}\mathcal{D}^\mu\mathcal{M}^{\alpha\beta}\\ &\quad - \frac{1}{4}\mathcal{M}_{ij}\mathcal{M}_{\alpha\beta}\mathcal{F}_{\mu\nu}{}^{i\alpha}\mathcal{F}^{\mu\nu j\beta} - \frac{1}{12}\mathcal{M}^{ij}\mathcal{H}_{\mu\nu\rho i}\mathcal{H}^{\mu\nu\rho}{}_j - \frac{1}{96}\mathcal{M}_{\alpha\beta}\mathcal{J}_{\mu\nu\rho\sigma}{}^{\alpha}\mathcal{J}^{\mu\nu\rho\sigma\beta}\,,\end{aligned} \tag{9.18}$$

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \frac{1}{4}\mathcal{M}^{MN}\partial_M\mathcal{M}^{kl}\partial_N\mathcal{M}_{kl} + \frac{1}{4}\mathcal{M}^{MN}\partial_M\mathcal{M}^{\alpha\beta}\partial_N\mathcal{M}_{\alpha\beta} - \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_K\mathcal{M}_{LN}\\ &\quad + \frac{1}{2}\partial_M\mathcal{M}^{MN}\partial_N \ln|g| + \frac{1}{4}\mathcal{M}^{MN}\left(\partial_M g_{\mu\nu}\partial_N g^{\mu\nu} + \partial_M \ln|g|\partial_N \ln|g|\right)\,.\end{aligned} \tag{9.19}$$

The topological (Chern-Simons) term can be defined via its variation:

$$\begin{aligned}\delta\mathcal{L}_{\text{top}} = \kappa\epsilon^{\mu_1\dots\mu_8}\Big(&- \delta\mathcal{A}_{\mu_1}{}^{i\alpha}\epsilon_{\alpha\beta}\mathcal{J}_{\mu_2\dots\mu_5}{}^{\beta}\mathcal{H}_{\mu_6\mu_7\mu_8 i}\\ &+ 6\Delta\mathcal{B}_{\mu_1\mu_2 i}\left(\epsilon_{\alpha\beta}\mathcal{F}_{\mu_3\mu_4}{}^{i\alpha}\mathcal{J}_{\mu_5\dots\mu_8}{}^{\beta} - \tfrac{4}{9}\epsilon^{ijk}\mathcal{H}_{\mu_3\mu_4\mu_5 j}\mathcal{H}_{\mu_6\mu_7\mu_8 k}\right)\\ &+ 4\Delta\mathcal{C}_{\mu_1\mu_2\mu_3}{}^{\alpha}\epsilon_{\alpha\beta}\left(\mathcal{D}_{\mu_4}\mathcal{J}_{\mu_5\dots\mu_8}{}^{\beta} + 4\mathcal{F}_{\mu_4\mu_5}{}^{i\beta}\mathcal{H}_{\mu_6\dots\mu_8 i}\right)\\ &- \partial_{i\alpha}\Delta\mathcal{D}_{\mu_1\dots\mu_4}{}^{i}\mathcal{J}_{\mu_5\dots\mu_8}{}^{\alpha}\Big)\,,\end{aligned} \tag{9.20}$$

where the 'improved' Δ variation includes by definition contributions of variations of lower rank gauge fields, for explicit expressions (which we do not require) see [54]. Finally, we must impose the constraint (9.14) after varying the above pseudo-action.

9.2 Obtaining the 11-dimensional Newton-Cartan theory via ExFT

In this subsection, we perform a dimensional decomposition of the 11-dimensional MNC variables, and use this to explain how exceptional field theory describes this theory.

Dimensional decomposition of 11-dimensional Newton-Cartan theory We start with the 11-dimensional coordinates labelled as $x^{\hat{\mu}} = (x^\mu, y^i)$ with $\mu = 1, \ldots, 11-d$ and $i = 1\ldots, d$. We keep all coordinate dependence on y^i throughout. Thus this is a decomposition rather than a reduction. In terms of the vielbein decomposition (8.16), we take $q = d-3$ and $n = 11-d$. The flat indices are $\mathsf{a} = 1, \ldots, 11-d$ and $\bar{\imath} = 1, \ldots, d-3$. Explicitly, we take the SO(8) vielbein to have the form

$$\hat{h}^a{}_{\hat{\mu}} = \begin{pmatrix} \Omega^{-\frac{1}{9-d}} e^{\mathsf{a}}{}_\mu & 0 \\ A_\mu{}^k h^{\bar{\imath}}{}_k & h^{\bar{\imath}}{}_i \end{pmatrix} , \quad \hat{h}^{\hat{\mu}}{}_a = \begin{pmatrix} \Omega^{\frac{1}{9-d}} e^\mu{}_{\mathsf{a}} & 0 \\ -\Omega^{\frac{1}{9-d}} e^\rho{}_{\mathsf{a}} A_\rho{}^k & h^i{}_{\bar{\imath}} \end{pmatrix} , \tag{9.21}$$

with $e^{\mathsf{a}}{}_\mu$ an invertible vielbein for an $(11-d)$-dimensional metric, $g_{\mu\nu} = e^{\mathsf{a}}{}_\mu e^{\mathsf{b}}{}_\nu \delta_{\mathsf{ab}}$. We also have to take

$$\hat{\tau}_{\hat{\mu}}{}^A = (A_\mu{}^i \tau_i{}^A, \tau_i{}^A) , \quad \hat{\tau}^{\hat{\mu}}{}_A = (0, \tau^i{}_A) . \tag{9.22}$$

where $\tau_{ij} = \tau_i{}^A \tau_j{}^B \eta_{AB}$, with $A = 0,1,2$ as before. The conformal factor Ω appearing in (9.21) is defined by

$$\Omega^2 = -\tfrac{1}{3!(d-3)!} \epsilon^{i_1\ldots i_d} \epsilon^{j_1\ldots j_d} \tau_{i_1 j_1} \tau_{i_2 j_2} \tau_{i_3 j_3} H_{i_4 j_4} \ldots H_{i_d j_d} , \tag{9.23}$$

and related to that of the 11-dimensional theory by $\hat{\Omega} = (\det e) \Omega^{-\frac{2}{9-d}}$. It is useful to write down the full transverse and longitudinal metrics:

$$\begin{aligned} \hat{H}_{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} \Omega^{-\frac{2}{9-d}} g_{\mu\nu} + H_{kl} A_\mu{}^k A_\nu{}^l & H_{jk} A_\mu{}^k \\ H_{ik} A_\nu{}^k & H_{ij} \end{pmatrix} , \quad \hat{\tau}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} A_\mu{}^k A_\nu{}^l \tau_{kl} & A_\mu{}^k \tau_{kj} \\ A_\nu{}^k \tau_{ki} & \tau_{ij} \end{pmatrix} , \\ \hat{H}^{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} \Omega^{\frac{2}{9-d}} g^{\mu\nu} & -\Omega^{\frac{2}{9-d}} g^{\mu\rho} A_\rho{}^j \\ -\Omega^{\frac{2}{9-d}} g^{\nu\sigma} A_\sigma{}^i & H^{ij} + \Omega^{\frac{2}{9-d}} g^{\rho\sigma} A_\rho{}^i A_\sigma{}^j \end{pmatrix} , \quad \hat{\tau}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & 0 \\ 0 & \tau^{ij} \end{pmatrix} . \end{aligned} \tag{9.24}$$

In this way all the degenerate structure is encoded in the d-dimensional part of the spacetime, with a degenerate d-dimensional metric $H_{ij} \equiv h^{\bar{\imath}}{}_i h^{\bar{\jmath}}{}_j \delta_{\bar{\imath}\bar{\jmath}}$. This ensures that the metric $g_{\mu\nu}$ can be identified with the metric appearing in exceptional field theory, while the degenerate Newton-Cartan metric structure will appear in the generalised metric. In addition, we redefine the three-form and its field strength according to (9.4) and (9.5), now without hats:

$$C_3 = C_3 + C_{2i} Dy^i + \tfrac{1}{2} C_{1ij} Dy^i Dy^j + \tfrac{1}{3!} C_{ijk} Dy^i Dy^j Dy^k , \tag{9.25}$$

$$F_4 = F_4 + F_{3i} Dy^i + \tfrac{1}{2} F_{2ij} Dy^i Dy^j + \tfrac{1}{3!} F_{1ijk} Dy^i Dy^j Dy^k + \tfrac{1}{4!} F_{ijkl} Dy^i Dy^j Dy^k Dy^l , \tag{9.26}$$

where again $Dy^i \equiv dy^i + A_\mu{}^i dx^\mu$. We carry out an analogous decomposition for $\widetilde{C}_3$ and $\widetilde{F}_4$, and for C_6 and F_7. Finally, we can consider the Newton-Cartan torsion: with $\hat{T}_{\hat{\mu}\hat{\nu}}{}^A \equiv 2\partial_{[\hat{\mu}} \hat{\tau}_{\hat{\nu}]}{}^A$ we have

$$\begin{aligned} T_{ij}{}^A &\equiv \hat{T}_{ij}{}^A = 2\partial_{[i} \tau_{j]}{}^A , \quad T_{\mu i}{}^A \equiv \hat{T}_{\mu i}{}^A - A_\mu{}^j \hat{T}_{ji} = D_\mu \tau_i{}^A , \\ T_{\mu\nu}{}^A &\equiv \hat{T}_{\mu\nu}{}^A - 2\hat{T}_{[\mu|i|}{}^A A_{\nu]}{}^i + A_\mu{}^i A_\nu{}^j \hat{T}_{ij} = F_{\mu\nu}{}^j \tau_j{}^A . \end{aligned} \tag{9.27}$$

Embedding the limit in ExFT Let's start by considering the expansions (**??**) and (**??**) of the original 11-dimensional metric and three-form. We make use of the decompositions (9.24) and (9.25) for the Newton-Cartan variables and three-form appearing in the decomposition, and then use these to work out the decomposition (9.2) of the 11-dimensional metric and that (9.4) of the three-form. The potentially singular terms as $c \to \infty$ then appear in the d-dimensional components of the metric and of the three-form, with

$$\phi_{ij} = c^2 \tau_{ij} + c^{-1} H_{ij} \,, \quad \hat{C}_{ijk} = -c^3 \epsilon_{ABC} \tau_i{}^A \tau_j{}^B \tau_k{}^C + C_{ijk} + c^{-3} \widetilde{C}_{ijk} \,. \tag{9.28}$$

The metric $g_{\mu\nu}$ and Kaluza-Klein vector $A_\mu{}^i$ appearing in (9.2) are then exactly those appearing in $\hat{H}_{\mu\nu}$ in (9.24). The redefined form components carrying an $(11-d)$-dimensional index are all non-singular, so $\hat{C}_{\mu ij} = C_{\mu ij} + \mathcal{O}(c^{-3})$, and so on. One point of danger is that $\hat{C}_{ijk}$ still appears in the field strengths (9.5) of these fields. However, consulting the more explicit expressions (10.17), one sees that the field strength $\mathcal{F}_{\mu\nu}{}^M$ appearing in ExFT in fact involves the combination $\mathcal{F}_{\mu\nu ij} = \hat{F}_{\mu\nu ij} - \hat{C}_{ijk} F_{\mu\nu}{}^k$, which is in fact independent of $\hat{C}_{ijk}$, such that $\hat{F}_{\mu\nu ij} - \hat{C}_{ijk} F_{\mu\nu}{}^k = F_{\mu\nu ij} - C_{ijk} F_{\mu\nu}{}^k$.

For the generalised metric (9.6), inserting the expressions (9.28) one finds that all terms at leading order in c cancel, and sending $c \to \infty$ one has a manifestly finite and boost invariant expression:

$$\mathcal{M}_{MN} = \Omega^{\frac{2}{9-d}} \begin{pmatrix} H_{ij} - \epsilon_{ABC} \tau_{(i|}{}^A C_{j)kl} \tau^{kB} \tau^{lC} + C_{ikl} C_{jmn} H^{km} \tau^{ln} & -\epsilon_{ABC} \tau_i{}^A \tau^{kB} \tau^{lC} + 2 C_{ipq} H^{p[k} \tau^{l]q} \\ -\epsilon_{ABC} \tau_k{}^A \tau^{iB} \tau^{jC} + 2 C_{kpq} H^{p[i} \tau^{l]j} & 2 H^{i[k} \tau^{l]j} + 2 \tau^{i[k} H^{l]j} \end{pmatrix} . \tag{9.29}$$

The parametrisation (9.29) can be viewed as a *non-Riemannian parametrisation* of the generalised metric, and viewed simply as an alternative possibility to taking the usual form (9.6). The reason why this is a *non-Riemannian parametrisation* is most clearly seen by looking at the inverse generalised metric $\mathcal{M}^{MN}$. In the Riemannian case, the parametrisation (9.6) implies that the $d \times d$ block $\mathcal{M}^{ij}$ is given by $\mathcal{M}^{ij} = |\hat{\phi}|^{-1/(9-d)} \hat{\phi}^{ij}$ and therefore corresponds to the inverse spacetime metric. Assuming this block is invertible then uniquely fixes (given the definition of the generalised metric as a particular coset element obeying certain properties) the rest of the parametrisation. In the non-Riemannian case, we instead have $\mathcal{M}^{ij} = \Omega^{-\frac{2}{9-d}} H^{ij}$, which is non-invertible. This leads instead to an alternative parametrisation. This is exactly as in the DFT case [35], which was generalised to ExFT in [40]. The expression (9.29) can be checked to be equivalent to the non-Riemannian SL(5) generalised metric worked out from first principles in [40]. In fact, from this point of view, one need not even go through the complications of taking the limit, but simply write down (9.29), insert it into the ExFT and study the resulting dynamics.

Returning to the embedding of the expansion in ExFT, we also need to worry about the singular pieces in the expansion of the dual gauge field $\hat{C}_6$. This inevitably appears in the tensor hierarchy for all exceptional field theories. From (6.43), we have $\hat{C}_6 \sim c^3 C_3 \wedge \tau \wedge \tau \wedge \tau + \ldots$, and so given the decomposition according to (9.22) and (9.25), any component of $\hat{C}_6$ carrying three d-dimensional indices will be singular, i.e. $\hat{C}_{\mu\nu\rho ijk}$, $\hat{C}_{\mu\nu ijkl}$, $\hat{C}_{\mu ijklm}$, $\hat{C}_{ijklmn}$. The claim is that, remarkably, all such singularities cancel automatically thanks to the precise combinations of $\hat{C}_6$ and $\hat{C}_3$ that appear in the ExFT fields. For $d = 3, 4$, this is most straightforwardly checked at the level of the ExFT field strengths. One sees from (9.15) for SL(3) × SL(2) (and from (11.12) for SL(5)) that the components of $\hat{F}_7$ always appear in the combinations $\hat{F}_{\mu\nu\rho\sigma ijk} - \hat{C}_{ijk} \hat{F}_{\mu\nu\rho\sigma}$ and $\hat{F}_{\mu\nu\rho ijkl} + 4 \hat{C}_{[ijk} \hat{F}_{|\mu\nu\rho\sigma|l]}$ exactly such that the singularity coming from $\hat{C}_{ijk}$ cancels that coming from $\hat{F}_7$, which was written down in (6.44). That the ExFT gauge potentials themselves are non-singular can further be verified by hunting down the correct field redefinitions that relate the ExFT gauge fields to the 11-dimensional ones. Note that for $d \geq 6$ the components $\hat{C}_{ijklmn}$ are present and appear in the generalised metric itself: we have not verified explicitly but the expectation would be that it does so in a way that ensures the generalised metric remains finite.

Summary From the above we can conclude that the fields used in ExFT are manifestly non-singular in the non-relativistic limit (equivalently this shows that the fields which are U-duality covariant in a genuine dimensional reduction are non-singular). We can also view the distinction between the relativistic and the non-relativistic 11-dimensional theory as being solely governed by the choice of parametrisation of the generalised metric. Having picked a generalised metric parametrisation, it is then consistent to directly identify the ExFT gauge fields and metric $g_{\mu\nu}$ with the gauge field components and metric of the decomposed relativistic *or* non-relativistic theory.

This is summarised in figure 3. The upper triangular half of this diagram corresponds to first embedding the relativistic fields in ExFT in the usual manner, with a Riemannian parametrisation of the generalised metric, and then taking the non-relativistic limit giving a non-Riemannian parametrisation. The lower triangular half corresponds to first taking the non-relativistic limit for the original 11-dimensional fields, and then embedding these into ExFT, giving the same non-Riemannian parametrisation. In both cases, one needs to make the appropriate dimensional decomposition of the fields of the Newton-Cartan theory, corresponding to fixing the local tangent space (non-Lorentzian) symmetry.

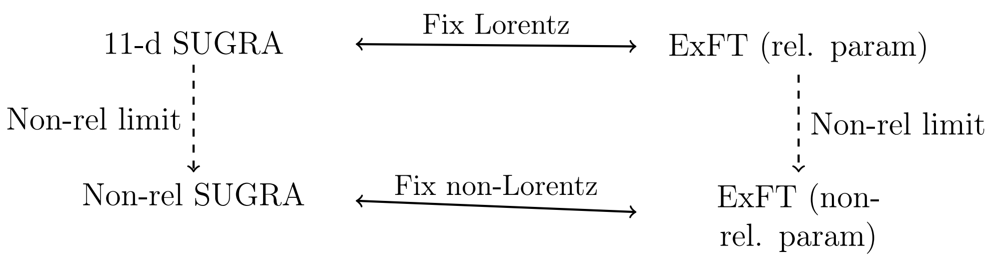


Figure 3: Relationship between non-relativistic limit and non-relativistic parametrisation of ExFT

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Inserting the non-Riemannian parametrisation into the ExFT action or equations of motion will then reproduce the finite action and equations of motion results from taking the limit, after decomposing. For the action, we calculate this decomposition in appendix 10. What we will show next is that, remarkably, the ExFT equations of motion also automatically reproduce the Poisson equation (7.41).

Generalised metric and equations of motion

We now take a closer look at the consequences of using the non-relativistic parametrisation of the generalised metric. We focus on the $d = 3$ SL(3)$\times$SL(2) ExFT. For the $d = 3$ Newton-Cartan geometry, H^{ij} and H_{ij} have rank zero and so are identically zero. The longitudinal metric τ_{ij} is a three-by-three matrix and in fact invertible, with $\Omega^2 = -\det\tau$. The resulting non-Riemannian parametrisation of the generalised metric (9.7) is

$$\mathcal{M}_{ij} = \Omega^{-2/3}\tau_{ij}\,, \quad \mathcal{M}_{\alpha\beta} = \begin{pmatrix} 2\varphi & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \varphi \equiv \tfrac{1}{3!}\epsilon^{ijk}C_{ijk}\,, \tag{9.30}$$

Comparing (9.30) and (9.8), we can note that (9.30) is the most general possible SL(2) non-Riemannian parametrisation (up to the sign of the off-diagonal components), as this is completely fixed by requiring $\mathcal{M}_{22} = 0$ which prevents us from interpreting that component as the determinant of a standard three-dimensional spacetime metric.

Normally, the generalised metric $\mathcal{M}_{\alpha\beta}$ encodes two degrees of freedom. It is clear that the non-Riemannian parametrisation given by (9.30) is restricted and is missing one degree of freedom. We may identify this missing degree of freedom with the overall scale of the longitudinal metric, as the latter only appears in the combination $|\det\tau|^{-1/3}\tau_{ij}$, which is conformally invariant. This makes the dilatation invariance trivial in this formulation.

If we insert this parametrisation into the SL(3) × SL(2) pseudo-action, with Lagrangian (9.16), we find that $\mathcal{L}_{\text{int}}$ as defined in (9.19) vanishes, while

$$\frac{1}{4}\mathcal{D}_\mu\mathcal{M}^{ij}\mathcal{D}^\mu\mathcal{M}_{ij}+\frac{1}{4}\mathcal{D}_\mu\mathcal{M}_{\alpha\beta}\mathcal{D}^\mu\mathcal{M}^{\alpha\beta}=\frac{1}{4}D_\mu(\Omega^{2/3}\tau^{ij})D^\mu(\Omega^{-2/3}\tau_{ij})\,. \tag{9.31}$$

This reproduces exactly the expected terms in the $d=3$ case of (10.27) and (10.28).

Notice that the kinetic terms for $\mathcal{M}_{\alpha\beta}$ completely drop out. So if we insert the non-relativistic parametrisation into the action, and then vary with respect to φ, we will never find an equation involving $\mathcal{D}^\mu\mathcal{D}_\mu\varphi$, i.e the Poisson equation. However, instead we can consider the equations of motion of the generalised metric, which can be evaluated independently of its choice of parametrisation. These will provide the missing Poisson equation. This is exactly analogous to the situation in DFT, see the discussions in [26,36]. One has to make a choice about whether you allow the equations of motion that follow from variations of the generalised metric that do not preserve the non-Riemannian parametrisation. In both the DFT SNC case, and the present case, there is exactly one such independent variation, which provides an additional equation of motion beyond what is obtained by varying the fields of the parametrisation themselves.

Let's see how this works. Naively, the result of varying the generalised metric $\mathcal{M}_{\alpha\beta}$ in the action is

$$\delta S=\int \mathrm{d}^8x\,\mathrm{d}^6Y\,\sqrt{g}\delta\mathcal{M}^{\alpha\beta}\mathcal{K}_{\alpha\beta}\,, \tag{9.32}$$

with

$$\begin{aligned}\mathcal{K}_{\alpha\beta}=&-\tfrac{1}{4}\tfrac{1}{\sqrt{g}}\left(\mathcal{D}_\mu(\sqrt{g}\mathcal{D}^\mu\mathcal{M}_{\alpha\beta})-\mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\mathcal{D}_\mu(\sqrt{g}\mathcal{D}^\mu\mathcal{M}^{\gamma\delta})\right)\\&+\frac{1}{4}\mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\mathcal{M}_{ij}\mathcal{F}_{\mu\nu}{}^{i\gamma}\mathcal{F}^{\mu\nu j\delta}+\frac{1}{96}\mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\mathcal{J}_{\mu\nu\rho\sigma}{}^{\gamma}\mathcal{J}^{\mu\nu\rho\sigma\delta}\\&+\tfrac{1}{4}\mathcal{M}^{ij}\left(\partial_{i(\alpha|}\mathcal{M}^{kl}\partial_{j|\beta)}\mathcal{M}_{kl}+\partial_{i(\alpha|}\mathcal{M}^{\gamma\delta}\partial_{j|\beta)}\mathcal{M}_{\gamma\delta}+\partial_{i(\alpha|}g_{\mu\nu}\partial_{j|\beta)}g^{\mu\nu}\right)\\&-\tfrac{1}{2}\mathcal{M}^{ij}\partial_{i\alpha}\partial_{j\beta}\ln g+\tfrac{1}{\sqrt{g}}\partial_{i(\alpha|}(\sqrt{g}\partial_{j|\beta)}\mathcal{M}^{ij})\\&-\tfrac{1}{2}\mathcal{M}^{ij}\left(\partial_{i(\alpha|}\mathcal{M}^{kl}\partial_{k|\beta)}\mathcal{M}_{lj}+\partial_{i(\alpha|}\mathcal{M}^{\gamma\delta}\partial_{j\gamma}\mathcal{M}_{|\beta)\delta}\right)\\&+\tfrac{1}{2\sqrt{g}}(\partial_{i\gamma}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\gamma\delta}\partial_{j(\alpha}\mathcal{M}_{\beta)\delta})-\mathcal{M}_{\gamma(\alpha}\mathcal{M}_{\beta)\delta}\partial_{j\kappa}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\epsilon\gamma}\partial_{i\epsilon}\mathcal{M}^{\kappa\delta})\\&-\tfrac{1}{4\sqrt{g}}(\partial_{i\gamma}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\gamma\delta}\partial_{j\delta}\mathcal{M}_{\alpha\beta})-\mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\partial_{i\epsilon}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\epsilon\kappa}\partial_{j\kappa}\mathcal{M}^{\gamma\delta})\,.\end{aligned} \tag{9.33}$$

Now, the variation $\delta\mathcal{M}^{\alpha\beta}$ cannot be arbitrary but must preserve that $|\det\mathcal{M}|=1$. This ensures that one gets two rather than three independent equations, corresponding to the usual two degrees of freedom encoded in $\mathcal{M}_{\alpha\beta}$. The true equation of motion taking this into account is:

$$\mathcal{R}_{\alpha\beta}\equiv\mathcal{K}_{\alpha\beta}-\tfrac{1}{2}\mathcal{M}_{\alpha\beta}\mathcal{M}^{\gamma\delta}\mathcal{K}_{\gamma\delta}=0\,. \tag{9.34}$$

This can be thought of as the vanishing of a generalised Ricci tensor, $\mathcal{R}_{\alpha\beta}$. For the non-Riemannian parametrisation (9.30), the two independent equations are

$$\mathcal{R}_{22}=\mathcal{K}_{22}=0\,,\quad \mathcal{R}_{11}-2\varphi\mathcal{R}_{22}=\mathcal{K}_{11}-2\varphi K_{12}=0\,. \tag{9.35}$$

Setting $\partial_{i1}\equiv\partial_i$, $\partial_{i2}=0$, we have explicitly that

$$\mathcal{K}_{22}=+\tfrac{1}{4}\mathcal{M}_{ij}F_{\mu\nu}{}^{i}F^{\mu\nu j}+\tfrac{1}{96}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}=0\,. \tag{9.36}$$

This is the equation of motion (7.15) arising as the totally longitudinal part of the equation of motion of the three-form. This is consistent with its appearance here as the equation of motion of φ, which is indeed the totally longitudinal part of the three-form.

The other equation of motion is (after using (9.36))

$$\begin{aligned}0 &= \mathcal{K}_{11} - 2\varphi K_{12} \\ &= -\tfrac{1}{\sqrt{g}}\tfrac{1}{6}\epsilon^{ijk} D_\mu(\sqrt{g} g^{\mu\nu} F_{\nu ijk}) \\ &\quad - \tfrac{1}{8}\mathcal{M}^{km}\mathcal{M}^{ln} F_{\mu\nu kl} F^{\mu\nu}{}_{mn} + \tfrac{1}{96} F_{\mu\nu\rho\sigma ijk} F^{\mu\nu\rho\sigma}{}_{lmn} \tfrac{1}{3!3!}\epsilon^{ijk}\epsilon^{lmn} \\ &\quad + \tfrac{1}{4}\mathcal{M}^{ij}\left(\partial_i \mathcal{M}^{kl}\partial_j \mathcal{M}_{kl} + \partial_i g_{\mu\nu}\partial_j g^{\mu\nu}\right) - \tfrac{1}{2}\mathcal{M}^{ij}\partial_i\mathcal{M}^{kl}\partial_k\mathcal{M}_{jl} \\ &\quad - \tfrac{1}{2}\mathcal{M}^{ij}\partial_i\partial_j \ln g - \tfrac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial_j\mathcal{M}^{ij})\,. \end{aligned} \tag{9.37}$$

Here we have $F_{\mu ijk} = D_\mu C_{ijk} - 3\partial_{[i} C_{|\mu|jk]}$, having used $\mathcal{D}_\mu \mathcal{M}_{11} = D_\mu \mathcal{M}_{11} - \epsilon^{ijk}\partial_i \mathcal{A}_{\mu jk}\mathcal{M}_{12}$. We can then identify (9.37) as the Poisson equation for $\varphi \equiv \frac{1}{6}\epsilon^{ijk} C_{ijk}$, as it has the form $\frac{1}{\sqrt{g}} D_\mu(\sqrt{g} D^\mu \varphi) + \cdots = 0$. It is conjugate to the variation $\delta\mathcal{M}^{11}$. For the non-Riemannian parametrisation, $\mathcal{M}^{11} = 0$, so allowing this variation corresponds to allowing variations that do not respect the parametrisation. In terms of the expansion of $\mathcal{M}^{\alpha\beta}$ in powers of $1/c$, this variation is subleading in origin. Finally, one can precisely check that this equation (9.37) is indeed exactly the Poisson equation (7.41), which we found at subleading order in the expansion of the relativistic theory, and here is rewritten in terms of ExFT variables after making the dimensional decomposition of all the fields. (It can be easily checked that the gauge field terms match, using (10.30) to relate the seven-form components appearing here to those of $\widetilde{F}_4$, and a patient calculation shows that inserting the dimensional decomposition of the eleven-dimensional fields matches perfectly.)

Structure of generalised Ricci tensor Geometrically, $\mathcal{R}_{\alpha\beta}$ should be thought of as (the SL(2) part of) a generalised Ricci tensor. It is a symmetric generalised tensor of weight 0 and obeys $\mathcal{M}^{\alpha\beta}\mathcal{R}_{\alpha\beta} = 0$. When we take the relativistic parametrisation (9.8) of the generalised metric, it can therefore be parametrised as

$$\mathcal{R}_{\alpha\beta} = \tfrac{1}{2}\begin{pmatrix}1 & \hat{C}\\ 0 & 1\end{pmatrix}\begin{pmatrix}|\phi|^{1/2}\mathcal{R}_\phi & \mathcal{R}_C \\ \mathcal{R}_C & |\phi|^{-1/2}\mathcal{R}_\phi\end{pmatrix}\begin{pmatrix}1 & 0\\ \hat{C} & 1\end{pmatrix} \tag{9.38}$$

with $\mathcal{R}_\phi$ and $\mathcal{R}_C$ tensors of three-dimensional weight 0, such that the variation of the action leads to

$$\delta S \supset -\int \mathrm{d}^8x\, \mathrm{d}^6y\, \sqrt{g}\left(\frac{\delta|\phi|^{1/2}}{|\phi|^{1/2}}\mathcal{R}_\phi + |\phi|^{-1/2}\delta\hat{C}\mathcal{R}_C\right) \tag{9.39}$$

Let's examine what happens to the components of $\mathcal{R}_{\alpha\beta}$ in the non-relativistic limit. We have $|\phi|^{1/2} = \Omega c^3$, $\hat{C} = -c^3\Omega + C + c^{-3}\widetilde{C}$. This leads to the expression

$$\mathcal{R}_{\alpha\beta} = \tfrac{1}{2}\begin{pmatrix}1 & C\\ 0 & 1\end{pmatrix}\begin{pmatrix}c^3\Omega(\mathcal{R}_\phi - \mathcal{R}_C) & \mathcal{R}_C - \mathcal{R}_\phi \\ \mathcal{R}_C - \mathcal{R}_\phi & c^{-3}\Omega^{-1}\mathcal{R}_\phi\end{pmatrix}\begin{pmatrix}1 & 0\\ C & 1\end{pmatrix} \tag{9.40}$$

So in principle the independent equations are still $\mathcal{R}_C$ and $\mathcal{R}_\phi$. However, we already know that this generalised Ricci tensor has no leading parts in c when we take the limit (because none of the ExFT fields contain singular terms). If we expand

$$\mathcal{R}_\phi = c^3\mathcal{R}_\phi^{(3)} + c^0\mathcal{R}_\phi^{(0)} + c^{-3}\mathcal{R}_\phi^{(-3)}\,, \quad \mathcal{R}_C = c^3\mathcal{R}_C^{(3)} + c^0\mathcal{R}_C^{(0)} + c^{-3}\mathcal{R}_C^{(-3)}\,, \tag{9.41}$$

it must be that we have $\mathcal{R}_\phi^{(3)} = \mathcal{R}_C^{(3)}$, $\mathcal{R}_\phi^{(0)} = \mathcal{R}_C^{(0)}$, viewed as off-shell identities, and the independent equations of motion, i.e. those appearing as the actual finite entries of $\mathcal{R}_{\alpha\beta}$, are actually

$$\mathcal{R}_\phi^{(3)} = 0\,, \quad \mathcal{R}_\phi^{(-3)} - \mathcal{R}_C^{(-3)} = 0\,. \tag{9.42}$$

The former is conjugate to $\delta\mathcal{M}^{22}$ and the latter to the $\delta\mathcal{M}^{11}$ that is forbidden if we insist on keeping a non-Riemannian parametrisation. We can go back to the variation (9.39) and expand that:

$$\delta S = -\int \mathrm{d}^8x\, \mathrm{d}^6y\, \sqrt{g}\left(\delta \ln \Omega(\mathcal{R}_\phi - \mathcal{R}_C) + \Omega^{-1}c^{-3}\delta C\mathcal{R}_C\right) , \tag{9.43}$$

hence the first non-zero variations are

$$\delta S = -\int \mathrm{d}^8x\, \mathrm{d}^6y\, \sqrt{g}\left(c^{-3}\delta \ln \Omega(\mathcal{R}_\phi^{(-3)} - \mathcal{R}_C^{(-3)}) + \Omega^{-1}\delta C\mathcal{R}_C^{(3)}\right) . \tag{9.44}$$

We see again that we get the longitudinal equation of motion for the three-form at finite order, and the extra Poisson equation of motion comes from a subleading variation associated to the variation of the volume factor Ω, which otherwise has no dynamics associated to it in this formulation.

Generating non-relativistic generalised metrics via U-duality

Non-trivial U-duality transformations act as SL(2) transformations on the generalised metric $\mathcal{M}_{\alpha\beta}$, via $\mathcal{M} \to \mathcal{M}' = U^T\mathcal{M}U$ with $\det U = 1$. Parametrising $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transformation of the non-relativistic parametrisation (9.30) gives

$$\mathcal{M}'_{\alpha\beta} = \begin{pmatrix} 2a(a\varphi + c) & 2ab\varphi + ad + bc \\ 2ab\varphi + ad + bc & 2b(b\varphi + d) \end{pmatrix} , \tag{9.45}$$

and this remains in the non-relativistic form only if $b = 0$, or else if φ is constant and $d = -b\varphi$. In the former case, the effect of the transformation is $\varphi \to a(a\varphi + c)$ and so amounts to a scaling and shift of the three-form.

The genuine non-geometric U-dualities correspond to the SL(2) inversion symmetry with $a = d = 0$, $bc = -1$. If $\varphi < 0$, this takes us from the non-relativistic parametrisation to a relativistic one with

$$\phi_{ij} = (-\tfrac{1}{2\varphi})^{2/3}(\det \tau)^{-1/3}\tau_{ij} , \quad C_{ijk} = -\tfrac{1}{2\varphi}\epsilon_{ijk} . \tag{9.46}$$

These obey $|\det \phi| = C^2$ which corresponds to a 'critical' three-form.

We can apply this to a real supergravity background along the lines of [33, 40], namely the M2 brane solution in the form

$$\mathrm{d}s^2 = f^{-2/3}\eta_{ij}dy^idy^j + f^{1/3}\delta_{\mu\nu}dx^\mu dx^\nu , \quad C_{ijk} = (f^{-1} + \gamma)\epsilon_{ijk} , \tag{9.47}$$

where the harmonic function f obeys $\partial_\mu\partial^\mu f = 0$ and γ is a constant. This has constant exceptional field theory 8-dimensional metric, $g_{\mu\nu} = \delta_{\mu\nu}$, while

$$\mathcal{M}_{ij} = \eta_{ij} , \quad \mathcal{M}_{\alpha\beta} = \begin{pmatrix} -\gamma(f + 2\gamma) & -(1 + \gamma f) \\ -(1 + \gamma f) & -f \end{pmatrix} . \tag{9.48}$$

Sending $f \to 0$ corresponds exactly to the original limit (**??**). Alternatively, we can formally U-dualise along the y^i directions (including time) to obtain a solution with

$$\mathcal{M}_{\alpha\beta} = \begin{pmatrix} -f & 1 + \gamma f \\ 1 + \gamma f & -\gamma(f + 2\gamma) \end{pmatrix} . \tag{9.49}$$

The standard M2 solution has $\gamma = -1$ and $f = 1 + \frac{q}{r^6}$, with $r^2 \equiv \delta_{\mu\nu}x^\mu x^\nu$. In this case, the generalised metric (9.49) corresponds to the *negative M2* solution [56]:

$$\mathrm{d}s^2 = \widetilde{f}^{-2/3}\eta_{ij}dy^idy^j + \widetilde{f}^{1/3}\delta_{\mu\nu}dx^\mu dx^\nu , \quad C_{ijk} = (\widetilde{f}^{-1} - 1)\epsilon_{ijk} , \quad \widetilde{f} = 1 - \tfrac{q}{r^6} . \tag{9.50}$$

This solution has a naked singularity at $\widetilde{f}=0 \Leftrightarrow f-2=0$. Evidently the generalised metric (9.49) is non-singular everywhere and at $\widetilde{f}=0$ becomes non-relativistic. This suggests [22] interpreting such backgrounds as containing a singular locus at which the geometry degenerates to a non-relativistic one.

If we alternatively take $\gamma = 0$ then the generalised metric (9.49) has the non-relativistic form everywhere, with $\varphi \equiv -\frac{1}{2}f$. If we now reconsider the equation of motion (9.37) which can only be found by varying the generalised metric before inserting the parametrisation, then this is exactly the equation $\nabla^2 f = 0$ obeyed by the harmonic function. Finally, we can reconstruct the full 11-dimensional MNC geometry:

$$\hat{\tau}_{\hat{\mu}}{}^{A} = (0, \delta_i{}^A)\,, \quad \hat{H}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \delta^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}\,, \quad C_{012} = -\tfrac{1}{2}f\,. \tag{9.51}$$

9.3 Gauge fields and self-duality in SL(3) × SL(2) ExFT

Now let's look at what happens in the gauge field sector of the SL(3) × SL(2) ExFT. Let's repeat the parametrisations (9.13) and (9.15) now for the field strength components of the non-relativistic theory:

$$\mathcal{F}_{\mu\nu}{}^{i1} \equiv F_{\mu\nu}{}^{i}\,, \quad \mathcal{F}_{\mu\nu}{}^{i2} \equiv \frac{1}{2}\epsilon^{ijk}(F_{\mu\nu jk} - C_{jkl}F_{\mu\nu}{}^{l})\,, \quad \mathcal{H}_{\mu\nu\rho i} \equiv -F_{\mu\nu\rho i}\,, \tag{9.52}$$

$$\mathcal{J}_{\mu\nu\rho\sigma}{}^{1} \equiv F_{\mu\nu\rho\sigma}\,, \quad \mathcal{J}_{\mu\nu\rho\sigma}{}^{2} \equiv \frac{1}{6}\epsilon^{ijk}(F_{\mu\nu\rho\sigma ijk} - C_{ijk}F_{\mu\nu\rho\sigma})\,. \tag{9.53}$$

Then the kinetic terms (9.18) in the SL(3) × SL(2) ExFT pseudo-action (9.16) are

$$-\tfrac{1}{4}\mathcal{M}_{ij}\mathcal{M}_{\alpha\beta}\mathcal{F}_{\mu\nu}{}^{i\alpha}\mathcal{F}^{\mu\nu j\beta} - \tfrac{1}{12}\mathcal{M}^{ij}\mathcal{H}_{\mu\nu\rho i}\mathcal{H}^{\mu\nu\rho}{}_{j} = -\tfrac{1}{4}\Omega^{-2/3}\tau_{ij}F^{\mu\nu i}\epsilon^{jkl}F_{\mu\nu kl} - \frac{1}{12}\Omega^{2/3}\tau^{ij}F_{\mu\nu\rho i}F^{\mu\nu\rho}{}_{j}\,, \tag{9.54}$$

which matches the corresponding terms in the decomposition (10.27) of the non-relativistic action.

To discuss the three-form gauge field, consider the SL(3) × SL(2) ExFT equation of motion obtained from the pseudo-action by varying $\mathcal{C}_{\mu\nu\rho}{}^{\alpha}$:

$$\begin{aligned} &\mathcal{D}_\sigma(\sqrt{|g|}\mathcal{M}_{\alpha\beta}\mathcal{J}^{\mu\nu\rho\sigma\beta}) - 2\partial_{i\alpha}(\sqrt{|g|}\mathcal{M}^{ij}\mathcal{H}^{\mu\nu\rho}{}_{j}) \\ &- 48\kappa\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_5}\left(\mathcal{D}_{\sigma_1}\mathcal{J}_{\sigma_2\ldots\sigma_5}{}^{\beta} + 4\mathcal{F}_{\sigma_1\sigma_2}{}^{i\beta}\mathcal{H}_{\sigma_3\sigma_4\sigma_5 i}\right) = 0\,. \end{aligned} \tag{9.55}$$

After varying, we must also impose the constraint (9.14). This constraint involves the generalised metric, and so it is sensitive to whether we are describing the relativistic or non-relativistic theory. However, in either case, using the constraint in the equation of motion of $\mathcal{C}_{\mu\nu\rho}{}^{2}$ in fact produces the Bianchi identity (9.12) for $\mathcal{J}_{\mu\nu\rho\sigma}{}^{1} = F_{\mu\nu\rho\sigma}$. In the relativistic case, with the Riemannian parametrisation (9.8) of the generalised metric (or its Euclidean version), we could go on to use the constraint to eliminate $\mathcal{J}_{\mu\nu\rho\sigma}{}^{2}$ from the equation of motion of $\mathcal{C}_{\mu\nu\rho}{}^{2}$. The result would be the equation of motion of the three-form $C_{\mu\nu\rho}$ following from the decomposition of 11-dimensional SUGRA.

Now let's consider the situation where the generalised metric admits the non-relativistic parametrisation (9.30). In this case, choosing the minus sign for κ, the constraint (9.14) implies that

$$\sqrt{g}\,F^{\mu\nu\rho\sigma} = -\tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma\lambda_1\ldots\lambda_4}F_{\lambda_1\ldots\lambda_4}\,, \quad \sqrt{g}\,F^{\mu\nu\rho\sigma}{}_{ijk} = +\tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma\lambda_1\ldots\lambda_4}F_{\lambda_1\ldots\lambda_4 ijk}\,. \tag{9.56}$$

So we can no longer eliminate $F_{\mu\nu\rho\sigma ijk}$ in favour of $F_{\mu\nu\rho\sigma}$. This is clearly as expected for the MNC theory for which the former indeed appears explicitly in the action and equations of motion (note it is related to $\widetilde{F}_{\mu\nu\rho\sigma}$ via (10.30)). We therefore see that the ExFT constraint gives not only the expected constraint (6.26) that the original four-form field strength becomes self-dual, but also the duality condition with opposite sign which is obeyed by the dual seven-form (6.47). Thus the SL(3) × SL(2) ExFT contains the expected degrees of freedom of the non-relativistic theory, and efficiently rearranges them into self-dual and anti-self-dual parts automatically on the non-Riemannian parametrisation.

10 Dimensional Decomposition of Non-Relativistic Action for ExFT

Decomposition of $R^{(0)}$ Consider the part of the scalar curvature $R^{(0)}$ as defined in (6.23) not involving the longitudinal metric, but just the transverse metrics $\hat{H}_{\hat{\mu}\hat{\nu}}$ and $\hat{H}^{\hat{\mu}\hat{\nu}}$ and the measure factor $\hat{\Omega}$. In the dimensional decomposition used in exceptional field theory, the latter two factorise as

$$\hat{H}_{\hat{\mu}\hat{\nu}} = U_{\hat{\mu}}{}^{\hat{\rho}} U_{\hat{\nu}}{}^{\hat{\sigma}} \bar{H}_{\hat{\rho}\hat{\sigma}} \,, \quad \hat{H}^{\hat{\mu}\hat{\nu}} = (U^{-1})_{\hat{\rho}}{}^{\hat{\mu}} (U^{-1})_{\hat{\sigma}}{}^{\hat{\nu}} \bar{H}^{\hat{\rho}\hat{\sigma}} \,, \tag{10.1}$$

with

$$U_{\hat{\mu}}{}^{\hat{\nu}} = \begin{pmatrix} \delta_\mu{}^\nu & A_\mu{}^j \\ 0 & \delta_i{}^j \end{pmatrix} \,, \quad \bar{H}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & H_{ij} \end{pmatrix} \,, \quad \bar{H}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G^{\mu\nu} & 0 \\ 0 & H^{ij} \end{pmatrix} \,. \tag{10.2}$$

Here $G^{\mu\nu}$ is the inverse of $G_{\mu\nu}$, but H^{ij} and H_{ij} are not invertible. The idea is to completely factor out the matrix U from derivatives of $\hat{G}$. Defining

$$\partial_{\hat{\mu}} \hat{H}_{\hat{\nu}\hat{\rho}} = U_{\hat{\mu}}{}^{\hat{\sigma}} U_{\hat{\nu}}{}^{\hat{\lambda}} U_{\hat{\rho}}{}^{\hat{\kappa}} \overline{\partial H}_{\hat{\sigma}\hat{\lambda}\hat{\kappa}} \,, \quad \partial_{\hat{\mu}} \hat{H}^{\hat{\nu}\hat{\rho}} = U_{\hat{\mu}}{}^{\hat{\sigma}} (U^{-1})_{\hat{\lambda}}{}^{\hat{\nu}} (U^{-1})_{\hat{\kappa}}{}^{\hat{\rho}} \overline{\partial h}_{\hat{\sigma}}{}^{\hat{\lambda}\hat{\kappa}} \tag{10.3}$$

we have the relatively simple expressions

$$\overline{\partial H}_{\mu\hat{\nu}\hat{\rho}} = \begin{pmatrix} \bar{D}_\mu G_{\nu\rho} & H_{kl} \bar{D}_\mu A_\nu{}^l \\ H_{jl} \bar{D}_\mu A_\rho{}^k & \bar{D}_\mu H_{jk} \end{pmatrix} \,, \quad \overline{\partial H}_{i\hat{\nu}\hat{\rho}} = \begin{pmatrix} \partial_i G_{\nu\rho} & H_{kl} \partial_i A_\nu{}^l \\ H_{jl} \partial_i A_\rho{}^k & \partial_i H_{jk} \end{pmatrix} \tag{10.4}$$

$$\overline{\partial H}_\mu{}^{\hat{\nu}\hat{\rho}} = \begin{pmatrix} \bar{D}_\mu G^{\nu\rho} & -G^{\nu\sigma} \bar{D}_\mu A_\sigma{}^k \\ -G^{\rho\sigma} \bar{D}_\mu A_\sigma{}^j & \bar{D}_\mu H^{jk} \end{pmatrix} \,, \quad \overline{\partial H}_i{}^{\hat{\nu}\hat{\rho}} = \begin{pmatrix} \partial_i G^{\nu\rho} & -G^{\nu\sigma} \partial_i A_\sigma{}^k \\ -G^{\rho\sigma} \partial_i A_\sigma{}^j & \partial_i H^{jk} \end{pmatrix} \tag{10.5}$$

where $\bar{D}_\mu \equiv \partial_\mu - A_\mu{}^i \partial_i$. For instance, consider the following terms in the scalar curvature:

$$\tfrac{1}{4} \bar{H}^{\hat{\mu}\hat{\nu}} \overline{\partial H}_{\hat{\mu}\hat{\rho}\hat{\sigma}} \overline{\partial H}_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} - \tfrac{1}{2} \bar{H}^{\mu\nu} \overline{\partial H}_\mu{}^{\rho\sigma} \overline{\partial H}_{\rho\nu\sigma} \,. \tag{10.6}$$

A fairly straightforward calculations shows that these equal

$$\begin{aligned} &\tfrac{1}{4} G^{\mu\nu} D_\mu G_{\rho\sigma} D_\nu G^{\rho\sigma} - \tfrac{1}{2} G^{\mu\nu} D_\mu G^{\rho\sigma} D_\rho G_{\nu\sigma} - \tfrac{1}{4} G^{\mu\nu} G^{\rho\sigma} H_{ij} F_{\mu\rho}{}^i F_{\nu\sigma}{}^j + \tfrac{1}{4} G^{\mu\nu} D_\mu H_{ij} D_\nu H^{ij} \\ &\quad + \tfrac{1}{4} H^{ij} (\partial_i G_{\rho\sigma} \partial_j G^{\rho\sigma} + \partial_i H_{kl} \partial_j H^{kl}) - \tfrac{1}{2} H^{ij} \partial_i H^{kl} \partial_k H_{jl} \\ &\quad - \tfrac{1}{2} (\delta^i_k + H^{ij} H_{jk}) \bar{D}_\mu A_\nu{}^k \partial_i G^{\mu\nu} + G^{\mu\nu} H^{ij} H_{jk} \partial_l A_\mu{}^k \partial_i A_\nu{}^l \end{aligned} \tag{10.7}$$

where $F_{\mu\nu}{}^i \equiv 2\bar{D}_{[\mu} A_{\nu]}{}^i$, $D_\mu = \partial_\mu - L_{A_\mu}$, and acting on $G_{\mu\nu}$ and $G^{\mu\nu}$, we have $D_\mu = \bar{D}_\mu$.

Next, consider the part of $R^{(0)}$ that involves τ:

$$\tfrac{1}{4} \hat{H}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \hat{\tau}_{\hat{\rho}\hat{\sigma}} \partial_{\hat{\nu}} \hat{\tau}^{\hat{\rho}\hat{\sigma}} + \tfrac{1}{4} \hat{\tau}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \tau_{\hat{\rho}\hat{\sigma}} \partial_{\hat{\nu}} \hat{H}^{\hat{\rho}\hat{\sigma}} - \tfrac{1}{2} \hat{\tau}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\nu}} H^{\hat{\rho}\hat{\sigma}} \partial_{\hat{\rho}} \hat{\tau}_{\hat{\mu}\hat{\sigma}} - \tfrac{1}{2} \hat{H}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\nu}} \hat{\tau}^{\hat{\rho}\hat{\sigma}} \partial_{\hat{\rho}} \hat{\tau}_{\hat{\mu}\hat{\sigma}} \tag{10.8}$$

Similar calculations to above give

$$\begin{aligned} &\tfrac{1}{4} G^{\mu\nu} D_\mu \tau_{ij} D_\nu \tau^{ij} + g^{\mu\nu} \tau^{ik} \tau_{kj} \partial_i A_\mu{}^l \partial_l A_\nu{}^j - \tfrac{1}{2} \tau^{ik} \tau_{kj} \bar{D}_\mu A_\nu{}^k \partial_i G^{\mu\nu} \\ &\quad + \tfrac{1}{4} H^{ij} \partial_i \tau_{kl} \partial_j \tau^{kl} + \tfrac{1}{4} \tau^{ij} \partial_i \tau_{kl} \partial_j H^{kl} - \tfrac{1}{2} \tau^{ij} \partial_j H^{kl} \partial_k \tau_{il} - \tfrac{1}{2} H^{ij} \partial_j \tau^{kl} \partial_k \tau_{il} \end{aligned} \tag{10.9}$$

The terms involving $\tau^{ik}\tau_{kj}$ on the first line here combine with the terms involving $H^{ik}H_{kj}$ in the last line of (10.7) and sum up to give $\delta^i_j = H^{ik}H_{kj} + \tau^{ik}\tau_{kj}$, after which point the rest of the calculation proceeds identically to that normally used in exceptional field theory.

Finally one has the terms

$$-\bar{G}^{\hat{\mu}\hat{\nu}} \bar{\partial}_{\hat{\mu}} \ln \hat{\Omega} \, \bar{\partial}_{\hat{\nu}} \ln \hat{\Omega} + 2\bar{\partial}_{\hat{\mu}} \ln \hat{\Omega} \, \overline{\partial G}_{\hat{\nu}}{}^{\hat{\mu}\hat{\nu}} - \partial_{\hat{\mu}} \partial_{\hat{\nu}} \hat{G}^{\hat{\mu}\hat{\nu}} - \hat{G}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \partial_{\hat{\nu}} \ln \hat{\Omega} \tag{10.10}$$

where $\hat{\Omega}$ has weight 1, and in the final two terms $\bar{\partial}_\mu \equiv \bar{D}_\mu$, $\bar{\partial}_i \equiv \partial_i$. Note $D_\mu \ln\hat{\Omega} = \bar{D}_\mu \ln\hat{\Omega} - \partial_i A_\mu{}^i$. We let $\hat{\Omega} = \Omega\sqrt{|G|}$, where Ω has weight 1 under internal diffeomorphisms. Straightforward manipulations allow one to rewrite (10.10) in the decomposition and combine with (10.7) and (10.9) After dropping a total derivative, the final result is:

$$
\begin{aligned}
R^{(0)}(\hat{H},\hat{\tau}) = R_{\text{ext}}(G) + R^{(0)}(H,\tau) - \tfrac{1}{4}F_{\mu\nu}{}^i F_{\rho\sigma}{}^j G^{\mu\rho}G^{\nu\sigma}H_{ij} \\
+ \tfrac{1}{4}G^{\mu\nu}(D_\mu H_{ij} D_\nu H^{ij} + D_\mu \tau_{ij} D_\nu \tau^{ij} + D_\mu \ln\Omega^2 D_\nu \ln\Omega^2) \\
+ \tfrac{1}{4}H^{ij}(\partial_i G_{\mu\nu}\partial_j G^{\mu\nu} + \partial_i \ln|G| \partial_j \ln|G|)
\end{aligned}
\tag{10.11}
$$

where

$$
\begin{aligned}
R_{\text{ext}}(g) =& \tfrac{1}{4}G^{\mu\nu}D_\mu G_{\rho\sigma}D_\nu G^{\rho\sigma} - \tfrac{1}{2}G^{\mu\nu}D_\mu G^{\rho\sigma}D_\rho G_{\nu\sigma} - \tfrac{1}{4}G^{\mu\nu}D_\mu \ln|G| D_\nu \ln|G| \\
& - D_\mu \ln|G| D_\nu G^{\mu\nu} - G^{\mu\nu}D_\mu D_\nu \ln|G| - D_\mu D_\nu G^{\mu\nu} \,,
\end{aligned}
\tag{10.12}
$$

$$
\begin{aligned}
R^{(0)}(H,\tau) = & + \tfrac{1}{4}H^{ij}\partial_i \tau_{kl}\partial_j \tau^{kl} + \tfrac{1}{4}\tau^{ij}\partial_i \tau_{kl}\partial_j H^{kl} - \tfrac{1}{2}\tau^{ij}\partial_j H^{kl}\partial_k \tau_{il} - \tfrac{1}{2}H^{ij}\partial_j \tau^{kl}\partial_k \tau_{il} \\
& + \tfrac{1}{4}H^{ij}\partial_i H_{kl}\partial_j H^{kl} - \tfrac{1}{2}H^{ij}\partial_j H^{kl}\partial_k H_{il} - \tfrac{1}{4}H^{ij}\partial_i \ln\Omega^2 \partial_j \ln\Omega^2 \\
& - \partial_i \ln\Omega^2 \partial_j H^{ij} - \partial_i\partial_j H^{ij} - H^{ij}\partial_i\partial_j \ln\Omega^2 \,.
\end{aligned}
\tag{10.13}
$$

The measure factor is $\hat{\Omega} = \Omega\sqrt{|G|}$. To obtain an Einstein frame action, we let

$$
G_{\mu\nu} = \Omega^{-\frac{2}{9-d}} g_{\mu\nu} \,. \tag{10.14}
$$

Gauge fields The compact expressions (9.4) and (9.5) are equivalent to

$$
C_{\hat{\mu}\hat{\nu}\hat{\rho}} = (U^{-1})^{\hat{\lambda}_1}{}_{\hat{\mu}}(U^{-1})^{\hat{\lambda}_2}{}_{\hat{\nu}}(U^{-1})^{\hat{\lambda}_3}{}_{\hat{\rho}} C_{\hat{\lambda}_1\dots\hat{\lambda}_3} \,, \quad F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = (U^{-1})^{\hat{\lambda}_1}{}_{\hat{\mu}}(U^{-1})^{\hat{\lambda}_2}{}_{\hat{\nu}}(U^{-1})^{\hat{\lambda}_3}{}_{\hat{\rho}}(U^{-1})^{\hat{\lambda}_4}{}_{\hat{\sigma}} F_{\hat{\lambda}_1\dots\hat{\lambda}_4} \,, \tag{10.15}
$$

giving in components

$$
\begin{aligned}
& C_{ijk} \equiv C_{ijk} \,, \quad C_{\mu ij} \equiv C_{\mu ij} - A_\mu{}^k C_{ijk} \,, \\
& C_{\mu\nu i} \equiv C_{\mu\nu i} - 2A_{[\mu}{}^j C_{\nu]ij} + A_\mu{}^j A_\nu{}^k C_{ijk} \,, \\
& C_{\mu\nu\rho} \equiv C_{\mu\nu\rho} - 3A_{[\mu}{}^i C_{\nu\rho]i} + 3A_{[\mu}{}^i A_\nu{}^j C_{\rho]ij} - A_\mu{}^i A_\nu{}^j A_\rho{}^k C_{ijk} \,,
\end{aligned}
\tag{10.16}
$$

$$
\begin{aligned}
& F_{mnpq} = 4\partial_{[m}C_{npq]} \,, \qquad F_{\mu mnp} = D_\mu C_{mnp} - 3\partial_{[m}C_{|\mu|np]} \\
& F_{\mu\nu mn} = 2D_{[\mu}C_{\nu]mn} + F_{\mu\nu}{}^p C_{pmn} + 2\partial_{[m}C_{|\mu\nu|n]} \,, \\
& F_{\mu\nu\rho m} = 3D_{[\mu}C_{\nu\rho]m} + 3F_{[\mu\nu}{}^n C_{\rho]mn} - \partial_m C_{\mu\nu\rho} \,, \\
& F_{\mu\nu\rho\sigma} = 4D_{[\mu}C_{\nu\rho\sigma]} + 6F_{[\mu\nu}{}^m C_{\rho\sigma]m} \,,
\end{aligned}
\tag{10.17}
$$

where $F_{\mu\nu}{}^i$ is as defined in (9.3). The original Bianchi identity $dF_4 = 0$ becomes a set of equations

$$
\begin{aligned}
D_\mu F_{mnpq} &= 4\partial_{[m}F_{npq]} \,, \\
2D_{[\mu}F_{\nu]mnp} &= -3\partial_{[m|}F_{\mu\nu|np]} - F_{\mu\nu}{}^q F_{qmnp} \,, \\
3D_{[\mu}F_{\nu\rho]mn} &= 2\partial_{[m|}F_{\mu\nu\rho|n]} + 3F_{[\mu\nu}{}^p F_{\rho]pmn} \,, \\
4D_{[\mu}F_{\nu\rho\sigma]m} &= -\partial_m F_{\mu\nu\rho\sigma} + 6F_{[\mu\nu}{}^p F_{\rho\sigma]mp} \,, \\
5D_{[\mu}F_{\nu\rho\sigma\lambda]} &= 10F_{[\mu\nu}{}^m F_{\rho\sigma\lambda]m} \,.
\end{aligned}
\tag{10.18}
$$

The above formulae are applicable to any dimensional reduction. In particular for the 11-dimensional MNC theory they allow us to easily decompose the terms in the action (6.27). For example, using the Einstein frame metric to raise indices, the kinetic terms for the field strength are:

$$
\begin{aligned}
& -\tfrac{1}{12}\hat{H}^{\hat{\mu}_1\hat{\nu}_1}\hat{H}^{\hat{\mu}_2\hat{\nu}_2}\hat{H}^{\hat{\mu}_3\hat{\nu}_3}\hat{\tau}^{\hat{\mu}_4\hat{\nu}_4}F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}F_{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4} \\
& \quad = -\tfrac{1}{12}\Omega^{6/(9-d)}\tau^{ij}F^{\mu\nu\rho}{}_i F_{\mu\nu\rho j} - \tfrac{1}{4}\Omega^{4/(9-d)}H^{ij}\tau^{kl}F_{\mu\nu ik}F^{\mu\nu}{}_{jl} \\
& \qquad - \tfrac{1}{4}\Omega^{2/(9-d)}H^{ij}H^{kl}\tau^{pq}F_{\mu ikp}F^\nu{}_{jlq} - \tfrac{1}{4}H^{ij}H^{kl}H^{mn}\tau^{pq}F_{ikmp}F_{jlnq} \,.
\end{aligned}
\tag{10.19}
$$

Similar manipulations apply to the rest of the action. Let us also indicate how the factorisation applies to an equation of the form $\partial_{\hat{\sigma}} X^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = \Theta^{\hat{\mu}\hat{\nu}\hat{\rho}}$ where X has weight 1, and both X and Θ admit a factorisation via U^{-1} in terms of quantities $\bar{X}$ and $\bar{\Theta}$ independent of bare ${A_\mu}^i$. This is of course the form of the gauge field equation of motion (6.21). After decomposing, one has the simple expression

$$D_\sigma \bar{X}^{\hat{\mu}\hat{\nu}\hat{\rho}\sigma} + \partial_l \bar{X}^{\hat{\mu}\hat{\nu}\hat{\rho}l} + \tfrac{3}{2} {F_{\kappa\lambda}}^l \delta_l^{[\hat{\mu}} \bar{X}^{\hat{\nu}\hat{\rho}]\kappa\lambda} = \bar{\Theta}^{\hat{\mu}\hat{\nu}\hat{\rho}} . \tag{10.20}$$

Constraint The constraint (6.26) decomposes in terms of the redefined strengths:

$$\begin{aligned}
\sqrt{g}\Omega^{\frac{6}{9-d}} g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4} F_{\nu_1\dots\nu_4} &= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_4\hat{\nu}_1\dots\hat{\nu}_4 ijk}\tfrac{1}{6}\epsilon_{ABC}{\tau_i}^A{\tau_j}^B{\tau_k}^C F_{\hat{\nu}_1\dots\hat{\nu}_4} ,\\
\sqrt{g}\Omega^{\frac{4}{9-d}} g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3} H^{ij} F_{\nu_1\nu_2\nu_3 j} &= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_3 i\hat{\nu}_1\dots\hat{\nu}_4 pqr}\tfrac{1}{6}\epsilon_{ABC}{\tau_p}^A{\tau_q}^B{\tau_r}^C F_{\hat{\nu}_1\dots\hat{\nu}_4} ,\\
\sqrt{g}\Omega^{\frac{2}{9-d}} g^{\mu_1\nu_1} g^{\mu_2\nu_2} H^{i_1j_1} H^{i_2j_2} F_{\nu_1\nu_2 j_1 j_2} &= -\tfrac{1}{4!}\epsilon^{\mu_1\mu_2 i_1 i_2\hat{\nu}_1\dots\hat{\nu}_4 pqr}\tfrac{1}{6}\epsilon_{ABC}{\tau_p}^A{\tau_q}^B{\tau_r}^C F_{\hat{\nu}_1\dots\hat{\nu}_4} ,\\
\sqrt{g} g^{\mu_1\nu_1} H^{i_1j_1}\dots H^{i_3j_3} F_{\nu j_1 j_2 j_3} &= -\tfrac{1}{4!}\epsilon^{\mu i_1\dots i_3\hat{\nu}_1\dots\hat{\nu}_4 pqr}\tfrac{1}{6}\epsilon_{ABC}{\tau_p}^A{\tau_q}^B{\tau_r}^C F_{\hat{\nu}_1\dots\hat{\nu}_4} ,\\
\sqrt{g}\Omega^{-\frac{2}{9-d}} H^{i_1j_1}\dots H^{i_4j_4} F_{j_1 j_2 j_3 j_4} &= -\tfrac{1}{4!}\epsilon^{i_1\dots i_4\hat{\nu}_1\dots\hat{\nu}_4 pqr}\tfrac{1}{6}\epsilon_{ABC}{\tau_p}^A{\tau_q}^B{\tau_r}^C \hat{F}_{\hat{\nu}_1\dots\hat{\nu}_4} .
\end{aligned} \tag{10.21}$$

For instance, when $d=3$ only the first of these is non-zero, giving:

$$\begin{aligned}
\sqrt{g}\Omega g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4} F_{\nu_1\dots\nu_4} &= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_4 ijk}\tfrac{1}{6}\epsilon_{ABC}{\tau_i}^A{\tau_j}^B{\tau_k}^C F_{\nu_1\dots\nu_4} ,\\
&= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_4}\Omega F_{\nu_1\dots\nu_4} .
\end{aligned} \tag{10.22}$$

When $d=4$ only the first two are non-zero:

$$\begin{aligned}
\sqrt{g}\Omega^{\frac{6}{5}} g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4} F_{\nu_1\dots\nu_4} &= -\tfrac{1}{3!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_3 lijk}\tfrac{1}{6}\epsilon_{ABC}{\tau_i}^A{\tau_j}^B{\tau_k}^C F_{\nu_1\nu_2\nu_3 l} ,\\
\sqrt{g}\Omega^{\frac{4}{5}} g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3} H^{ij} F_{\nu_1\nu_2\nu_3 j} &= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_3 i\nu_1\dots\nu_4 pqr}\tfrac{1}{6}\epsilon_{ABC}{\tau_p}^A{\tau_q}^B{\tau_r}^C F_{\nu_1\dots\nu_4} ,
\end{aligned} \tag{10.23}$$

or if we take $\frac{1}{6}\epsilon^{ijkl}\epsilon_{ABC}{\tau_i}^A{\tau_j}^B{\tau_k}^C h_l = \Omega$ these are

$$\begin{aligned}
\sqrt{g}\Omega^{\frac{1}{5}} g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4} F_{\nu_1\dots\nu_4} &= \tfrac{1}{3!}\eta\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_3} h^l F_{\nu_1\nu_2\nu_3 l} ,\\
\sqrt{g} g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3} H^{ij} F_{\nu_1\nu_2\nu_3 j} &= \tfrac{1}{4!}\eta\epsilon^{\mu_1\dots\mu_3\nu_1\dots\nu_4} h^i \Omega^{\frac{1}{5}} F_{\nu_1\dots\nu_4} .
\end{aligned} \tag{10.24}$$

Here $H^{ij} = h^i h^j$ (as it has rank 1), and so both of these are equivalent.

Result Putting everything together, the dimensional decomposition of the finite action $S^{(0)}$ is:

$$S^{(0)} = \int \mathrm{d}^{11-d}x\,\mathrm{d}^d y \sqrt{g}(R_{\text{ext}}(g) + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\tilde{F}} + \sqrt{g}^{-1}\mathcal{L}_{\text{CS}}) . \tag{10.25}$$

Here, using $g^{\mu\nu}$ to raise $(11-d)$-dimensional indices, we have

$$R_{\text{ext}}(g) = \frac{1}{4} g^{\mu\nu} D_\mu g_{\rho\sigma} D_\nu g^{\rho\sigma} - \frac{1}{2} g^{\mu\nu} D_\mu g^{\rho\sigma} D_\rho g_{\nu\sigma} + \frac{1}{4} g^{\mu\nu} D_\mu \ln g D_\nu \ln g + \frac{1}{2} D_\mu \ln g D_\nu g^{\mu\nu} , \tag{10.26}$$

$$\begin{aligned}
\mathcal{L}_{\text{kin}} = &\tfrac{1}{4}(D_\mu H^{ij} D^\mu H_{ij} + D_\mu \tau^{ij} D^\mu \tau_{ij} - \tfrac{1}{9-d} D_\mu \ln\Omega^2 D^\mu \ln\Omega^2) + \tfrac{1}{2} D_\mu {\tau_k}^A {\tau_A}^k D^\mu {\tau_l}^B {\tau_B}^l \\
&+ \tfrac{1}{2} H^{ij} F_{\mu ikl}\epsilon_{ABC} D^\mu {\tau_j}^A \tau^{kB}\tau^{lC} - \tfrac{1}{4} H^{ij} H^{kl} \tau^{pq} F_{\mu ikp} {F^\mu}_{jlq} \\
&+ \tfrac{1}{4}\Omega^{\frac{2}{9-d}}(- {F_{\mu\nu}}^i F^{\mu\nu j} H_{ij} + F_{\mu\nu kl} F^{\mu\nu m}\epsilon_{ABC}\tau^A_m \tau^{kB}\tau^{lC} - H^{ij}\tau^{kl} F_{\mu\nu ik} {F^{\mu\nu}}_{jl}) \\
&- \tfrac{1}{12}\Omega^{\frac{4}{9-d}} \tau^{ij} F_{\mu\nu\rho i} {F^{\mu\nu\rho}}_j
\end{aligned} \tag{10.27}$$

and

$$
\begin{aligned}
\Omega^{\frac{2}{9-d}}\mathcal{L}_{\rm int} = &\tfrac{1}{4}H^{ij}\left(\partial_i g^{\mu\nu}\partial_j g_{\mu\nu} + \partial_i \ln g \partial_j \ln g\right) + \tfrac{1}{2}\Omega^{\frac{2}{9-d}}\partial_i(H^{ij}\Omega^{-\frac{2}{9-d}})\partial_j \ln g \\
&+ \tfrac{1}{4}H^{ij}\partial_i\tau_{kl}\partial_j\tau^{kl} + \tfrac{1}{4}\tau^{ij}\partial_i\tau_{kl}\partial_j H^{kl} - \tfrac{1}{2}\tau^{ij}\partial_j H^{kl}\partial_k\tau_{il} - \tfrac{1}{2}H^{ij}\partial_j\tau^{kl}\partial_k\tau_{il} \\
&+ \tfrac{1}{4}H^{ij}\partial_i H_{kl}\partial_j H^{kl} - \tfrac{1}{2}H^{ij}\partial_j H^{kl}\partial_k H_{il} \\
&+ \tfrac{1}{4}\tfrac{d-7}{(9-d)^2}H^{ij}\partial_i \ln\Omega^2\,\partial_j\ln\Omega^2 - \tfrac{1}{9-d}\partial_i\ln\Omega^2\,\partial_j H^{ij} \\
&- \tfrac{1}{4}H^{ij}H^{kl}H^{mn}\tau^{pq}F_{ikmp}F_{jlnq} + \tfrac{1}{4}H^{im}H^{jn}F_{ijkl}\epsilon_{ABC}T_{mn}{}^{A}\tau^{kB}\tau^{lC} \\
&+ \tfrac{1}{2}H^{ij}T_{ik}{}^{A}\tau^{k}{}_{A}T_{jl}{}^{B}\tau^{l}{}_{B}\,.
\end{aligned}
\tag{10.28}
$$

The term $\mathcal{L}_{\widetilde{F}}$ consists of a sum of contractions of $\widetilde{F}_{\mu\nu\rho\sigma}$, $\widetilde{F}_{\mu\nu\rho i}$, etc. (following analogous redefinition of the components) with the constraints as decomposed in (10.21). For instance, when $d=3$,

$$
\mathcal{L}_{\widetilde{F}} = -\tfrac{1}{4!}\widetilde{F}_{\mu_1\dots\mu_4}(\sqrt{g}\Omega g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4}F_{\nu_1\dots\nu_4} + \tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_4}\Omega F_{\nu_1\dots\nu_4})\,,
\tag{10.29}
$$

In this case the relationship between the dual seven-form field strength and $\widetilde{F}_{\mu\nu\rho\sigma}$ gives

$$
\tfrac{1}{6}\epsilon^{ijk}F_{\mu_1\dots\mu_4 ijk} = \Omega(\widetilde{F}_{\mu_1\dots\mu_4} + \tfrac{1}{4!}\sqrt{g}\epsilon_{\mu_1\dots\mu_4\nu_1\dots\nu_4}\widetilde{F}^{\nu_1\dots\nu_4})\,.
\tag{10.30}
$$

When $d=4$,

$$
\begin{aligned}
\mathcal{L}_{\widetilde{F}} = -\tfrac{1}{3!}&\left(\widetilde{F}_{\mu_1\mu_2\mu_3 i}h^i - \Omega^{1/5}\epsilon^{\lambda_1\dots\lambda_4\sigma_1\dots\sigma_3}\tfrac{1}{4!}\tfrac{1}{\sqrt{g}}g_{\sigma_1\mu_1}\dots g_{\sigma_3\mu_3}\widetilde{F}_{\lambda_1\dots\lambda_4}\right)\times \\
&\times\left(\sqrt{g}\Omega^{\frac{4}{5}}g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3}h^j F_{\nu_1\nu_2\nu_3 j} - \Omega\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_3\nu_1\dots\nu_4}F_{\nu_1\dots\nu_4}\right)\,,
\end{aligned}
\tag{10.31}
$$

Using (6.46) we can rewrite (10.31) in terms of the dual seven-form field strength directly as

$$
\mathcal{L}_{\widetilde{F}} = +\tfrac{1}{3!4!}F_{\mu_1\dots\mu_3 ijkl}\epsilon^{ijkl}\left(\sqrt{g}\Omega^{-\frac{1}{5}}g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3}h^j F_{\nu_1\nu_2\nu_3 j} - \tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_3\nu_1\dots\nu_4}F_{\nu_1\dots\nu_4}\right)\,.
\tag{10.32}
$$

Finally, the Chern-Simons term can be worked out by taking wedge products of (9.5) and (9.4), we do not display this explicitly.

11 The SL(5) ExFT and its Non-Relativistic Parametrisation

In the $d=4$ case, more of the degenerate Newton-Cartan structure is preserved.

Elements of SL(5) **ExFT** For $d=4$, generalised vectors $V^M=(V^i,V_{ij})$ transform in the **10** of SL(5), with $i,j,\dots$ now four-dimensional. This representation is the antisymmetric representation, and we can see this more clearly as follows. Let $\mathcal{M},\mathcal{N},\dots$ denote fundamental five-dimensional indices of SL(5). Then we can equivalently write a generalised vector as carrying an antisymmetric pair of such indices, $V^M\equiv V^{\mathcal{MN}}=-V^{\mathcal{NM}}$, and on writing $\mathcal{M}=(i,5)$ we can identify $V^{i5}\equiv V^i$, and $V^{ij}\equiv\frac{1}{2}\epsilon^{ijkl}V_{kl}$. The generalised Lie derivative acting on vectors of weight λ_V is explicitly

$$
\mathcal{L}_\Lambda V^{\mathcal{MN}} = \tfrac{1}{2}\Lambda^{\mathcal{PQ}}\partial_{\mathcal{PQ}}V^{\mathcal{MN}} + 2\partial_{\mathcal{PQ}}\Lambda^{\mathcal{P}[\mathcal{M}}V^{\mathcal{N}]\mathcal{Q}} + \tfrac{1}{2}(1+\lambda_V+\omega)\partial_{\mathcal{PQ}}\Lambda^{\mathcal{PQ}}V^{\mathcal{MN}}\,.
\tag{11.1}
$$

The section condition is $\epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\partial_{\mathcal{PQ}}=0$, and below we work with the M-theory solution, where splitting $\mathcal{M}=(i,5)$ the derivatives ∂_{ij} are viewed as identically zero, and the derivatives ∂_{i5} are identified with the 4-dimensional partial derivatives.

In this case, the generalised metric admits a factorisation

$$\mathcal{M}_{\mathcal{MN},\mathcal{PQ}} = -(m_{\mathcal{MP}}m_{\mathcal{QN}} - m_{\mathcal{MQ}}m_{\mathcal{PN}}) \tag{11.2}$$

where the 'little metric' $m_{\mathcal{MN}}$ is symmetric and has unit determinant. The overall sign in this expression needed for the ExFT action to reproduce SUGRA correctly when we include timelike signatures in the generalised metric, according to the conventions of [40].

The gauge fields, $\mathcal{A}_\mu{}^M$, $\mathcal{B}_{\mu\nu\mathcal{M}}$, $\mathcal{C}_{\mu\nu\rho}{}^{\mathcal{M}}$ and $\mathcal{D}_{\mu\nu\rho\sigma M}$ have weights 1/5, 2/5, 3/5 and 4/5 respectively, with field strengths denoted $\mathcal{F}_{\mu\nu}{}^M$, $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$, $\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{N}}$ and $\mathcal{K}_{\mu\nu\rho\sigma\lambda M}$. Under generalised diffeomorphisms, $\mathcal{F}^M$ transforms as a generalised vector of weight 1/5, while $\mathcal{H}$ and $\mathcal{J}$ transform via the generalised Lie derivative acting as

$$\mathcal{L}_\Lambda \mathcal{H}_{\mathcal{M}} = \tfrac{1}{2}\Lambda^{\mathcal{PQ}}\partial_{\mathcal{PQ}}\mathcal{H}_{\mathcal{M}} + \mathcal{H}_{\mathcal{P}}\partial_{\mathcal{MQ}}\Lambda^{\mathcal{PQ}}\,, \quad \mathcal{L}_\Lambda \mathcal{J}^{\mathcal{M}} = \partial_{\mathcal{PQ}}(\tfrac{1}{2}\Lambda^{\mathcal{PQ}}\mathcal{J}^{\mathcal{M}}) - \partial_{\mathcal{PQ}}\Lambda^{\mathcal{PM}}\mathcal{J}^{\mathcal{Q}}\,. \tag{11.3}$$

They obey Bianchi identities:

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{\mathcal{MN}} = \tfrac{1}{2}\epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{PQ}}\mathcal{H}_{\mu\nu\rho\mathcal{K}}\,, \tag{11.4}$$

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]\mathcal{M}} + \tfrac{3}{4}\epsilon_{\mathcal{MNPKL}}\mathcal{F}_{[\mu\nu}{}^{\mathcal{NP}}\mathcal{F}_{\rho\sigma]}{}^{\mathcal{KL}} = \partial_{\mathcal{NM}}\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{N}}\,, \tag{11.5}$$

$$5\mathcal{D}_{[\mu}\mathcal{J}_{\nu\rho\sigma\lambda]}{}^{\mathcal{M}} + 10\mathcal{F}_{[\mu\nu}{}^{\mathcal{MN}}\mathcal{H}_{\rho\sigma\lambda]\mathcal{N}} = \tfrac{1}{2}\epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{NP}}\mathcal{K}_{\mu\nu\rho\sigma\lambda\mathcal{QK}}\,. \tag{11.6}$$

The dynamics follow from the variation of an action $S = \int \mathrm{d}^7x\,\mathrm{d}^{10}y\mathcal{L}_{\text{ExFT}}$ where $\mathcal{L}_{\text{ExFT}}$ has the same form as (9.16), with R_{ext} again as defined in (9.17), and [65]

$$\mathcal{L}_{\text{kin}} = +\tfrac{1}{12}\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{D}^\mu\mathcal{M}^{MN} - \tfrac{1}{4}\mathcal{M}_{MN}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu N} - \tfrac{1}{12}m^{\mathcal{MN}}\mathcal{H}_{\mu\nu\rho\mathcal{M}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{N}} \tag{11.7}$$

$$\begin{aligned}\mathcal{L}_{\text{int}}(m,g) = {}& \tfrac{1}{12}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} - \tfrac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_K\mathcal{M}_{LN} + \tfrac{1}{2}\partial_M\mathcal{M}^{MN}\partial_N\ln|g|\\ &+\tfrac{1}{4}\mathcal{M}^{MN}\left(\partial_M g_{\mu\nu}\partial_N g^{\mu\nu} + \partial_M\ln|g|\partial_N\ln|g|\right)\,.\end{aligned} \tag{11.8}$$

The topological term can be defined via its variation (again up to a choice of sign equivalent to changing the sign of $\hat{C}_3$ in eleven-dimensional SUGRA):

$$\begin{aligned}\delta\mathcal{L}_{\text{top}} = -\tfrac{1}{6\cdot4!}\epsilon^{\mu_1\ldots\mu_7}\Big(&2\delta\mathcal{A}_{\mu_1}{}^{\mathcal{MN}}\mathcal{H}_{\mu_2\mu_3\mu_4\mathcal{M}}\mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}} + 6\mathcal{F}_{\mu_1\mu_2}{}^{\mathcal{MN}}\Delta\mathcal{B}_{\mu_3\mu_4\mathcal{M}}\mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}}\\ &\partial_{\mathcal{NM}}\Delta\mathcal{C}_{\mu_1\mu_2\mu_3}{}^{\mathcal{N}}\mathcal{J}_{\mu_4\ldots\mu_7}{}^{\mathcal{M}}\Big)\,.\end{aligned} \tag{11.9}$$

We refer to the original paper [65] or the review [45] for explicit details.

Review of 11-dimensional SUGRA embedding We start with the little metric, $m_{\mathcal{MN}}$. The parametrisation reproducing (9.6) is

$$m_{\mathcal{MN}} = |\phi|^{1/10}\begin{pmatrix} |\phi|^{-1/2}\phi_{ij} & -|\phi|^{-1/2}\phi_{ik}\hat{C}^k \\ -|\phi|^{-1/2}\phi_{jk}\hat{C}^k & |\phi|^{1/2}(-1)^t + |\phi|^{-1/2}\phi_{kl}\hat{C}^k\hat{C}^l\end{pmatrix}\,, \quad \hat{C}^i \equiv \tfrac{1}{3!}\epsilon^{ijkl}\hat{C}_{jkl}\,. \tag{11.10}$$

For the gauge fields, we can again identify $\mathcal{A}_\mu^M = (A_\mu{}^i, \hat{C}_{\mu ij})$. However, we already require dualisations when treating the two-forms. We get four 7-dimensional two-forms, $\hat{C}_{\mu\nu i}$ and a single three-form $\hat{C}_{\mu\nu\rho}$. The latter can be dualised into an extra two-form, $\widetilde{C}_{\mu\nu}$ (identifiable with the components $\hat{C}_{\mu\nu ijkl}$ of the six-form in eleven-dimensions) such that $\mathcal{B}_{\mu\nu\mathcal{M}} \sim (\hat{C}_{\mu\nu i}, \widetilde{C}_{\mu\nu})$ gives a five-dimensional representation of SL(5). Meanwhile, we can view $\hat{C}_{\mu\nu\rho}$ together with the four four-forms $\hat{C}_{\mu\nu\rho ijk}$ as comprising the conjugate five-dimensional representation. The equations of motion of the SL(5) ExFT then imply that the field strengths of these two- and three-forms are related by duality. This involves the seven-dimensional Hodge star acting on the seven-dimensional indices and the generalised metric acting on the SL(5) indices:

$$\sqrt{|g|}m^{\mathcal{M}\mathcal{P}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{P}} = -\tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_4}\mathcal{J}_{\sigma_1\dots\sigma_4}{}^{\mathcal{M}} \tag{11.11}$$

Again, the field strengths are all tensors under generalised diffeomorphisms, we may make the (usual) identifications consistent with the Bianchi identities [45]

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^{i5} &= F_{\mu\nu}{}^{i}\,, & \mathcal{F}_{\mu\nu}{}^{ij} &= \tfrac{1}{2}\epsilon^{ijkl}(\hat{F}_{\mu\nu kl} - \hat{C}_{klm}\hat{F}_{\mu\nu}{}^{m})\,,\\ \mathcal{H}_{\mu\nu\rho i} &= -\hat{F}_{\mu\nu\rho i}\,, & \mathcal{H}_{\mu\nu\rho 5} &= -\tfrac{1}{4!}\epsilon^{ijkl}(\hat{F}_{\mu\nu\rho ijkl} - 4\hat{F}_{\mu\nu\rho i}\hat{C}_{jkl})\,,\\ \mathcal{J}_{\mu\nu\rho\sigma}{}^{5} &= -\hat{F}_{\mu\nu\rho\sigma}\,, & \mathcal{J}_{\mu\nu\rho\sigma}{}^{i} &= +\tfrac{1}{3!}\epsilon^{ijkl}(\hat{F}_{\mu\nu\rho\sigma jkl} - \hat{C}_{jkl}\hat{F}_{\mu\nu\rho\sigma})\,. \end{aligned} \tag{11.12}$$

Generalised metric The distinction between Riemannian and non-Riemannian parametrisations can be seen at the level of the unit-determinant five-by-five little generalised metric. For an M-theory parametrisation, this can be written as:

$$m_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} m_{ij} & m_{i5} \\ m_{j5} & m_{55}\end{pmatrix}\,,\quad m_{55}\det(m_{ij}) - \tfrac{1}{6}m_{i5}m_{j5}\epsilon^{iklm}\epsilon^{jpqr}m_{kp}m_{lq}m_{mr} = 1\,. \tag{11.13}$$

If $\det(m_{ij}) \neq 0$ this leads to the Riemannian parametrisation (11.10) encoding a four-dimensional metric, g_{ij}, and a three-form, $\hat{C}_{ijk}$. However, we can also have $\det(m_{ij}) = 0$ with m_{ij} of rank 3 and this leads to a non-Riemannian parametrisation which was worked out in [40]. We can rediscover this parametrisation by taking the non-relativistic limit of (11.10) using (9.28). The resulting expression for $m_{\mathcal{M}\mathcal{N}}$ is

$$m_{\mathcal{M}\mathcal{N}} = \Omega^{-4/5}\begin{pmatrix} \tau_{ij} & \tfrac{1}{6}H_{ik}\epsilon^{klmn}\epsilon_{ABC}\tau_l{}^A\tau_m{}^B\tau_n{}^C - \tau_{ik}C^k \\ \tfrac{1}{6}H_{jk}\epsilon^{klmn}\epsilon_{ABC}\tau_l{}^A\tau_m{}^B\tau_n{}^C - \tau_{jk}C^k & \tau_{ij}C^iC^j - \tfrac{1}{3}\epsilon^{jklm}\epsilon_{ABC}H_{ij}\tau_k{}^A\tau_l{}^B\tau_m{}^CC^i\end{pmatrix}\,, \tag{11.14}$$

in terms of four-dimensional Newton-Cartan variables and $C^i \equiv \tfrac{1}{3!}\epsilon^{ijkl}C_{jkl}$. The unit determinant constraint implies that

$$-\tfrac{1}{3!}\epsilon^{i_1\dots i_4}\epsilon^{j_1\dots j_4}\tau_{i_1j_1}\tau_{i_2j_2}\tau_{i_3j_3}H_{i_4j_4} = \Omega^2\,, \tag{11.15}$$

which is the definition of Ω^2 in this case. As H_{ij} has rank 1, we can introduce a projective vielbein h_i such that $H_{ij} = h_ih_j$ and we take

$$\tfrac{1}{6}\epsilon^{ijkl}\epsilon_{ABC}\tau_i{}^A\tau_j{}^B\tau_k{}^Ch_l = \Omega\,, \tag{11.16}$$

choosing to fix an arbitrary sign (by sending $\tau_i{}^A \to -\tau_i{}^A$ if necessary) which could appear here (Ω is assumed positive). Then (11.14) can be written as

$$m_{\mathcal{M}\mathcal{N}} = \Omega^{-4/5}\begin{pmatrix} \tau_{ij} & -\Omega h_i - \tau_{ik}C^k \\ -\Omega h_j - \tau_{jk}C^k & \tau_{ij}C^iC^j + 2\Omega h_iC^i\end{pmatrix}\,, \tag{11.17}$$

which in this form can be checked to correspond to the parametrisation written down in [40] from first principles. Note that the boost invariance, acting as

$$\delta h_i = h^j\Lambda_j{}^A\tau_{iA}\,,\quad \delta C^i = -\Omega\Lambda_j{}^Ah^j\tau^i{}_A\,,\quad \tau^i{}_A\Lambda_i{}^B = 0\,, \tag{11.18}$$

corresponds to a shift symmetry of the parametrisation (11.17) pointed out in [40]. This generalises the Milne shift redundancy of the DFT non-Riemannian parametrisation [35]. Here we introduced the inverse vielbeins h^i and $\tau^i{}_A$ obeying the obvious relations

$$h_ih^i = 1\,,\quad \tau^i{}_A\tau_j{}^A + h^ih_j = \delta^i_j\,,\quad \tau^i{}_Ah_i = 0\quad \tau_i{}^Ah^i = 0\,,\quad \tau^i{}_A\tau_i{}^B = \delta^B_A\,. \tag{11.19}$$

The generalised metric in the 10×10 representation followng from the little metric (11.14) can be seen to take the form (9.29), after rewriting in the basis where generalised indices run over vector and two-form indices, and using the identities

$$\begin{aligned}\epsilon^{i_1\dots i_3 k}\epsilon^{j_1\dots j_3 l}\tau_{kl} &= -3!\,\Omega^2\big(\tau^{j_1[i_1}\tau^{i_2|j_2|}H^{i_3]j_3} + \tau^{j_2[i_1}\tau^{i_2|j_3|}H^{i_3]j_1} + \tau^{j_3[i_1}\tau^{i_2|j_1|}H^{i_3]j_2}\big)\,,\\ \epsilon^{i_1\dots i_3 k}\epsilon^{j_1\dots j_3 l}H_{kl} &= -3!\,\Omega^2\tau^{i_1[j_1|}\tau^{i_2|j_2|}\tau^{i_3|j_3]}\,.\end{aligned} \tag{11.20}$$

It is useful to record the explicit expression for the inverse little metric:

$$m^{\mathcal{M}\mathcal{N}} = \Omega^{4/5}\begin{pmatrix}\tau^{ij} - 2\Omega^{-1}h^{(i}C^{j)} & -\Omega^{-1}h^i\\ -\Omega^{-1}h^j & 0\end{pmatrix}\,. \tag{11.21}$$

Clearly, variations $\delta m^{\mathcal{M}\mathcal{N}}$ with $\delta m^{55} \neq 0$ do not preserve this parametrisation. This means that if we look at the equations of motion $\mathcal{R}_{\mathcal{M}\mathcal{N}} = 0$ of the generalised metric, we expect that $\mathcal{R}_{55} = 0$ provides an additional equation of motion that we would not find by varying the action evaluated on the non-relativistic parametrisation.

Field strengths and self-duality in SL(5) **ExFT** Our field strengths (11.12) are now

$$\begin{aligned}\mathcal{F}_{\mu\nu}{}^{i5} &= F_{\mu\nu}{}^i\,, & \mathcal{F}_{\mu\nu}{}^{ij} &= \tfrac{1}{2}\epsilon^{ijkl}(F_{\mu\nu kl} - C_{klm}F_{\mu\nu}{}^m)\,,\\ \mathcal{H}_{\mu\nu\rho i} &= -F_{\mu\nu\rho i}\,, & \mathcal{H}_{\mu\nu\rho 5} &= -\tfrac{1}{4!}\epsilon^{ijkl}(F_{\mu\nu\rho ijkl} - 4F_{\mu\nu\rho i}C_{jkl})\,,\\ \mathcal{J}_{\mu\nu\rho\sigma}{}^5 &= -F_{\mu\nu\rho\sigma}\,, & \mathcal{J}_{\mu\nu\rho\sigma}{}^i &= +\tfrac{1}{3!}\epsilon^{ijkl}(F_{\mu\nu\rho\sigma jkl} - C_{jkl}F_{\mu\nu\rho\sigma})\,.\end{aligned} \tag{11.22}$$

The kinetic terms (11.7) in the SL(5) ExFT action are:

$$\begin{aligned}&-\tfrac{1}{4}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N - \tfrac{1}{12}m^{\mathcal{M}\mathcal{N}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{M}}\mathcal{H}_{\mu\nu\rho\mathcal{N}}\\ &= -\tfrac{1}{4}\Omega^{2/5}\big(H_{ij}F^{\mu\nu i}F_{\mu\nu}{}^j - \epsilon^{ABC}\tau_{iA}\tau_B{}^j\tau_C{}^kF^{\mu\nu i}F_{\mu\nu jk} + \tau_C^i\tau^{jC}H^{kl}F^{\mu\nu}{}_{ik}F_{\mu\nu jl}\big)\\ &\quad - \tfrac{1}{12}\Omega^{4/5}\tau^{ij}F^{\mu\nu\rho}{}_iF_{\mu\nu\rho j} + \tfrac{1}{6}\Omega^{-1/5}h^iF^{\mu\nu\rho}{}_i\tfrac{1}{4!}\epsilon^{jklm}F_{\mu\nu\rho jklm}\end{aligned} \tag{11.23}$$

which match exactly the corresponding terms in (10.27) and (10.31), including the appearance of components of the dual seven-form field strength.

We see again that the ExFT description automatically contains the correct dual fields to reproduce the non-relativistic action immediately. It's worthwhile to go into some detail about the appearance of dual fields in the relativistic case. As mentioned above, the decomposition of the 11-dimensional three-form in the $(7+4)$-dimensional split produces four two-forms, $\hat{C}_{\mu\nu i}$ and a single three-form, $\hat{C}_{\mu\nu\rho}$. We exchange the latter for an additional two-form, $\hat{C}_{\mu\nu}$, in order to obtain the five-dimensional SL(5) multiplet $\mathcal{B}_{\mu\nu\mathcal{M}} = (\hat{C}_{\mu\nu i}, \hat{C}_{\mu\nu})$. This is normally done by introducing the two-form into the action as a Lagrange multiplier enforcing the Bianchi identity for $\hat{F}_{\mu\nu\rho\sigma}$. When this is done, the terms involving $\hat{F}_4$ in the action are schematically $\hat{F}_4 \wedge \star_7\hat{F}_4 - \hat{C}_2 \wedge (d\hat{F}_4 + \dots) + \hat{F}_4 \wedge X_3$, where X_3 denotes whatever appears alongside $\hat{F}_4$ in the decomposition of the Chern-Simons term. Integrating by parts one defines a field strength $H_3 \sim d\hat{C}_2 + X_3$ and treating $\hat{F}_4$ then as an independent field, one can integrate that out of the action to produce a kinetic term for H_3. The latter is then the $\mathcal{M} = 5$ component of the ExFT field strength $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$, and in this way the ExFT action matches the partially dualised SUGRA action.

In the non-relativistic theory, there is already no kinetic term for F_4 in the decomposed action, as seen from (10.27). It only appears (linearly) in the constraint term (10.31), schematically in the form $F_4 \wedge (\star_7\widetilde{F}_4 + \widetilde{F}_{3i}h^i)$. So instead if we carry out the same procedure, we find that F_4 equation of motion sets $H_3 = \star_7\widetilde{F}_4 + \widetilde{F}_{3i}h^i$, which in this case exactly corresponds to the relationship between the dual seven-form and $\widetilde{F}_4$ as expressed by (6.46). Hence now it is this H_3 that we identify with $\mathcal{H}_{\mu\nu\rho ijkl}$ via the above arguments. All this exactly mirrors what happened for the SL(3) × SL(2) case.

We finish with a brief look at the equations of motion. The field strength $\mathcal{J}_{\mu\nu\rho\sigma}$ of the gauge field $\mathcal{C}_{\mu\nu\rho}$ only appears in the topological term. This gauge field also appears in the field strength $\mathcal{H}_{\mu\nu\rho}$. Its equation of motion has the form $\partial_{\mathcal{MN}}\theta^{\mu\nu\rho\mathcal{N}} = 0$ where

$$\theta^{\mu\nu\rho\mathcal{M}} \equiv \sqrt{g}m^{\mathcal{MP}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{P}} + \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_4}\mathcal{J}_{\sigma_1\dots\sigma_4}{}^{\mathcal{M}} \,. \tag{11.24}$$

Meanwhile the equation of motion of $\mathcal{B}_{\mu\nu\mathcal{M}}$ is

$$\mathcal{D}_\rho(\sqrt{g}m^{\mathcal{MN}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{N}}) + \tfrac{1}{8}\epsilon^{\mathcal{MPQKL}}\partial_{\mathcal{PQ}}(\sqrt{g}\mathcal{M}_{\mathcal{KL},\mathcal{K}'\mathcal{L}'}\mathcal{F}^{\mu\nu\mathcal{K}'\mathcal{L}'}) - \tfrac{2}{4!}\epsilon^{\mu\nu\lambda_1\dots\lambda_5}\mathcal{F}_{\lambda_1\lambda_2}{}^{\mathcal{MN}}\mathcal{H}_{\lambda_3\dots\lambda_5\mathcal{N}} = 0\,. \tag{11.25}$$

The $\mathcal{M} = 5$ component combines with the $\mathcal{M} = 5$ component of the Bianchi identity (11.6) to give $\mathcal{D}_\rho\theta^{\mu\nu\rho5} = 0$. Hence we integrate and set $\theta^{\mu\nu\rho\mathcal{M}} = 0$. Let's examine the content of this constraint. Firstly, the $\theta^{\mu\nu\rho5}$ component implies

$$\Omega^{-1/5}\sqrt{g}h^jF^{\mu\nu\rho}{}_j - \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_4}F_{\sigma_1\dots\sigma_4} = 0 \tag{11.26}$$

This is the 11-dimensional self-duality constraint (6.26) on the transverse part of the four-form field strength, here decomposed as in (10.23). Secondly, setting $\theta^{\mu\nu\rho i} - C^i\theta^{\mu\nu\rho5} = 0$ and projecting gives

$$\begin{aligned}\sqrt{g}\Omega^{-1/5}F^{\mu\nu\rho}{}_{ijkl} + \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_4}4h_{[i|}F_{\sigma_1\dots\sigma_4|jkl]} = 0\,,\\ \sqrt{g}\Omega^{4/5}\tau^{iA}F^{\mu\nu\rho}{}_i - \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_4}\tau_i^A\tfrac{1}{3!}\epsilon^{ijkl}F_{\sigma_1\dots\sigma_4jkl} = 0\,.\end{aligned} \tag{11.27}$$

The first of these is part of the self-duality condition (6.47) obeyed by the totally longitudinal part of the dual-seven form. The second is part of the duality between the partly longitudinal four-form and the rest of the seven-form. We see again that the ExFT rearrangement of degrees of freedom exactly captures the novel features of the eleven-dimensional non-relativistic limit.

12 The Extremal Nature of Exotic Branes Actions

In this research article, assuming the existence of some isometry directions, we construct effective actions for various mixed-symmetry tensors that couple to exotic branes. We consider the cases of the exotic 5_2^2-brane, the 1_4^6-brane, and the Dp_{7-p}-brane, and argue that these exotic branes are the magnetic sources of the non-geometric fluxes associated with polyvectors β^{ij}, $\beta^{i_1\cdots i_6}$, and $\gamma^{i_1\cdots i_{7-p}}$, respectively. As it is well-known, an exotic-brane background written in terms of the usual background fields is not single-valued and has a U-duality monodromy. However, with a suitable redefinition of the background fields, the U-duality monodromy of the exotic-brane background simply becomes a gauge transformation associated with a shift in a polyvector, which corresponds to a natural extension of the *β-transformation* known in the generalized geometry. Here we study the case of exotic super p-brane. The contribution of the boundary terms in the variation of S_p is given by

$$\delta S_p|_\Gamma = \oint ds_\mu\rho^\mu Y^\Lambda G_{\Lambda\Xi}\delta Y^\Xi, \tag{12.1}$$

where $ds^\nu = \frac{1}{p!}\varepsilon^{\nu\mu_1\mu_2\dots\mu_p}dS_{\mu_1\mu_2\dots\mu_p}$. Here, we consider the variational problem with the fix initial ($\tau = \tau_i$) and final ($\tau = \tau_f$) data, so the integral along the super p-brane profile for $\tau = (\tau_i, \tau_f)$ does not contribute to $\delta S_p|_\Gamma$

$$\int_{s_\tau} ds_\tau\rho^\tau Y^\Lambda G_{\Lambda\Xi}\delta Y^\Xi|_{\tau_i}^{\tau_f} = 0. \tag{12.2}$$

As a result, the variation $\delta S_p|_\Gamma$ is filled out by the integrals along the p-dimensional boundaries of the brane worldvolume containing the τ-direction

$$\delta S_p|_\Gamma = \Sigma_{i=1}^{i=p} \int_{s_i} ds_i \rho^i Y^\Lambda G_{\Lambda\Xi} \delta Y^\Xi|_{\sigma^i=0}^{\sigma^i=\pi}. \tag{12.3}$$

In the case of variational problem with free ends, when the field variations on the p-brane boundaries are arbitrary, the vanishing of these hypersurface terms in $\delta S_p|_\Gamma$ gives the super p-brane boundary conditions. As shown in the current literature, the Wess-Zumino term of the $5_2^2(34567, 89)$-brane action (smeared in the isometry directions, x^8 and x^9) can be written as

$$\begin{aligned} S_{\rm WZ}^{5_2^2} &= -\mu_{5_2^2}\, n^{89} \int_{\mathcal{M}_6 \times T_{89}^2} \iota\beta_{89}^{(8)} \wedge \frac{\mathrm{d}x^8 \wedge \mathrm{d}x^9}{(2\pi R_8)(2\pi R_9)} = -\frac{\mu_{5_2^2}\, n^{89}}{(2\pi R_8)(2\pi R_9)} \int_{\mathcal{M}_6 \times T_{89}^2} \beta_{89}^{(8)} \\ &= -\mu_{5_2^2} \int \beta_{89}^{(8)} \wedge \delta^{89}(x - X(\xi)) \quad (n^{89}\text{: number of the } 5_2^2(34567, 89)\text{-branes}) \\ &\Big(\delta^{\mathbf{p}_1\cdots\mathbf{p}_n}(x - X(\xi)) \equiv \frac{n^{\mathbf{p}_1\cdots\mathbf{p}_n}\, \delta^2(x - X(\xi))}{(2\pi R_{\mathbf{p}_1})\cdots(2\pi R_{\mathbf{p}_n})}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\,, \quad n^{\mathbf{p}_1\cdots\mathbf{p}_n} \in \mathbb{Z}\Big)\,, \end{aligned} \tag{12.4}$$

where we used $\mathcal{M}_6$ the worldvolume of the 5_2^2-brane, and the Ramond-Ramond fields and the worldvolume gauge fields are turned off for simplicity. Now, let us consider the dual action which additionally includes the Wess-Zumino term

$$\begin{aligned} S[\,\tilde{g}_{ij},\, \tilde{\phi},\, \beta_{ij}^{(8)}\,] &= \frac{1}{2\kappa_{10}^2} \int \Big[\mathrm{e}^{-2\tilde{\phi}}\, (\tilde{*}\, \tilde{R} + 4\, \mathrm{d}\tilde{\phi} \wedge \tilde{*}\, \mathrm{d}\tilde{\phi}) - \frac{1}{4}\, \mathrm{e}^{2\tilde{\phi}}\, \tilde{g}^{ik}\, \tilde{g}^{jl}\, Q_{ij}^{(9)} \wedge \tilde{*}\, Q_{kl}^{(9)}\Big] \\ &\quad - \mu_{5_2^2} \int \frac{1}{2}\, \beta_{\mathbf{pq}}^{(8)} \wedge \delta^{\mathbf{pq}}(x - X(\xi))\,. \end{aligned} \tag{12.5}$$

Taking a variation with respect to $\beta_{\mathbf{pq}}^{(8)}$, we obtain the following equation of motion:

$$\frac{1}{2\kappa_{10}^2}\, \mathrm{d}Q^{(1)\,\mathbf{pq}} = \frac{\mu_{5_2^2}}{(2\pi R_{\mathbf{p}})(2\pi R_{\mathbf{q}})}\, n^{\mathbf{pq}}\, \delta^2(x - X(\xi))\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\,. \tag{12.6}$$

From (12.6), we conclude that the current for the $5_2^2(n_1 \cdots n_5, m_1 m_2)$-brane (in the absence of the Ramond-Ramond fields) is given by

$$\tilde{*}\, j_{5_2^2(n_1\cdots n_5,\, m_1 m_2)} = \frac{(2\pi R_{m_1})(2\pi R_{m_2})}{2\kappa_{10}^2\, \mu_{5_2^2}}\, \mathrm{d}Q^{(1)\, m_1 m_2}\,. \tag{12.7}$$

According to the Wess-Zumino term of the $5_3^2(34567, 89)$-brane action (smeared in the isometry directions, x^8 and x^9) is written as

$$S_{\rm WZ}^{5_3^2} = -\mu_{5_3^2} \int \gamma_{89}^{(8)} \wedge \delta^{89}(x - X(\xi))\,, \tag{12.8}$$

where the B-field, the Ramond-Ramond 0- and 4-forms, and the worldvolume gauge fields are turned off for simplicity, and $\delta^{89}(x - X(\xi))$ is defined in (12.4). As in the case of the 5_2^2-brane, if we consider the action

$$\begin{aligned} S[\,\tilde{g}_{ij},\, \tilde{\phi},\, \gamma_{ij}^{(8)}\,] &= \frac{1}{2\kappa_{10}^2} \int \Big[\mathrm{e}^{-2\tilde{\phi}}\, (\tilde{*}\tilde{R} + 4\, \mathrm{d}\tilde{\phi} \wedge \tilde{*}\, \mathrm{d}\tilde{\phi}) - \frac{1}{4}\, \mathrm{e}^{4\tilde{\phi}}\, \tilde{g}^{ik}\, \tilde{g}^{jl}\, P_{ij}^{(9)} \wedge \tilde{*}\, P_{kl}^{(9)}\Big] \\ &\quad - \mu_{5_3^2} \int \frac{1}{2}\, \gamma_{\mathbf{pq}}^{(8)} \wedge \delta^{\mathbf{pq}}(x - X(\xi))\,, \end{aligned} \tag{12.9}$$

and take a variation with respect to $\gamma^{(8)}_{\mathbf{pq}}$, we obtain the Bianchi identity for the P-flux with a source term:

$$\frac{1}{2\kappa_{10}^2}\,\mathrm{d}P^{(1)\,\mathbf{pq}} = \frac{\mu_{5_3^2}}{(2\pi R_{\mathbf{p}})(2\pi R_{\mathbf{q}})}\,n^{\mathbf{pq}}\,\delta^2(x-X(\xi))\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,. \tag{12.10}$$

As in the case of the β-supergravity, we can further find a solution corresponding to the (Euclidean) background of an instanton that couples to γ^{ij} electrically. The explicit form of the background fields is presented.

We have presented various actions with the following form:

$$S[\tilde{g}_{ij},\,\tilde{\phi},\,\mathcal{A}^{i_1\cdots i_{7-p}}] = \frac{1}{2\kappa_{10}^2}\int\Big[\mathrm{e}^{-2\phi}\,(\tilde{*}\,\tilde{R}+4\,\mathrm{d}\phi\wedge\tilde{*}\,\mathrm{d}\phi)-\frac{\mathrm{e}^{2\,(\alpha+1)\,\tilde{\phi}}}{2\,(7-p)!}\,\tilde{g}_{i_1j_1}\cdots\tilde{g}_{i_{7-p}j_{7-p}}\,\mathcal{Q}^{(1)\,i_1\cdots i_{7-p}}\wedge\tilde{*}\,\mathcal{Q}^{(1)\,j_1\cdots j_{7-p}}\Big]\,, \tag{12.11}$$

where $\mathcal{Q}^{(1)\,i_1\cdots i_{7-p}}\equiv\mathrm{d}\mathcal{A}^{i_1\cdots i_{7-p}}$ is a non-geometric flux of which an exotic brane acts as the magnetic source, and α is an integer. A list of non-geometric fluxes and their magnetic/electric sources.

The equations of motion are given by

$$\tilde{R}+4(\tilde{\nabla}^i\partial_i\tilde{\phi}-\tilde{g}^{ij}\,\partial_i\tilde{\phi}\,\partial_j\tilde{\phi})+\tfrac{(\alpha+1)\,\mathrm{e}^{2\,(\alpha+2)\,\tilde{\phi}}}{2\,(7-p)!}\,\mathcal{Q}_i{}^{j_1\cdots j_{7-p}}\,\mathcal{Q}^i{}_{j_1\cdots j_{7-p}}=0\,, \tag{12.12}$$

$$\begin{aligned}\tilde{R}_{ij}+2\tilde{\nabla}_i\partial_j\tilde{\phi}-\tfrac{\mathrm{e}^{2\,(\alpha+2)\,\tilde{\phi}}}{2\,(7-p)!}\Big(&\mathcal{Q}_i{}^{k_1\cdots k_{7-p}}\,\mathcal{Q}_{jk_1\cdots k_{7-p}}-(7-p)\,\mathcal{Q}_{k_1}{}^{k_2\cdots k_{7-p}}\,\mathcal{Q}^{k_1}{}_{jk_2\cdots k_{7-p}}\\ &-\tfrac{\alpha+2}{2}\,\mathcal{Q}_k{}^{l_1\cdots l_{7-p}}\,\mathcal{Q}^k{}_{l_1\cdots l_{7-p}}\,\tilde{g}_{ij}\Big)=0\,,\end{aligned} \tag{12.13}$$

$$\mathrm{d}\mathcal{Q}^{(9)}_{i_1\cdots i_{7-p}}=0\,,\quad \mathcal{Q}^{(9)}_{i_1\cdots i_{7-p}}\equiv\mathrm{e}^{2\,(\alpha+1)\,\tilde{\phi}}\,\tilde{g}_{i_1j_1}\cdots\tilde{g}_{i_{7-p}j_{7-p}}\,\tilde{*}\,\mathcal{Q}^{(1)\,j_1\cdots j_{7-p}}\equiv\mathrm{d}\mathcal{A}^{(8)}_{i_1\cdots i_{7-p}}\,. \tag{12.14}$$

If we regard the dual potential $\mathcal{A}^{(8)}_{i_1\cdots i_{7-p}}$ as a fundamental field, the dual action is given by

$$\begin{aligned}S[\tilde{g}_{ij},\,\tilde{\phi},\,\mathcal{A}^{(8)}_{i_1\cdots i_{7-p}}]=\frac{1}{2\kappa_{10}^2}\int\Big[&\mathrm{e}^{-2\phi}\,(\tilde{*}\,\tilde{R}+4\,\mathrm{d}\phi\wedge\tilde{*}\,\mathrm{d}\phi)\\ &-\frac{\mathrm{e}^{2\,(\tilde{\alpha}+1)\,\tilde{\phi}}}{2\,(7-p)!}\,\tilde{g}^{i_1j_1}\cdots\tilde{g}^{i_{7-p}j_{7-p}}\,\mathrm{d}\mathcal{A}^{(8)}_{i_1\cdots i_{7-p}}\wedge\tilde{*}\,\mathrm{d}\mathcal{A}^{(8)}_{j_1\cdots j_{7-p}}\Big]\,,\end{aligned} \tag{12.15}$$

where we defined $\tilde{\alpha}\equiv-\alpha-2$. We can add the Wess-Zumino term of the exotic $p^{7-p}_{-\alpha}$-brane extending in the $x^{\mathbf{r}_1},\cdots,x^{\mathbf{r}_p}$-directions and smeared over the $x^{\mathbf{s}_1},\cdots,x^{\mathbf{s}_{7-p}}$-directions:

$$\begin{aligned}S_{\mathrm{WZ}}&=-\mu_{p^{7-p}_{-\alpha}}\sum_{\mathbf{s}_1,\cdots,\mathbf{s}_{7-p}}\int_{\mathcal{M}_{p+1}\times T^{7-p}_{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}}\frac{n^{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}}{(7-p)!}\,\iota\mathcal{A}^{(8)}_{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}\wedge\frac{\mathrm{d}x^{\mathbf{s}_1}\wedge\cdots\wedge\mathrm{d}x^{\mathbf{s}_{7-p}}}{(2\pi R_{\mathbf{s}_1})\cdots(2\pi R_{\mathbf{s}_{7-p}})}\\ &=-\mu_{p^{7-p}_{-\alpha}}\int\frac{1}{(7-p)!}\,\mathcal{A}^{(8)}_{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}\wedge\delta^{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}(x-X(\xi))\,.\end{aligned} \tag{12.16}$$

Then, taking variation, we obtain the following Bianchi identity as the equation of motion:

$$\mathrm{d}^2\mathcal{A}^{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}=2\kappa_{10}^2\,\mu_{p^{7-p}_{-\alpha}}\,\frac{n^{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}}{(2\pi R_{\mathbf{s}_1})\cdots(2\pi R_{\mathbf{s}_{7-p}})}\,\delta^2(x-X(\xi))\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,. \tag{12.17}$$

If we choose $n^{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}=1$ and integrate the equation, we obtain

$$\sigma=\int\mathrm{d}^2\mathcal{A}^{\mathbf{s}_1\cdots\mathbf{s}_{7-p}}=\frac{2\kappa_{10}^2\,\mu_{p^{7-p}_{-\alpha}}}{(2\pi R_{\mathbf{s}_1})\cdots(2\pi R_{\mathbf{s}_{7-p}})}\,, \tag{12.18}$$

where we used $\mathcal{A}^{\mathbf{s}_1\cdots\mathbf{s}_{7-p}} = \rho_1$. From this relation and the value of σ given, we can confirm that $\mu_{p^{7-p}_{-\alpha}}$ is indeed equal to the tension of the exotic brane:

$$\mu_{p^{7-p}_{-\alpha}} = \frac{\sigma\,(2\pi R_{\mathbf{s}_1})\cdots(2\pi R_{\mathbf{s}_{7-p}})}{(2\pi l_s)^7\, l_s\, g_s^2} = \frac{M_{p^{7-p}_{-\alpha}}}{(2\pi R_{\mathbf{r}_1})\cdots(2\pi R_{\mathbf{r}_{p+1}})}\,, \tag{12.19}$$

where we used $2\kappa_{10}^2 = (2\pi l_s)^7\, l_s\, g_s^2$. It will be also important to investigate a reformulation of the effective worldvolume theory of exotic branes by using the newly introduced background fields ($\tilde{g}_{ij}$, $\tilde{\phi}$, $\mathcal{A}^{i_1\cdots i_{7-p}}$). More generally, it will be important to find a manifestly U-duality covariant formulation for the effective worldvolume theory of exotic branes.

13 Wess-Zumino Actions of Exotic Branes

We considered the general solutions of the equations of motion in the simple model of closed and open tensionless superstring and exotic p-branes. Using the $OSp(1,2M)$ invariant character of the differential one-form $Y^\Lambda G_{\Lambda\Xi} dY^\Xi$ and two-form $dY^\Lambda G_{\Lambda\Xi} dY^\Xi$ one can construct more general $OSp(1,2M)$ invariant super p-brane actions with enhanced supersymmetry. At first, we note that the closed $2n$-differential form $\Omega_{2n} = (G_{\Lambda\Xi} dY^\Lambda \wedge dY^\Xi)^n$

$$\Omega_{2n} = d \wedge \Omega_{(2n-1)} \equiv G_{\Lambda_1\Xi_1} dY^{\Lambda_1} \wedge dY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} dY^{\Lambda_n} \wedge dY^{\Xi_n} \tag{13.1}$$

which is not equal to zero, because of the symplectic character of the supertwistor metric $G_{\Lambda\Xi}$, can be used to generate the Dirichlet boundary terms for the open super p-brane ($p = 2n-1$) described by the generalized action

$$S = S_{2n-1} + \beta_{(2n-1)} \int_{M_{2n}} \Omega_{2n}. \tag{13.2}$$

Similarly to the open superstring case, the Wess-Zumino integral in (13.2) is transformed to the integral along the $(2n-1)$-dimensional boundary M_{2n-1} of the super $(2n-1)$-brane worldvolume

$$\int_{M_{2n}} \Omega_{2n} = \oint_{M_{2n-1}} G_{\Lambda_1\Xi_1} Y^{\Lambda_1} \wedge dY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} dY^{\Lambda_n} \wedge dY^{\Xi_n}. \tag{13.3}$$

The sufficient conditions for the vanishing of the variations of the integral (13.3) with the fix initial and final data are the conditions

$$\partial_\tau Y^\Lambda(\tau,\sigma)|_{\sigma^i=0,\pi} = 0, \quad (i = 1,2,...,2n-1) \tag{13.4}$$

generalizing the Dirichlet boundary condition. Therefore, this open super p-brane is described by the pure static solution

$$Y^\Lambda(\tau,\sigma) = Y_0^\Lambda(\sigma^i), \quad (i = 1,2,...,2n-1) \tag{13.5}$$

generalizing the superstring static solution. On the other hand the integrals (13.3)

$$S_{(2n-2)} = \beta_{(2n-2)} \int_{M_{2n-1}} \Omega_{2n-1},$$
$$\Omega_{2n-1} \equiv G_{\Lambda_1\Xi_1} Y^{\Lambda_1} dY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} dY^{\Lambda_n} \wedge dY^{\Xi_n} \tag{13.6}$$

can be considered as the $OSp(1,2M)$ invariant actions for the new models of super p-branes ($p = 2n-2$) with enhanced supersymmetry. For $n = 1$ we get the known action for superparticles, but for $n = 2,3$ we find the new actions for the supermembrane

$$S_2 = \beta_2 \int_{M_3} \Omega_3 = \tilde{\beta}_2 \int d\tau d^2\sigma\, \varepsilon^{\mu\nu\rho} Y^\Lambda \partial_\mu Y_\Lambda \partial_\nu Y^\Xi \partial_\rho Y_\Xi, \tag{13.7}$$

or a domain wall in the symplectic superspace, and for the super four-brane

$$S_4 = \beta_4 \int_{M_5} \Omega_5 = \tilde{\beta}_4 \int d\tau d^4\sigma\, \varepsilon^{\mu\nu\rho\lambda\phi} Y^\Lambda \partial_\mu Y_\Lambda \partial_\nu Y^\Xi \partial_\rho Y_\Xi \partial_\lambda Y^\Sigma \partial_\phi Y_\Sigma. \tag{13.8}$$

When the Wess-Zumino terms are considered as the boundary terms generating the Dirichlet boundary conditions for the superstring and super p-branes (13.4) the breaking of the Weyl symmetry is localized at the boundaries. It shows that the spontaneous breaking of the $OSp(1,2M)$ symmetry on the boundaries is accompanied by the explicit breakdown of the Weyl gauge symmetry on the boundaries. Because the Dirichlet boundary conditions are associated with the Dp-branes attached on their boundaries, a question on the action of Dp-branes in the symplectic superspaces considered here appears. It implies the correspondent generalization of the proposed Wess-Zumino actions. One of the posssible generalizations is rather natural and is based on the observation that the Weyl invariance of the considered Wess-Zumino actions may be restored by the minimal lengthening of the differentials $d \rightarrow D = (d - A)$, where the worldvolume one-form A is the gauge field associated with the Weyl symmetry. The covariant differentials DY^Σ are homogeneously transformed under the Weyl symmetry transformations

$$(DY^\Sigma)' \equiv ((d-A)Y^\Sigma)' = e^\lambda DY^\Sigma, \quad A' = A + d\lambda. \tag{13.9}$$

Then the generalized $OSp(1,2M)$ invariant two and one-forms

$$\begin{aligned} (e^\phi DY^\Sigma G_{\Sigma\Xi} DY^\Xi)' &= e^\phi DY^\Sigma G_{\Sigma\Xi} DY^\Xi, \\ (e^\phi Y^\Sigma G_{\Sigma\Xi} DY^\Xi)' &= e^\phi Y^\Sigma G_{\Sigma\Xi} DY^\Xi \end{aligned} \tag{13.10}$$

become the invariants of the Weyl symmetry also, where the compensating scalar field ϕ, with the transformation low

$$\phi' = \phi - 2\lambda, \tag{13.11}$$

was introduced. Then the closed $2n$-differential form $\Omega_{2n} = (G_{\Lambda\Xi} dY^\Lambda \wedge dY^\Xi)^n$ may be changed by the Weyl invariant $2n$-differential form $\tilde{\Omega}_{2n} = (e^\phi G_{\Lambda\Xi} DY^\Lambda \wedge DY^\Xi)^n$

$$\tilde{\Omega}_{2n} \equiv e^{n\phi} G_{\Lambda_1\Xi_1} DY^{\Lambda_1} \wedge DY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} DY^{\Lambda_n} \wedge DY^{\Xi_n}, \tag{13.12}$$

and Ω_{2n-1} by $\tilde{\Omega}_{2n-1}$

$$\tilde{\Omega}_{2n-1} \equiv e^{n\phi} Y^{\Lambda_1} \wedge DY_{\Lambda_1} \wedge ... \wedge DY^{\Lambda_n} \wedge DY_{\Lambda_n}. \tag{13.13}$$

As a result, the actions (13.3) is transformed to the new super $(2n-1)$-brane action

$$\tilde{S}_{(2n-1)} = \beta_{(2n-1)} \int_{M_{2n}} \tilde{\Omega}_{2n} = \beta_{(2n-1)} \int e^{n\phi} G_{\Lambda_1\Xi_1} DY^{\Lambda_1} \wedge DY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} DY^{\Lambda_n} \wedge DY^{\Xi_n} \tag{13.14}$$

invariant under the $OSp(1,2M)$ and Weyl symmetries. Respectively, the action

$$\tilde{S}_{(2n-2)} = \beta_{(2n-2)} \int_{M_{2n-1}} \tilde{\Omega}_{2n-1} = \beta_{(2n-2)} \int e^{n\phi} Y^{\Lambda_1} \wedge DY_{\Lambda_1} \wedge ... \wedge DY^{\Lambda_n} \wedge DY_{\Lambda_n} \tag{13.15}$$

will describe a new $OSp(1,2M)$ and Weyl invariant super $(2n-2)$-brane.
These actions may be presented in the Dp-brane like form

$$\tilde{S}_p = \tilde{\beta}_p \int d\tau d^p\sigma\, e^{\frac{(p+1)}{2}\phi} \sqrt{|det[(\partial_\mu - A_\mu) Y^\Lambda G_{\Lambda\Xi} (\partial_\nu - A_\nu) Y^\Xi]|},\ (p = 2n-1), \tag{13.16}$$

where $\tilde{\beta}_p$ is the Dp-brane tension. We generalized this model to the higher orders in the derivatives of the Goldstone fields and constructed the new Wess-Zumino like actions supposed to describe tensile exotic

p-branes. It was shown in deep detail, that the bosonic couplings described above were consistent with all the linear couplings of closed superstring background fields with higher-dimensional supergravity theory including exceptional degrees of freedom of multiple D-branes. These couplings were originally computed in the current literature and then extended to Dp-branes with using T-duality symmetries. We will review the illustration of the general formalism with presentation of the Wess-Zumino term for multiple D-branes that is required to do such matching

$$
\begin{aligned}
S_{\mathcal{WZ}} &= \Xi_\Sigma \int \mathrm{Tr}\Big[\mathcal{P}\wedge(\mathcal{D}^{(R)}+\Delta(\Xi_\Sigma)\big(\mathcal{D}^{(V)}\wedge B\big)-\Delta(\Xi_\Sigma)\Big(\mathcal{D}^{(U)}\wedge B+\frac{1}{2}D^{(R)}\wedge B\wedge B\Big)\\
&-\Delta(\Xi_\Sigma)\Big(\mathcal{D}^{(Z)}+D^{(T)}\wedge B+\frac{1}{2}D^{(V)}\wedge B\wedge B+\frac{1}{6}D^{(R)}\wedge B\wedge B\wedge B\Big)\wedge\mathcal{B}\wedge\mathcal{D}^{(Z)}\\
&-\Delta(\Xi_\Sigma)\Big(\mathcal{D}^{(Z)}+\frac{1}{2}D^{(T)}\wedge B+\frac{1}{6}D^{(R)}\wedge B\wedge B+\frac{1}{24}D^{(Z)}\wedge B\wedge B\wedge B\Big)\wedge\mathcal{B}\wedge\mathcal{D}^{(Z)}\\
&+\Delta(\Xi_\Sigma)\Big(\mathcal{D}^{(U)}\wedge G+D^{(U)}\wedge G\wedge\mathcal{K}^{(R)}-\mathcal{D}^{(T)}\wedge K^{(R)}-D^{(T)}\wedge K^{(R)}\wedge G\wedge K^{(T)}\Big)\\
&+\Delta(\Xi_\Sigma)\big(\mathcal{D}^{(T)}+B\wedge D^{(V)}-D^{(V)}\wedge K^{(R)}\wedge G-\mathcal{B}\wedge G+\mathcal{D}^{(V)}\wedge K^{(R)}\big)\wedge\mathcal{K}^{(\mathcal{V})}\\
&-(\mathcal{D}^{(W)}+D^{(S)}\wedge B)+B\wedge L^{(Z)}+D^{(S)}\wedge B-B\wedge L^{(R)}\wedge G+\mathcal{D}^{(S)}\wedge G\wedge L^{(W)}\\
&-\Delta(\Xi_\Sigma)\big(\mathcal{B}-\mathcal{D}^{(Z)}\wedge D^{(S)}\wedge\mathcal{L}^{(W)}\wedge L^{(V)}\big)+\Big(\mathcal{B}^{(X)}-\frac{1}{2}\mathcal{B}\wedge\mathcal{D}^{(Z)}\wedge D^{(S)}\wedge L^{(Z)}\Big)\\
&+\big(\mathcal{D}^{(W)}+D^{(S)}\wedge B\big)\wedge\mathcal{L}^{(R)}+\big(\mathcal{D}^{(W)}+D^{(S)}\wedge B\wedge\mathcal{L}^{(W)}\wedge L^{(V)}\big)\wedge\mathcal{L}^{(Z)}\wedge\mathcal{G}\\
&+\Big(\mathcal{D}^{(W)}+D^{(S)}\wedge B\wedge\mathcal{L}^{(\mathcal{R})}\wedge G+\mathcal{D}^{(W)}+D^{(S)}\wedge B\wedge\mathcal{L}^{(Z)}\wedge\mathcal{L}^{(R)}\wedge G\Big)\wedge\mathcal{L}^{(\mathcal{W})}\\
&-\Delta(\Xi_\Sigma)\Big(\mathcal{D}^{(W)}+D^{(S)}\wedge B-\mathcal{B}\wedge\mathcal{L}^{(\mathcal{R})}+\mathcal{B}\wedge\mathcal{L}^{(\mathcal{R})}\wedge\mathcal{G}+\mathcal{D}^{(S)}\wedge\mathcal{B}\wedge\mathcal{G}\Big)\wedge\mathcal{L}^{(\mathcal{R})}\\
&+\big(\mathcal{D}^{(W)}+D^{(S)}\wedge B\big)\wedge\mathcal{L}^{(R)}+\big(\mathcal{D}^{(W)}+D^{(S)}\wedge B\wedge\mathcal{L}^{(W)}\wedge L^{(V)}\big)\wedge\mathcal{L}^{(Z)}\wedge\mathcal{G}\\
&+\Big(\mathcal{D}^{(W)}+D^{(S)}\wedge B\wedge\mathcal{L}^{(\mathcal{R})}\wedge G+\mathcal{D}^{(W)}\wedge B\wedge\mathcal{L}^{(Z)}\wedge G\Big)\wedge\mathcal{L}^{(\mathcal{W})}\wedge\mathcal{L}^{(\mathcal{V})}\Big] \qquad (13.17)
\end{aligned}
$$

14 The Higher Dimensional Effective Actions of Supergravity with Fundamental Newton-Cartan Membranes

In the current consideration and review, we aim to build a higher dimensional theory of exceptional supergravity with included backgrounds, superstrings and fundamental supermembranes existing in D-dimensional spacetime supermanifolds. In addition, there is a real hope of exploring the interplay between higher-dimensional supergravities, superstrings and membrane theory. These are strong motivations to understand the fundamental nature including extremal interactions of supergravity theories plus their associated theoretical framework and mathematics more deeply. In this Chapter, we describe the construction of higher-dimensional effective actions of exceptional supergravity in backgrounds with superstrings and membrane models in the variant of bulk and brane systems. The special type low-energy effective interactions in D-spacetime dimensions are considered included in supergravity backgrounds with the participation of superstrings together in superfield representations for the construction of global dual symmetries presented in the theoretical framework.

The higher-dimensional effective action of bulk and brane system is

$$
S_{\mathcal{T}} = \int d^D x\sqrt{-\mathcal{G}}\mathcal{R}\mathcal{L}_{\mathcal{BULK}} + \int d^D x\sqrt{-\mathcal{G}}\mathcal{R}\mathcal{L}_{\mathcal{BRANE}} \qquad (14.1)
$$

The local moduli space $\widetilde{\mathcal{M}}_{\mathcal{T}}$ for the special higher-dimensional supergravity with an exceptional bulk and brane system is

$$\widetilde{\mathcal{M}}_{\mathcal{T}}(\Delta_{[\Xi]}) \hookrightarrow \left\{ \widetilde{\mathcal{M}}_{\mathcal{VM}}(\Delta_{[\Xi]}) \otimes \widetilde{\mathcal{M}}_{\mathcal{TM}}(\Delta_{[\Xi]}) \otimes \widetilde{\mathcal{M}}_{\mathcal{HM}}(\Delta_{[\Xi]}) \right\} \oplus \widetilde{\mathcal{M}}_{\mathcal{BRANE}}(\Delta_{[\Xi]}) \tag{14.2}$$

Analogous to the previous solutions we present the higher-dimensional effective action of an exceptional supergravity with a Newton-Cartan fundamental membrane

$$S_{\mathcal{D}-\mathcal{MNC}} \rightsquigarrow S_{\mathcal{D}} + \Delta^{\mathcal{N}}\Big(\Pi^{\mathcal{A}}\Big) S_{\mathcal{MNC}} \rightsquigarrow \int d^D x \sqrt{-\mathcal{G}} \mathcal{R} \mathcal{L}_{\mathcal{SUGRA}} + \Delta^{\mathcal{N}}\Big(\Pi^{\mathcal{A}}\Big) \int d^D x \sqrt{-\mathcal{G}} \mathcal{R} \mathcal{L}_{\mathcal{MNC}} \tag{14.3}$$

The bulk and brane system in the supergravity moduli space is combination of the bulk lagrangian $\mathcal{L}_{\mathcal{B}}$, brane $\mathcal{L}_{\mathcal{BR}}$, hidden brane lagrangian $\mathcal{L}_{\mathcal{HBR}}$ and the brane fields coupling action $\mathcal{L}_{\mathcal{BFC}}$ expressed with the equation

$$S_{\mathcal{M}_{\mathcal{D}}} = \int d^D x \sqrt{-\mathcal{G}} \mathcal{R} \mathcal{L}_{\mathcal{B}} + \int d^D x \sqrt{-\mathcal{G}} \mathcal{R} \mathcal{L}_{\mathcal{BR}} + \int d^D x \sqrt{-\mathcal{G}} \mathcal{R} \mathcal{L}_{\mathcal{HBR}} + \int d^D x \sqrt{-\mathcal{G}} \mathcal{R} \mathcal{L}_{\mathcal{BFC}} \tag{14.4}$$

The main goals of this article are to clarify, on general superstring-theoretical grounds, which duality symmetry we should expect for the effective spacetime theory of the massless fields to any higher order and to exhibit this symmetry in a manifest form. We shall consider a general set of external states subject to the condition of independence of the D spatial coordinates, we shall work in a special supergravity background left invariant by a large subset of the duality symmetries.

The background solution of the field equation for the metric depends essentially on the presence of the leading quantum correction to the CJS action, so the presence of that term has to be taken into account in studying the Kaluza-Klein modes of the metric. Then we have the tensor

$$\Delta_I{}^J{}_K \hookrightarrow \bar{\Gamma}_I{}^J{}_K - \Gamma_I{}^J{}_K \hookrightarrow \bar{\bar{\mathcal{G}}}^{JL} \left(\mathcal{D}_I \Sigma_{KL} + \mathcal{D}_K \Sigma_{IL} - \mathcal{D}_L \Sigma_{IK} \right) \tag{14.5}$$

where the standard Christoffel connection is $\Gamma_I{}^J{}_K = \frac{1}{2}\mathcal{G}^{JL}\left(\mathcal{D}_I \mathcal{G}_{LK} + \mathcal{D}_K \mathcal{G}_{LI} - \mathcal{D}_L \mathcal{G}_{IK}\right)$, and the modified Riemann tensor for the metric $\mathcal{G}_{IJ}$ is defined by

$$\bar{\bar{\mathcal{R}}}_{IJ}{}^K{}_L \hookrightarrow \mathcal{R}_{IJ}{}^K{}_L + \mathcal{D}_I \Delta_J{}^K{}_L - \mathcal{D}_J \Delta_I{}^K{}_L + \Delta_I{}^K{}_M \Delta_J{}^M{}_L - \Delta_J{}^K{}_M \Delta_I{}^M{}_L, \tag{14.6}$$

We then find

$$\begin{aligned}
\Xi_\Sigma \bar{\bar{\mathcal{R}}} \quad \hookrightarrow \quad & \Xi_\Sigma \bar{\bar{\mathcal{G}}}^{IJ} \bar{\bar{\mathcal{R}}}_{KI}{}^K{}_J \rightsquigarrow \Xi_\Sigma \Big(\mathcal{R} + \Sigma_{II}\mathcal{R} - 2\Sigma_{IJ}\mathcal{R}_{IJ} + 2\Sigma_{IK}\Sigma_{KJ}\mathcal{R}_{IJ} + 2\Sigma_{IK}\Sigma_{JL}\mathcal{R}_{IJKL} \\
& -2\Sigma_{KK}\Sigma_{IJ}\mathcal{R}_{IJ} - \Sigma_{IJ}\Sigma_{JI}\mathcal{R} + \frac{1}{2}\Sigma_{II}\Sigma_{JJ}\mathcal{R} - \mathcal{D}_K\Sigma_{IJ}\mathcal{D}_K\Sigma_{IJ} + 2\mathcal{D}_I\Sigma_{IK}\mathcal{D}_J\Sigma_{JK} \\
& -2\mathcal{D}_I\Sigma_{IJ}\mathcal{D}_J\Sigma_{KK} + \mathcal{D}_K\Sigma_{II}\mathcal{D}_K\Sigma_{JJ} + 2\,\Xi_\Sigma\Sigma_{IK}\Sigma_{JL}\mathcal{R}_{IJ}\mathcal{R}_{IJKL} - \mathcal{D}_A\mathcal{D}_B\mathcal{P}_{CD} \\
& -2\mathcal{R}_{ABCD}\mathcal{P}_{CD} + \frac{1}{2}\mathcal{H}_{ACEF}\mathcal{H}_{BDEF}\mathcal{P}_{CD} - \frac{1}{2}\mathcal{G}_{AB}\mathcal{H}_{CEFG}\mathcal{H}_{DEFG}\mathcal{P}_{CD} \\
& -\frac{1}{2}\,\Xi_\Sigma\Sigma_{IM}\Sigma_{JN}\mathcal{H}_{IJKL}\mathcal{H}_{MNKL} + \Xi_\Sigma\Sigma_{NN}\left[\mathcal{B}_\Lambda\right]_{\Delta_\xi} + \Xi_\Sigma\Sigma_{KK}\left[\mathcal{B}_\Lambda\right]_{\Delta_\eta} \Big),
\end{aligned} \tag{14.7}$$

where designation $\rightsquigarrow$ means up to the addition of total derivative terms, and the identity used in the previous expression is

$$\begin{aligned}
\Xi_\Sigma\mathcal{D}_I\Sigma_{JK}\mathcal{D}_J\Sigma_{IK} \quad \rightsquigarrow \quad & \Xi_\Sigma\mathcal{D}_I\Sigma_{IK}\mathcal{D}_J\Sigma_{JK} - \Xi_\Sigma\Sigma_{IK}\Sigma_{KJ}\mathcal{R}_{IJ} + \Xi_\Sigma\Sigma_{IK}\Sigma_{JL}\mathcal{R}_{IJKL} \\
& -\Xi_\Sigma\mathcal{D}_I\Sigma_{JK}\mathcal{D}_I\Sigma_{JK} + 2\,\Xi\mathcal{D}_I\Sigma_{IK}\mathcal{D}_J\Sigma_{JK} - 2\,\Xi_\Sigma\mathcal{D}_I\Sigma_{IJ}\mathcal{D}_J\Sigma_{KK} \\
& +\Xi_\Sigma\mathcal{D}_I\Sigma_{JJ}\mathcal{D}_I\Sigma_{KK} + 2\,\Xi_\Sigma\Sigma_{IK}\Sigma_{JL}\mathcal{R}_{IJ}\mathcal{R}_{IJKL} \\
& -\frac{1}{2}\,\Xi_\Sigma\Sigma_{IM}\Sigma_{JN}\mathcal{H}_{IJKL}\mathcal{H}_{MNKL}
\end{aligned} \tag{14.8}$$

The new definition of the three-form field strength is

$$\begin{aligned}
\mathcal{H}_{\mu\nu M} &= \partial_\mu \mathcal{B}_{\nu M} - \partial_\nu \mathcal{B}_{\mu M} + 3\Lambda_{MNP}\mathcal{V}^N{}_\mu \mathcal{V}^P{}_\nu + 4\Delta^P_{MN}\mathcal{B}_{[\mu P}\mathcal{V}^N{}_{\nu]} + 4\Omega^I_{MN}A^I{}_{[\mu}\mathcal{V}^N{}_{\nu]} \\
&-\frac{1}{2}A^I{}_\mu \mathcal{F}^I{}_{\nu\lambda} - \frac{1}{2}\mathcal{V}^M{}_\mu \mathcal{H}_{\nu\lambda M} - \frac{1}{2}\mathcal{B}_{\mu M}\mathcal{V}^M{}_{\nu\lambda} + \frac{1}{2}\Lambda_{MNP}\mathcal{V}^M{}_\mu \mathcal{V}^N{}_\nu \mathcal{V}^P{}_\lambda \\
&-\Omega^I_{MN}A^I{}_\mu \mathcal{V}^M{}_\nu \mathcal{V}^N{}_\lambda - \Delta^M_{NP}\mathcal{B}_{\mu M}\mathcal{V}^N{}_\nu \mathcal{V}^P{}_\lambda - \mathcal{A}^I{}_M \mathcal{F}^I{}_{\mu\nu} - \mathcal{P}_{MN}\mathcal{V}^N{}_{\mu\nu} \\
&-\frac{1}{2}\mathcal{F}^I{}_{\mu\nu}\mathcal{A}^I{}_M + \frac{1}{2}\mathcal{A}^I{}_\mu \partial_\nu \mathcal{A}^I{}_M + \frac{1}{2}\mathcal{A}^I{}_\mu \Omega^I_{MN} + \frac{1}{2}\mathcal{A}^I{}_P \Delta^P_{MN}
\end{aligned} \tag{14.9}$$

The elegant extension allows that the deformed gauge transformations indeed lead to the required non-abelian gauge transformations in the higher-dimensional construction of heterotic supergravity. The commutator of two deformed gauge transformations of the vector $\mathcal{V}^M$ is given by the equation

$$\begin{aligned}
[\Delta_{\xi_\Xi}, \Delta_{\xi_\Sigma}]\mathcal{V}^M &= \Delta_{\xi_\Xi}(\xi^N_\Sigma \mathcal{D}_N \mathcal{V}^M + (\mathcal{D}^M \xi_{\Sigma N} - \mathcal{D}_N \xi^M_\Sigma)\mathcal{V}^N - \xi^K_\Sigma \Gamma^M{}_{KN}\mathcal{V}^N) \\
&+\widehat{\mathcal{L}}_{\xi_{\Xi\Sigma}}\mathcal{V}^M - \xi^N_{\Xi\Sigma}\Gamma^M{}_{NK}\mathcal{V}^K - (\Xi \leftrightarrow \Sigma) \\
&= [\widehat{\mathcal{L}}_{\xi_\Xi}, \widehat{\mathcal{L}}_{\xi_\Sigma}]\mathcal{V}^M - \xi^N_\Sigma \mathcal{D}_N(\xi^K_\Xi \Gamma^M{}_{KP}\mathcal{V}^P) - (\mathcal{D}^M \xi_{\Sigma N} - \mathcal{D}_N \xi^M_\Sigma)\xi^K_\Xi \Gamma^N{}_{KP}\mathcal{V}^P \\
&-\xi^K_\Sigma \Gamma^M{}_{KN}(\xi^P_\Xi \mathcal{D}_P \mathcal{V}^N + (\mathcal{D}^N \xi_{\Xi P} - \mathcal{D}_P \xi^N_\Xi)\mathcal{V}^P - \xi^P_\Xi \Gamma^N{}_{PQ}\mathcal{V}^Q) \\
&+\widehat{\mathcal{L}}_{\xi_{\Xi\Sigma}}\mathcal{V}^M - \xi^N_{\Xi\Sigma}\Gamma^M{}_{NK}\mathcal{V}^K - (\Xi \leftrightarrow \Sigma)\ .
\end{aligned} \tag{14.10}$$

We build this exclusive section by introducing the modified or deformed gauge transformations. Each $\mathcal{O}(\mathcal{D},\mathcal{D})$ index will give rise to rotation with the structure constants $\Gamma^M{}_{NK}$. The $\mathcal{O}(\mathcal{D},\mathcal{D})$ indices make the transformation properties exhibit and in the general case manifest. For an $\mathcal{O}(\mathcal{D},\mathcal{D})$ vector $\mathcal{V}^M$ and an $\mathcal{O}(\mathcal{D},\mathcal{D})$ elements including in the transformation we get elegant expressions with the solutions. The transformation of $\mathcal{H}^I{}_J$ in involves an operation similar to generalized Lie derivative differs from the conventional Lie derivative by terms that involve explicitly the $\mathcal{O}(\mathcal{D},\mathcal{D})$ metric in heterotic supergravity. For multiple indices the gauge transformation with generalized Lie derivative is defined and constructed as

$$\begin{aligned}
\Delta_\xi \mathcal{H}^I{}_J &= \hat{\mathcal{L}}_\xi \hat{\mathcal{G}}^I{}_J = \xi^K \mathcal{D}_K \mathcal{H}^I{}_J - \mathcal{D}^P \xi^I \mathcal{H}_{PJ} + \Big(\mathcal{D}_J \xi^P - \mathcal{D}^P \xi_I\Big)\mathcal{H}^I{}_P + \xi^K \mathcal{D}_k \mathcal{H}^I{}_J \\
&-\mathcal{D}_K \xi^I \mathcal{H}^K{}_J + \mathcal{D}_J \xi^K \mathcal{H}^I{}_K + \mathcal{D}_J \tilde{\xi}_K \mathcal{H}^{IK} + \mathcal{D}_J \xi^B \mathcal{H}^I{}_B - \mathcal{D}_K \tilde{\xi}_J \mathcal{H}^{IK} + \hat{\mathcal{L}}_\xi \mathcal{H}^I{}_J \\
&+\Big(\mathcal{D}_J \tilde{\xi}_K - \mathcal{D}_K \tilde{\xi}_J\Big)\mathcal{H}^{IK} + \mathcal{D}_J \xi^B \mathcal{H}^I{}_B + \hat{\mathcal{L}}_\xi \Xi_{IJ} + \mathcal{A}_{IB}\mathcal{D}_J \Lambda^B + \frac{1}{2}\mathcal{F}_{IJ}{}^A \Lambda_A \\
&-\frac{1}{2}\Big(\Lambda_A \mathcal{D}_I \Lambda^A - \Lambda_A \mathcal{D}_I \Lambda^A\Big) + \xi^J \mathcal{D}_J \Lambda^A - \xi^J \mathcal{D}_J \Lambda^A + \Gamma^M{}_{NK}\Big(\Pi^P \mathcal{D}_P \Xi^N \Sigma^K\Big) \\
&-\Big(\Xi^P \mathcal{D}_P \Sigma^N - \Sigma^P \mathcal{D}_P \Xi^N\Big)\Pi^K\Big) - \frac{1}{2}\Gamma^N{}_{KL}\Big(\Xi^K \Sigma^L \mathcal{D}^M \Xi^N \Pi_N - \Pi_N \mathcal{D}^M \Big(\Xi^K \Sigma^L\Big) \\
&-\frac{1}{2}\mathcal{D}^M \Gamma_{NKL}\Xi^N \Sigma^K \Pi_L\Big)
\end{aligned} \tag{14.11}$$

In the current literature exist various contributions to the modified gauge transformations of the Γ-dependent terms. The resulting non-covariant terms can be accounted by assigning a fictitious non-covariant variation of the structure constants to

$$\begin{aligned}
\Delta_\xi \Gamma^M{}_{NK} &= -\hat{\mathcal{L}}_\xi \hat{\Gamma}^M{}_{NK} = -\mathcal{D}^M \xi_P \Gamma^P{}_{NK} - \mathcal{D}_N \xi^P \Gamma^M{}_{PK} - \mathcal{D}_K \xi^P \Gamma^M{}_{NP} \\
&-\xi^P \Gamma^N{}_{PK}\Xi^K \mathcal{D}_N \Sigma^M + \frac{1}{2}\xi^P \Gamma^N{}_{PK}\Xi^K \mathcal{D}^M \Sigma_N - \Xi^N \mathcal{D}_N \Big(\xi^P \Gamma^M{}_{PK}\Sigma^K\Big) \\
&+\frac{1}{2}\Xi^N \mathcal{D}^M \Big(\xi^P \Gamma_{NP}{}^K \Sigma_K\Big) - \xi^N \Gamma^M{}_{NK}\Big[\Xi, \Sigma\Big]^K_{\mathrm{C}} - \Big(\widehat{\mathcal{L}}_\xi \Gamma^M{}_{NK}\Big)\Xi^N \Sigma^K \\
&-\xi^N \Gamma^M{}_{NK}\Big[\Xi, \Sigma\Big]^K_\Gamma - \{\Xi \leftrightarrow \Sigma\}
\end{aligned} \tag{14.12}$$

We construct the associated supergravity background-independent action that is T-duality invariant and realizes the gauge superalgebra. The action is the sum of a standard action for supergravity, antisymmetric tensor, and dilaton fields written with ordinary derivatives, a similar action for dual superfields with dual derivatives, and a mixed term that is needed for gauge invariance. Superstring field theory provides the constructions of complete gauge-invariant formulation of superstring dynamics in higher spacetime dimensions around any consistent supergravity curved background, and provides a platform for studying of the special symmetry called T-duality. The higher-dimensional bulk and brane action for exceptional supergravity theory, can be organised and the solution is

$$
\begin{aligned}
S_T = & \int \mathrm{d}^N x\, \mathrm{d}^D Y \sqrt{|\mathcal{G}|}\, \Big(\mathcal{R}_{EXT}(\mathcal{G}) + \mathcal{L}_{KIN} + \mathcal{L}_{KR} + \mathcal{L}_{NS} + \mathcal{L}_{GFC} + \mathcal{L}_{INT} + \sqrt{|\mathcal{G}|}^{-1}\mathcal{L}_{CS}\Big) \\
& + \mathcal{T}_{\mathcal{P}} \int e^{n\phi} G_{\Lambda_1\Xi_1} \mathcal{D}Y^{\Lambda_1} \wedge \mathcal{D}Y^{\Xi_1} \wedge \ldots \wedge G_{\Lambda_n\Xi_n} \mathcal{D}Y^{\Lambda_n} \wedge \mathcal{D}Y^{\Xi_n} \\
& = \int \mathrm{d}^N x\, \mathrm{d}^D Y \sqrt{|\mathcal{G}|}\Big(4\mathcal{H}^{MN}\mathcal{D}_M\mathcal{D}_N\Xi - \mathcal{D}_M\mathcal{D}_N\mathcal{H}^{MN} - 4\,\mathcal{H}^{MN}\mathcal{D}_M\Xi\,\mathcal{D}_N\Xi + 4\mathcal{D}_M\mathcal{H}^{MN}\,\mathcal{D}_N\Xi \\
& + \frac{1}{8}\,\mathcal{H}^{MN}\mathcal{D}_M\mathcal{H}^{KL}\mathcal{D}_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\mathcal{D}_M\mathcal{H}^{KL}\,\mathcal{D}_K\mathcal{H}_{NL} - \frac{1}{2}\Gamma^M{}_{NK}\mathcal{H}^{NP}\mathcal{H}^{KQ}\mathcal{D}_P\mathcal{H}_{QM}\Big) \\
& + \frac{1}{4}\mathcal{G}^{MN}\Big(\partial_\mu\mathcal{B}_{\nu M} - \partial_\nu\mathcal{B}_{\mu M} + 3\Lambda_{MNP}\mathcal{V}^N{}_\mu\mathcal{V}^P{}_\nu + 4\Delta^P_{MN}\mathcal{B}_{[\mu P}\mathcal{V}^N{}_{\nu]} + 4\Omega^I_{MN}\mathcal{A}^I{}_{[\mu}\mathcal{V}^N{}_{\nu]} \\
& - \mathcal{A}^I{}_M\mathcal{F}^I{}_{\mu\nu} - \mathcal{P}_{MP}\mathcal{V}^P{}_{\mu\nu}\Big)\Big(\partial_\mu\mathcal{B}_{\nu M} - \partial_\nu\mathcal{B}_{\mu M} + 3\Lambda_{MNP}\mathcal{V}^N{}_\mu\mathcal{V}^P{}_\nu + 4\Delta^P_{MN}\mathcal{B}_{[\mu P}\mathcal{V}^N{}_{\nu]} \\
& + 4\Omega^I_{MN}\mathcal{A}^I{}_{[\mu}\mathcal{V}^N{}_{\nu]} - \mathcal{A}^I{}_N\mathcal{F}^{I\ \mu\nu} - \mathcal{P}_{NQ}\mathcal{V}^{Q\mu\nu}\Big) + \frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\mathcal{D}_{\mu MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\Big) \\
& + \frac{1}{4}\mathcal{G}^{MN}\Big(\mathcal{B}_{\mu\nu M} - \mathcal{A}^I{}_M\mathcal{F}^I{}_{\mu\nu} - \mathcal{P}_{MP}\mathcal{V}^P{}_{\mu\nu}\Big)\Big(\mathcal{B}^{\mu\nu}{}_N - \mathcal{A}^I{}_N\mathcal{F}^{I\ \mu\nu} - \mathcal{P}_{NQ}\mathcal{V}^{Q\mu\nu}\Big) \\
& + \frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\mathcal{D}_\mu\mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\Big)\Big(\mathcal{D}^\mu B_{PQ} + \mathcal{A}^J{}_{[P}\mathcal{D}^\mu\mathcal{A}^J{}_{Q]}\Big) \\
& + \frac{3}{4}\mathcal{G}^{MQ}\mathcal{G}^{NR}\mathcal{G}^{PS}\Big(\Lambda_{MNP} + 2\mathcal{A}^I{}_{[M}\Omega^I_{NP]} - 2\mathcal{P}_{T[M}\Delta^G_{NP]} \\
& - \frac{1}{4}\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{G}_{MN}\mathcal{G}^{PQ}\mathcal{D}^\mu\mathcal{G}_{PQ} + \frac{1}{4}\mathcal{D}^\mu\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{G}_{MN} - \frac{1}{2}\mathcal{D}_\mu\Big(\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{G}_{MN}\Big) \\
& - \mathcal{G}_{MN}\mathcal{G}^{PQ}\mathcal{G}^{RS}\Delta^M_{PR}\Delta^N_{QS} - 2\mathcal{G}^{MN}\Delta^P_{MQ}\Delta^Q_{NP} + \mathcal{D}^\mu\Phi\mathcal{D}_\mu\Phi + \frac{1}{4}\Big(\mathcal{D}_\mu\mathcal{G}_{MN}\Big)\Big(\mathcal{D}^\mu\mathcal{G}^{MN}\Big) \\
& - \frac{1}{2}\mathcal{G}^{MN}\Big(\mathcal{D}_\mu\mathcal{A}^I{}_M\Big)\Big(\mathcal{D}^\mu\mathcal{A}^I{}_N\Big) - \frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\mathcal{D}_\mu\mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\Big)\Big(\mathcal{D}^\mu\mathcal{B}_{PQ} \\
& + \mathcal{A}^J{}_{[P}\mathcal{D}^\mu\mathcal{A}^J{}_{Q]}\Big) + \frac{1}{4}\Big(\partial_\mu\mathcal{B}_{\nu\lambda} - \frac{1}{2}\mathcal{A}^I{}_\mu\mathcal{F}^I{}_{\nu\lambda} - \frac{1}{2}\mathcal{V}^M{}_\mu\mathcal{H}_{\nu\lambda M} - \frac{1}{2}\mathcal{B}_{\mu M}\mathcal{V}^M{}_{\nu\lambda} \\
& + \frac{1}{2}\Lambda_{MNP}\mathcal{V}^M{}_\mu\mathcal{V}^N{}_\nu\mathcal{V}^P{}_\lambda - \Omega^I_{MN}A^I{}_\mu\mathcal{V}^M{}_\nu\mathcal{V}^N{}_\lambda - \Delta^M_{NP}\mathcal{B}_{\mu M}\mathcal{V}^N{}_\nu\mathcal{V}^P{}_\lambda\Big)\Big(\partial^\mu\mathcal{B}^{\nu\lambda} \\
& - \frac{1}{2}\mathcal{A}^I_\mu\mathcal{F}^I_{\nu\lambda} - \frac{1}{2}\mathcal{V}^{M\mu}\mathcal{H}^{\nu\lambda M} - \frac{1}{2}\mathcal{B}^{\mu M}\mathcal{V}^{M\nu\lambda} + \frac{1}{2}\Lambda_{MNP}\mathcal{V}^{M\mu}\mathcal{V}^{N\nu}\mathcal{V}^{P\lambda} \\
& - \Omega^I_{MN}A^{I\mu}\mathcal{V}^{M\nu}\mathcal{V}^{N\lambda} - \Delta^M_{NP}\mathcal{B}^{\mu M}\mathcal{V}^{N\nu}\mathcal{V}^{P\lambda}\Big) + \frac{1}{2}\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{A}^I{}_M\mathcal{D}^\mu\mathcal{A}^I{}_N \\
& + \mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\Omega^I_{MN} + \mathcal{A}^I{}_R\Delta^R_{MN}\Big)\Big(\Omega^I_{PQ} + \mathcal{A}^I{}_S\Delta^S_{PQ}\Big) + \Big(\xi^K\mathcal{D}_K\mathcal{H}^I{}_J \\
& - \mathcal{D}^P\xi^I\,\mathcal{H}_{PJ} + \Big(\mathcal{D}_J\xi^P - \mathcal{D}^P\xi_I\Big)\mathcal{H}^I{}_P + \xi^K\mathcal{D}_k\mathcal{H}^I{}_J - \mathcal{D}_K\xi^I\,\mathcal{H}^K{}_J + \mathcal{D}_J\xi^K\,\mathcal{H}^I{}_K \\
& + \mathcal{D}_J\tilde{\xi}_K\,\mathcal{H}^{IK} + \mathcal{D}_J\xi^B\,\mathcal{H}^I{}_B - \mathcal{D}_K\tilde{\xi}_J\,\mathcal{H}^{IK} + \hat{\mathcal{L}}_\xi\,\mathcal{H}^I{}_J + \Big(\mathcal{D}_J\tilde{\xi}_K - \mathcal{D}_K\tilde{\xi}_J\Big)\mathcal{H}^{IK} \\
& + \mathcal{T}_{\mathcal{P}} \int e^{n\phi} G_{\Lambda_1\Xi_1} \mathcal{D}Y^{\Lambda_1} \wedge \mathcal{D}Y^{\Xi_1} \wedge \ldots \wedge G_{\Lambda_n\Xi_n} \mathcal{D}Y^{\Lambda_n} \wedge \mathcal{D}Y^{\Xi_n}
\end{aligned}
$$

We now rewrite the higher-dimensional supergravity action in a form that is covariant under $\mathcal{O}(\mathcal{D}, \mathcal{D}+\mathcal{K})$, where $\mathcal{K}$ is the dimension of the gauge algebra in the construction. This extremal action is of the same structural form as that obtained by Scherk-Schwarz compactification of heterotic supergravity truncated to the Cartan subalgebra. Moreover, it is closely related to that given in the scientific literature, which considers group manifold reductions of heterotic supergravity including non-abelian gauge fields and also displays the action with a formal $\mathcal{O}(\mathcal{D}, \mathcal{D}+\mathcal{K})$ symmetry. These are the situations we have in mind, and we will simply speak of $\mathcal{O}(\mathcal{D}, \mathcal{D}+\mathcal{K})$ as the duality group. We finally note that the action reduces to that found by Maharana-Schwarz, in which case the theory is properly invariant under a global $\mathcal{O}(\mathcal{D}, \mathcal{D}+\mathcal{K})$ symmetry.

We apply our universal recipe of the preceding section to write the supergravity corresponding action based on the moduli superspace construction. For the exceptional case of Membrane Newton-Cartan fundamental system the appropriate higher-dimensional theory includes special bulk action $\widetilde{\mathcal{L}}_{\mathcal{B}}(\Sigma_{\hat{\Delta}})$, the brane $\widetilde{\mathcal{L}}_{\mathcal{BR}}(\Sigma_{\hat{\Delta}})$ and hidden brane lagrangian $\widetilde{\mathcal{L}}_{\mathcal{HBR}}(\Sigma_{\hat{\Delta}})$, the brane fields coupling action $\widetilde{\mathcal{L}}_{\mathcal{BFC}}(\Sigma_{\hat{\Delta}})$ and hidden brane couplings term $\widetilde{\mathcal{L}}_{\mathcal{HBC}}(\Sigma_{\hat{\Delta}})$. The extremal solution of the higher-dimensional corresponding action of exceptional supergravity for the Membrane Newton-Cartan fundamental system is

$$\begin{aligned}
S_{\mathcal{MNC}} &= \hat{\Delta}\int_{\Sigma_{\hat{\Delta}}} d^D x\sqrt{\mathcal{G}_{\hat{\Delta}}}\,\mathcal{R}_{\hat{\Delta}}\,\widetilde{\mathcal{L}}_{\mathcal{MNC}}(\Sigma_{\hat{\Delta}}) + \sum_{\hat{\Delta}}\Big\{\int_{\Sigma_{\hat{\Delta}}} d^D x\sqrt{-\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{\mathcal{B}}(\Sigma_{\hat{\Delta}}) + \int_{\Sigma_{\hat{\Delta}}} d^D x\sqrt{\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{\mathcal{BR}}(\Sigma_{\hat{\Delta}}) \\
&+ \int_{\Sigma_{\hat{\Delta}}} d^D x\sqrt{\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{\mathcal{HBR}}(\Sigma_{\hat{\Delta}}) + \int_{\Sigma_{\hat{\Delta}}} d^D x\sqrt{-\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{\mathcal{BFC}}(\Sigma_{\hat{\Delta}}) + \int_{\Sigma_{\hat{\Delta}}} d^D x\sqrt{\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{\mathcal{HBC}}(\Sigma_{\hat{\Delta}})\Big\} \\
&+ \hat{\Delta}\int_{\Sigma_{\hat{\Delta}}} d^{D+1}x\,\sqrt{-\mathcal{G}}\,\mathcal{R}_{\hat{\Delta}}\Big\{\mathcal{T}_{\Sigma} + \frac{1}{2}\,\mathcal{X}_{\Sigma}\,\mathcal{D}_M\Sigma\,\mathcal{D}^M\Sigma + \mathcal{D}_M\mathcal{X}^M\Xi^A_M(\mathcal{X})\mathcal{D}_N\mathcal{X}^N\Xi^C_N(\mathcal{X})\,\mathcal{B}_{AC}(\mathcal{Z}) \\
&+ \frac{1}{2}\,\mathcal{U}_{\Sigma}\,\mathcal{D}_M\Sigma\,\mathcal{D}^M\Sigma + \mathcal{D}_M\,\mathcal{U}^M\Xi^A_M(\mathcal{U})\mathcal{D}_N\,\mathcal{U}^N\Xi^C_N(\mathcal{U})\,\mathcal{D}_{AC}(\mathcal{B}) + \frac{1}{2}\,\mathcal{Z}_{\Sigma}\,\mathcal{D}_M\Sigma\,\mathcal{D}^M\Sigma \\
&+ \mathcal{D}_M\mathcal{Z}^M\Xi^A_M(\mathcal{Z})\mathcal{D}_N\mathcal{Z}^N\Xi^C_N(\mathcal{Z})\,\mathcal{P}_{AC}(\mathcal{Z}) + \cdots\Big\}
\end{aligned} \tag{14.13}$$

We construct a fully consistent and gauge invariant actions in higher-dimensional exceptional supergravity with presence of backgrounds, superstrings and membrane interpretations in D- dimensional spacetime supermanifolds realized in the theoretical framework. We discuss and surrendered the challenges involved in the advanced construction of the full higher-dimensional supergravities in modern and constructive fashion. Our main results are both of purely fundamental and mathematical interest and lead, from the physical point of view, to the construction of new realistic superstring theories in supergravity backgrounds. We performed dimensional reduction of the higher-dimensional effective actions and displayed the expected global symmetry on the reduced theory of exceptional supergravity. Nowadays, searching for superstrings in supergravity backgrounds directly related to fundamental supermembranes has become a dogma for the theoretical physicists involved. The future of modern theoretical and mathematical physics is dependent on the creation of higher-dimensional models in the theoretical framework used in theories such as supergravity, superstrings and supersymmetric membranes. Based on the methods developed in this advanced research, an alternative to the dimensional reduction procedure has been presented in exceptional supergravities in D-dimensional spacetime supermanifolds with availability of curved backgrounds and a huge number of superfields in the presented fundamental interactions. We have provided the general technical tools for the computation of higher-dimensional heterotic supergravity theories with inclusion of supermanifolds, superstrings, backgrounds plus fundamental bulk and brane systems. The main objectives of the current research in supergravity theories are associated with the creation of a unified theoretical framework to explain and improving the current state of knowledge regarding deep understanding of our elegantly designed world.

15 Conclusion

Beyond the intrinsic interest in obtaining a higher-dimensional perspective on duality, ExFT is a very powerful tool in understanding the geometry of superstring and M-theory backgrounds, both with a view towards reductions – where it offers a way to efficiently characterise the properties of geometries with flux, and leads to new methods to obtain consistent truncations to gauged supergravities in lower-dimensions and towards expansions as it can be employed to obtain complicated higher-derivative corrections in an efficient manner. Underlying these successes is the fact that the geometry of ExFT treats the metric and form-fields of supergravity on the same footing, rearranging all degrees of freedom into multiplets of $E_{d(d)}$ in a form which is perfectly adapted to general dimensional reductions but completely general so that it works regardless of background. Exceptional field theory is by now a well-developed field with numerous interesting applications and outcrops. The selection of topics in this review is of course based by the author own interests and ignorances and further by the limitations of space, for all of which we ask for the understanding and patience of the reader. We hope to be able to describe the general concepts and technical tools needed to understand and make of ExFT. From a philosophical point of view, one might then wonder about the nature of geometry in superstring and M-theory. Is the standard Riemannian geometry that physicists have lived in since Einstein the most convenient language to capture the features of the backgrounds of superstring theory and M-theory? Is there a better organisational principle that takes into account the menagerie of p-form gauge fields and the branes to which they couple? In this review, we will try to answer these questions using exceptional field theory. In exceptional field theory (ExFT), an $E_{d(d)}$ symmetry is manifest acting on an extended or generalised geometry. Depending on how one chooses to identify the physical geometry with the extended geometry of ExFT, for each d, the $E_{d(d)}$ ExFT is equivalent to the full 11- or 10-dimensional maximal supergravities. It therefore provides a higher-dimensional origin of U-duality, in which no reduction is assumed, and on identifying the novel coordinates of the extended geometry as conjugate to brane winding modes, ExFT offers a glimpse towards the geometry of M-theory beyond supergravity.

Comparison with the Gomis-Ooguri or SNC superstring The extremal behaviour we found in eleven-dimensional supergravity can be seen to be extremely similar to that which happens on the worldsheet for the Gomis-Ooguri or SNC string. This is exactly analogous to the result of the expansion of the 11-dimensional supergravity action. Here the Wess-Zumino coupling to the B-field plays the role of the Chern-Simons term, and the singular piece can be cancelled by imposing a sort of twisted self-duality constraint. Normally one derives the finite part of the supergravity action by rewriting the action in an equivalent form using auxiliary degrees of freedom, such that the limit can be performed without singularities. After the limit, one finds these auxiliary degrees of freedom correspond to $\widetilde{F}_{\alpha}{}^{\mathsf{A}}$, and impose the chirality/anti-chirality conditions on the longitudinal degrees of freedom. This is also what happens in the doubled sigma model approach, which starts with coordinates X and duals $\widetilde{X}$, related by a self-duality constraint involving the generalised metric of double field theory. Taking the SNC limit in this set-up then leads to the situation as above where the longitudinal X and $\widetilde{X}$ are no longer related, but separately obey chiral/anti-chirality constraints. The doubled sigma model action then reproduces the finite terms. This then is analogous to the exceptional field theory description of the limit of 11-dimensional supergravity.

It could be conjectured that the appearance of (self)-duality constraints is a generic feature of non-relativistic limits of theories with topological or Chern-Simons terms, as a requirement for cancelling singular terms arising from the topological term against those arising from the kinetic term. Schematically given a Lagrangian $\mathcal{L} \sim F \wedge \star F + F \wedge G$ with a non-relativistic expansion leading to a term $c^n F \star (\star F + G)$, then we would take $\star F + G = 0$ as a constraint. It would be interesting to explore this mechanism in other contexts.

Subleading terms Our derivation of the MNC geometry made use of a field redefinition involving the parameter c which we then sought to send to ∞ and interpret as a non-relativistic limit. This could be extended to a full non-relativistic expansion, including first of all further subleading terms in the metric, with $\hat{g}_{\mu\nu} = c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu} + c^{-4}X_{\mu\nu} + \dots$. It is possible to check that doing so does not affect the expansion of the action up to order c^0, and it would be expected on general grounds [58] that the first appearance of the first subleading terms simply re-imposes the equations of motion already encountered (as we saw with $\tilde{C}_3$ and the equations of motion of C_3). In addition, we could reformulate the expansion by introducing additional one-form gauge fields (as for this case in [48]), accompanied by a shift symmetry, such that the three-form $C_{\mu\nu\rho}$ does not transform under boosts. The resulting more general expansion could then be attacked order-by-order without necessarily sending $c \to \infty$ or truncating as we did. Here it would be interesting to compare with the approach of [53], inputting the eleven-dimensional three-form as matter. A complicating feature, relative to usual $1/c$ expansions of general relativity leading to Newton-Cartan [51, 52, 58] for example, is that the longitudinal vielbein appears in both the metric and three-form and does so at different orders in c.

Supersymmetry and non-uniqueness of non-relativistic 11-dimensional supergravity We limited ourselves to an analysis of the bosonic geometry in this paper. The supersymmetric extension presumably exists and should be constructed. At the level of supersymmetric double and exceptional field theory, the logic would again be that changing the parametrisation of the generalised vielbein is all that is needed to arrive at the desired theory, and this seems to be possible without obstacles [57]. Note that in this paper we started with a non-relativistic expansion tailored to the M2. There should be a similar expansion based on the M5, in which we have six longitudinal and five transverse directions. (This should reduce to the dual NSNS six-form expansion discussed in the conclusions of [27].) This would then give a *second* non-relativistic version of 11-dimensional supergravity, so although this is the unique maximal supergravity in eleven dimensions, this uniqueness would then no longer hold in the non-relativistic setting.

Duality web and branes An obvious goal for which this paper should be useful is the study of the spacetime actions for the non-relativistic duality web in 11- and 10-dimensions. This can proceed both by applying standard dimensional reduction and dualisation to our 11-dimensional action, and by applying similar methods to individual supergravities by taking covariant non-relativistic limits associated to each p-brane present in the theory. Here, we performed a dimensional reduction to type IIA, but we did not discuss the expected T-duality relationship to type IIB, for example. Similarly, there is presumably a heterotic SNC which could be obtained by reducing non-relativistic M-theory on a longitudinal interval, although it is not immediately obvious what the result of reducing on a transverse interval should be. Note that the appearance of the original and dual field strength together in the 11-dimensional theory suggests that the appropriate formalism for describing generalisations of Newton-Cartan geometries in type II should be the formalism where the RR p-forms are treated 'democratically' [59], accompanied by a self-duality constraint. Here the double and exceptional field theory formulation may again prove a useful guide. Beyond the usual suspects, exceptional field theory also offers a way to handle the vast number of mixed symmetry tensors that appear coupling to exotic branes [60, 61]. It may not be unreasonable to suggest using the E_{11} 'master' ExFT recently constructed in [55], as this presumably provides scope to construct an infinite number of brane scaling limits. Here there is no need to artificially split the coordinates and one can work with 11/10-dimensional quantities throughout, albeit at the obvious price of dealing with a very infinite algebra. The ExFT description in this paper demonstrates that the non-relativistic theory is also controlled by the same exceptional Lie algebraic symmetries that appear in the relativistic case. A distinction can be made between these symmetries as they are used in ExFT and the actual U-duality symmetries present on toroidal reduction. As we saw in section (9.2), U-duality transformations can 'rotate' between relativistic and non-relativistic theories.

This is also the case for T-duality of non-relativistic strings [16]. A non-trivial U-duality, corresponding to an SL(2) inversion transformation in the SL(3) $\times$ SL(2) case, acts on three directions in spacetime. To make a systematic study of U-duality of non-relativistic theories, it would therefore be necessary to consider U-duality transformations acting on 0,1, 2 or 3 longitudinal directions and to check which of these do or do not take you back to a relativistic theory. The SL(3) $\times$ SL(2) ExFT description of section (9.2) only allowed for U-duality transformations acting on all three longitudinal directions, while the SL(5) ExFT description presented in appendix 11 would allow for transformations acting on two or three longitudinal directions. A precise group to consider would then be the $E_{6(6)}$ case which can accommodate all possible types of U-dualities acting on the MNC geometry, with some subgroup corresponding to the strict U-duality symmetries of the non-relativistic theory. This analysis is left for future work. Another interesting question is to understand the consequences of the non-relativistic limit on the brane spectrum of M-theory and hence also of type IIA, after reducing. The 'decoupling' of the transverse components of F_4 and the longitudinal components F_7 presumably means something at the level of the M2 and M5 branes coupling to the three- and six-form: the analysis of [62] should be pertinent here. One could similarly enquire about whether the duality constraint in the type IIA SNC theory can be seen at the level of the string spectrum resulting from the quantisation of the non-relativistic superstring [63] Obtaining brane solutions of the non-relativistic theory, whether by directly solving the equations of motion or using U-duality as in section 9.2, is also an interesting question. Interestingly, membrane solutions of 11-dimensional SUGRA with transverse self-dual field strength were constructed in the research literature and perhaps can be adapted or used in the non-relativistic setting. Even the 'flat' spacetime solution may have interesting properties including infinite-dimensional isometries as for the superstring case. S-duality and T-duality transformations in the general case do not commute. Combining them, we generate a larger group of dualities of the type II theories. This is known as U-duality. It is a non-perturbative duality of the type II superstrings on a torus, and hence also a duality of M-theory. The latter can be motivated by considering the strong coupling limit of the type IIA superstring. As the IIA superstring coupling goes to infinity, an eleventh dimension decompactifies, and we are led to conjecture the existence of an 11-dimensional M-theory, which when compactified on a circle reduces to the IIA superstring in the zero radius limit. The 11-dimensional radius R_{11} and Planck length l_p are related to the 10-dimensional string coupling constant g_s and string length The action of U-duality on the backgrounds of type IIA superstring theory follow on applying the reduction rules **??**. Then we can further T-dualise to identify the corresponding transformations in type IIB. In particular, the geometric SL(2) appearing when $d = 2$, for M-theory on a two-torus, becomes the S-duality of type IIB. For M-theory on a three-torus, the type IIB S-duality is likewise embedded in the SL(3) factor of the full U-duality group. Acting in more than three directions, there are further shift symmetries possible. The U-duality group acting on a d-dimensional torus in M-theory is then determined to be $E_{d(d)}(\mathbb{Z})$. This sequence of U-duality symmetries was first found in the context of reductions of eleven-dimensional supergravity on a torus. They are the global symmetries of maximal supergravity in n dimensions. As mentioned before, in terms of the supergravity action, these global symmetry groups are real-valued. This is an important feature of duality: the reduced supergravity will have a moduli space and there will be a continuous set of symmetries acting on the moduli that take one vacuum into another inequivalent vacuum. However, an arithmetic subgroup will leave the reduction space invariant and this will coincide with the duality group when taking into account quantum charge preservation. Exceptional field theory is by now a well-developed field with numerous interesting applications and outcrops. The selection of topics in this review is of course based by the author own interests and ignorances and further by the limitations of space, for all of which we ask for the understanding and patience of the reader. We hope to be able to describe the general concepts and technical tools needed to understand and make of ExFT with supergravity and membrane constructions. After developing the general theory, we will discuss some of the applications mentioned above and refer to the literature for further details when necessary.

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