Peacocks and the Zeta distributions

Imad El ghazi

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Abstract

We prove in this short paper that the stochastic process defined by:

\[ Y_t := \frac{X_{t+1}}{\mathbb{E}[X_{t+1}]}, \quad t \geq a > 1, \]

is an increasing process for the convex order, where \( X_t \) a random variable taking values in \( \mathbb{N} \) with probability \( \mathbb{P}(X_t = n) = \frac{n - t}{\zeta(t)} \) and
\[ \zeta(t) = \sum_{k=1}^{+\infty} \frac{1}{k^t}, \quad \forall t > 1. \]

1 Introduction

The notion of increasing process for the convex order, (PCOC, acronym of the french expression, Processus Croissant pour l’Ordre Convexe) has been deeply studied in \[2\]. This type of stochastic processes is quiet interesting in the financial options markets.

The main example of PCOC was introduced by Carr, Ewald and Xiao in \[1\]. Let \((B_s, s \geq 0)\) be a Brownian motion started from 0 and \((N_s := \exp^{B_s - \frac{s}{2}}, s \geq 0)\) then,

\[ X_t := \frac{1}{t} \int_0^t N_s ds, \quad t \geq 0 \]

is a PCOC.

The other attractive property satisfied by the PCOCs is siultristied by the Kellerer Theorem \[3\] establishing the relationship with the martingales theory.

2 Peacocks and 1-martingales
Definition 2.1. A process \((X_t, t \geq 0)\) is a peacock if the following conditions are verified:

i) \(|X_t|\) is integrable, i.e., for every \(t \geq 0\), \(\mathbb{E}[|X_t|] < \infty\).

ii) For every convex \(C^2\)-function \(\Psi : \mathbb{R} \to \mathbb{R}\), such that \(\Psi''\) has a compact support, the function \(\mathbb{E}[\Psi(X_t)]\) is increasing with respect to \(t\).

Proposition 2.1. (Proposition 1.3 [2])
Let \((X_t, t \geq 0)\) be a real valued process satisfying the following hypotheses:

i) the process \((X_t, t \geq 0)\) is a.s. continuous on \([0, +\infty[\) and differentiable on \(]0, +\infty[\), its derivative being denoted by \(\frac{\partial X_t}{\partial t}\).

ii) for every \(a > 0\),
\[
\mathbb{E}\left[\sup_{t \in [0,a]} |X_t|\right] < \infty
\]
and for every \(0 < a < b\),
\[
\mathbb{E}\left[\sup_{t \in [a,b]} \left|\frac{\partial X_t}{\partial t}\right|\right] < \infty.
\]

Then, the process \((X_t, t \geq 0)\) is a peacock if and only if the two following properties hold:

a) \(\mathbb{E}[X_t]\) does not depend on \(t \geq 0\),

b) for every real \(c\) and \(t > 0\):
\[
\mathbb{E}\left[1_{\{X_t \geq c\}} \frac{\partial X_t}{\partial t}\right] \geq 0.
\]

Definition 2.2. A process \((X_t, t \geq 0)\) is a 1-martingale if there exists a martingale \((M_t, t \geq 0)\), not necessarily defined on the same probability space, such that for every fixed \(t \geq 0\):

\[
X_t \overset{\text{law}}{=} M_t
\]

Theorem 2.1. (H.G. Kellerer [3]). The following properties are equivalent:

1) \((X_t, t \geq 0)\) is a peacock.

2) \((X_t, t \geq 0)\) is a 1-martingale.
3 Peacocks and the Zeta laws

Let \((N, \mathcal{P}(N), \mathbb{P}_t)\) a probability space, such that \(\mathbb{P}_t\) is the Zeta probability law of parameter \(t > 1\) the law on \(\mathbb{N}^*\) wich assigns the mass \(\frac{n^{-t}}{\zeta(t)}\) to the point \(n\), i.e, \(\mathbb{P}_t(x = n) = \frac{n^{-t}}{\zeta(t)}\) where

\[
\zeta(t) = \sum_{k=1}^{+\infty} \frac{1}{k^t}
\]

is the Riemann Zeta function.

Let suppose the hypothetical experience consiting of picking a number \(n \in \mathbb{N}^*\) at each instant \(t\) (supposed to be strictly superior to 1), with probability \(\frac{n^{-t}}{\zeta(t)}\). The resulting process will be denoted \((X_t)_{t>1}\).

**Remarque 3.1.** The results of the experience are supposed to be independant, this implies that the resulting process \((X_t)_{t>1}\) is not a martingale.

**Theorem 3.1.** Let \((Y_t, t \geq a), a > 1\), be the process defined by:

\[
Y_t := \frac{X_{t+1}}{\mathbb{E}[X_{t+1}]}
\]

such that, \(\mathbb{P}_t(X_t = n) = \frac{n^{-t}}{\zeta(t)}, n \in \mathbb{N}^*\) and \(\zeta(t) = \sum_{k=1}^{+\infty} \frac{1}{k^t}\) for every \(t > 1\). Then \((Y_t, t \geq a)\) is a peacock.

**Proof.** We will prove that \((Y_t, t \geq a), a > 1\), verifies the above Proposition.

Remark first that for every \(t \geq a\) one has \(\mathbb{E}[Y_t] = 1\) which means that \(\mathbb{E}[Y_t]\) does not depend on \(t\).

Recall that \(t \rightarrow n^{-t}\) and \(t \rightarrow \zeta(t)\) are \(C^\infty\)-continuous functions and that \(\frac{1}{\zeta(t)}\) is well defined on \([a, +\infty[\) for every \(a > 1\).

The continuity of \((Y_t, t \geq 1)\) follows from the Colmogorov criterion:

\[
|Y_t - Y_s| = \left| \frac{X_{t+1}}{\mathbb{E}[X_{t+1}]} - \frac{X_{s+1}}{\mathbb{E}[X_{s+1}]} \right|
\]

\[
|Y_t - Y_s| \leq \max(X_{t+1}, X_{s+1}) \left| \frac{1}{\mathbb{E}[X_{t+1}]} - \frac{1}{\mathbb{E}[X_{s+1}]} \right|
\]

\[
|Y_t - Y_s| \leq \max(X_{t+1}, X_{s+1}) \left| \frac{\zeta(t+1)}{\zeta(t)} - \frac{\zeta(s+1)}{\zeta(s)} \right|
\]
since $\frac{\zeta(t+1)}{\zeta(t)}$ is $C^\infty$ then it is Lipschitzen and hence there exists $K_1 > 0$ such that,

$$\left| \frac{\zeta(t+1)}{\zeta(t)} - \frac{\zeta(s+1)}{\zeta(s)} \right| \leq K_1 |t - s|$$

let’s choose $0 < \gamma < a - 1$ then $t - \gamma > 1$ and $s - \gamma > 1$ and we have,

$$|Y_t - Y_s|^{1+\gamma} \leq \max(X_{t+1}, X_{s+1})^{1+\gamma}K_2|t - s|^{1+\gamma}$$

and

$$E[|Y_t - Y_s|^{1+\gamma}] \leq E(\max(X_{t+1}, X_{s+1})^{1+\gamma})K_2|t - s|^{1+\gamma}$$

$$E(|Y_t - Y_s|^{1+\gamma}) \leq \left( \frac{n^{1+\gamma}n^{-t-1}}{\zeta(t+1)} + \frac{m^{1+\gamma}m^{-s-1}}{\zeta(s+1)} \right) K_2|t - s|^{1+\gamma}$$

$$E(|Y_t - Y_s|^{1+\gamma}) \leq 2K_2|t - s|^{1+\gamma}$$

because $\frac{n^{1+\gamma}n^{-t-1}}{\zeta(t+1)} < 1$.

For the differenciability of $(Y_t, t \geq 1)$ we use again the Kolmogorov Criterion. To do that we define $(\frac{\partial Y_t}{\partial t}, t \geq a)$ by,

$$\frac{\partial Y_t}{\partial t} = \lim_{\partial t \to 0} \frac{Y_{t+\partial t} - Y_t}{\partial t}$$

$$|Y'_t - Y'_s| \leq \lim_{\partial t \to 0} \frac{1}{\partial t} |Y_{t+\partial t} - Y_t - Y_{s+\partial t} + Y_s|$$

(1)

we denote $\phi(t) = (\frac{\zeta(t+1)}{\zeta(t)})'$ and hence

$$|Y'_t - Y'_s| \leq \max(X_{t+1}, X_{s+1})|\phi(t) - \phi(s)|$$

$$|Y'_t - Y'_s| \leq \max(X_{t+1}, X_{s+1})K_3|t - s|.$$
and,
\[ E(|Y'_t - Y'_s|^{1+\gamma}) \leq E(\max(X_{t+1}, X_{s+1})^{1+\gamma})K_4|t-s|^{1+\gamma} \]
\[ E(|Y'_t - Y'_s|^{1+\gamma}) \leq \left( \frac{n^{1+\gamma}n-t-1}{\zeta(t+1)} + \frac{m^{1+\gamma}m-s-1}{\zeta(s+1)} \right)K_4|t-s|^{1+\gamma} \]

because \( \frac{n^{1+\gamma}n-t-1}{\zeta(t+1)} < 1 \).

Let \( c = \sup_{t \in [a,b]} (Y_t) \), then,
\[ c = \frac{m}{E(X_{t_0+1})} \quad \text{for some } t_0 \in [a,b] \]
it comes that,
\[ E(\sup_{t \in [a,b]} (Y_t)) = \frac{m \times m-t_0-1}{\zeta(t_0)} \leq \frac{m^{-a}}{\zeta(t_0)} < +\infty \]
because \( \lim_{m \to +\infty} E(X_t = m) = 0 \)

For \( E(\sup_{t \in [a,b]} |\frac{\partial Y_t}{\partial t}|) \) we have,
\[ \sup_{t \in [a,b]} \left| \lim_{\partial t \to 0} \frac{\partial Y_t}{\partial t} \right| = \sup_{t \in [a,b]} \left| \lim_{\partial t \to 0} \frac{Y_{t+\partial t} - Y_t}{\partial t} \right| \]
\[ \sup_{t \in [a,b]} \left| \lim_{\partial t \to 0} \frac{\partial Y_t}{\partial t} \right| \leq \sup_{t \in [a,b]} \lim_{\partial t \to 0} \left| \frac{Y_{t+\partial t} - Y_t}{\partial t} \right| \]
\[ \sup_{t \in [a,b]} \left| \lim_{\partial t \to 0} \frac{\partial Y_t}{\partial t} \right| \leq \sup(X_{t+1}) \phi(t) \]
\[ \sup_{t \in [a,b]} \left| \lim_{\partial t \to 0} \frac{\partial Y_t}{\partial t} \right| \leq \sup(X_{t+1}) \max_{t \in [a,b]} \phi(t) \]
\[ \sup_{t \in [a,b]} \left| \lim_{\partial t \to 0} \frac{\partial Y_t}{\partial t} \right| \leq K_5 \sup_{t \in [a,b]} (X_{t+1}) \]

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\[ \mathbb{E} \left( \sup_{t \in [a,b]} \left| \frac{\partial Y_t}{\partial t} \right| \right) \leq K_5 \frac{m^{-t_0}}{\zeta(t_0 + 1)} < +\infty \]

because

\[ \lim_{m \to +\infty} \mathbb{E}(X_t = m) = 0 \]

and therefor,

\[ \mathbb{E} \left( \sup_{t \in [a,b]} \left| \lim_{\partial t \to 0} \frac{\partial Y_t}{\partial t} \right| \right) \leq K_5 \mathbb{E} \left( \sup_{t \in [a,b]} \left( X_{t+1} \right) \right) < +\infty. \]

Finally since \( Y_t > 0 \) then for every \( c > 0 \) we have,

\[ \mathbb{E} \left( \frac{\partial Y_t}{\partial t} 1_{Y_t \geq c} \right) = \mathbb{E} \left( \lim_{\partial t \to 0} \frac{1}{\partial t} Y_t(\partial t) - Y_t 1_{Y_t \geq c} \right) \]

\[ \mathbb{E} \left( \frac{\partial Y_t}{\partial t} 1_{Y_t \geq c} \right) = \lim_{\partial t \to 0} \frac{1}{\partial t} \left( 1 - \sum_{n \geq c} ^{\infty} n^{-t} \mathbb{E}(X_{t+1}) \zeta(t+1) \right) \geq 0 \]

for every \( c \in \mathbb{R} \) and every \( t \geq a > 1 \).

\[ \square \]

**Corollary 3.1.** The process \((Y_t, t \geq a), a > 1\) is a 1-martingale, i.e., there exists a martingale \((M_t, t \geq a), a > 1\), not necessarily defined on the same probability space, such that for every fixed \( t \):

\[ Y_t \overset{\text{law}}{=} M_t \]

**Proof.** According to the above Theorem and the Kereller Theorem.

\[ \square \]

**References**
