Another Values of the Barnes Function and Formulas

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Abstract

In the continuity of my precedent paper Values of Barnes Function (1), this time I talk about for the Barnes function at unusual points as \( G(1/8) \), \( G(3/8) \) or \( G(5/12) \) for example.

In the same time, I give several formulas and so we can evaluate easily elementary values of Barnes function. I give eight conjectural integral formulas and we see several applications, in particular Wallis product.

1 Definition

The Barnes function is defined as the following Weierstrass product:

\[
G(1 + z) = (2\pi)^{z/2} \exp\left(-z(1 + z)\right) \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{k} \exp\left(-z + \frac{z^2}{2}\right)
\]  

where gamma is the Euler-Mascheroni constant.

The following properties of \( G \) are well-known.

2 Properties

\[
G(1) = 1 \tag{3}
\]
\[
G(1 + z) = G(z) \Gamma(z) \tag{4}
\]
\[
\log\left(G(1 + z)\right) = \frac{z \log(2\pi)}{2} - \frac{z(1 + z)}{2} + z \log(\Gamma(1 + z)) - \int_{0}^{z} \log(\Gamma(t + 1)) \, dt \tag{5}
\]
\[
\int_{0}^{z} \log(\Gamma(t + 1)) \, dt = \frac{z \log(2\pi)}{2} - \frac{z(1 + z)}{2} + z \log(\Gamma(1 + z)) - \log(G(z)) - \log(\Gamma(z)) \tag{6}
\]
3 List of Formulas

Let \( A \) be the Glaisher–Kinkelin’s constant (7), \( K \) be the Catalan’s constant (8) and \( \Psi \left(1, \frac{1}{3}\right) \) is the trigamma function at 1/3 (9).

If \( t \) is positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) - \left[\sum_{v=2}^{t} [- \log \left(-1 + v\right) + \log \left(-1 + v\right)]\right]
\]

If \( t = k + \frac{1}{2} \) where \( k \) is 0 or positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) + \frac{1}{8} - \frac{3 \log(A)}{2} + \frac{\log(2)}{24}
\]

If \( t = k + \frac{1}{4} \) where \( k \) is 0 or positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) + \frac{3}{32} - \frac{9 \log(A)}{8} - \frac{K}{4\pi}
\]

If \( t = k + \frac{3}{4} \) where \( k \) is 0 or positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) + \frac{9}{32} - \frac{9 \log(A)}{8} + \frac{K}{4\pi}
\]

If \( t = k + \frac{1}{3} \) where \( k \) is 0 or positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) + \frac{\pi\sqrt{3}}{36} - \frac{\sqrt{3}\Psi \left(1, \frac{1}{3}\right)}{24\pi} + \frac{1}{9} - \frac{4 \log(A)}{3} + \frac{\log(3)}{72} - \left[\sum_{v=1}^{t} \left(v - \frac{2}{3}\right) \log \left(v - \frac{2}{3}\right)\right]
\]

If \( t = k + \frac{2}{3} \) where \( k \) is 0 or positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) - \frac{\pi\sqrt{3}}{36} + \frac{\sqrt{3}\Psi \left(1, \frac{1}{3}\right)}{24\pi} + \frac{1}{9} - \frac{4 \log(A)}{3} + \frac{\log(3)}{72} - \left[\sum_{v=1}^{t} \left(v - \frac{1}{3}\right) \log \left(v - \frac{1}{3}\right)\right]
\]

If \( t = k + \frac{1}{6} \) where \( k \) is 0 or positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) + \frac{\pi\sqrt{3}}{36} - \frac{\sqrt{3}\Psi \left(1, \frac{1}{3}\right)}{24\pi} + \frac{5}{72} - \frac{5 \log(A)}{6} - \frac{\log(2)}{72} - \frac{\log(3)}{144} - \left[\sum_{v=1}^{t} \left(v - \frac{5}{6}\right) \log \left(v - \frac{5}{6}\right)\right]
\]

If \( t = k + \frac{5}{6} \) where \( k \) is 0 or positive integer then
\[
\log \left(G\left(t\right)\right) = (t - 1) \log \left(\Gamma \left(t\right)\right) - \frac{\pi\sqrt{3}}{36} + \frac{\sqrt{3}\Psi \left(1, \frac{1}{3}\right)}{24\pi} + \frac{5}{72} - \frac{5 \log(A)}{6} - \frac{\log(2)}{72} - \frac{\log(3)}{144} - \left[\sum_{v=1}^{t} \left(v - \frac{1}{6}\right) \log \left(v - \frac{1}{6}\right)\right]
\]
4 Hypothesis around the integral \( \int_0^t \log(\Gamma(t+1)) \, dt \) and the sum \( \sum_{k=1}^{\infty} \frac{(1-\zeta(2k+1))q^{2k+2}}{(k+1)(2k+1)} \)

We consider the integral \( \int_0^t \log(\Gamma(t+1)) \, dt \)

So the closed form in general is \( \int_0^t \log(\Gamma(t+1)) \, dt = \text{constant} + y \cdot \log(A) + x \cdot \log(\pi) + f + \text{several terms in logarithms} + z \cdot \frac{K}{\pi} \),

Where \( f \) is a complex function in terms of trigamma and constant is a real number.

When the bound \( t \) is between -1 and 0, we can directly calculate some terms and we have

\[
\int_0^t \log(\Gamma(t+1)) \, dt = (-6t^2 - 6t) \cdot \log(A) + \frac{t}{2} \cdot \log(\pi) + f + \text{several terms in logarithms} + z \cdot \frac{K}{\pi}.
\]

(There are no constant)

About the polynom \(-6t^2 - 6t\): I see successively \( \int_0^{-1} \log(\Gamma(t+1)) \, dt \)

then \( \int_0^{-1/2} \log(\Gamma(t+1)) \, dt \) then \( \int_0^{-1/3} \log(\Gamma(t+1)) \, dt \) then \( \int_0^{-2/3} \log(\Gamma(t+1)) \, dt \) then \( \int_0^{-1/4} \log(\Gamma(t+1)) \, dt \)

and just I see that the polynom give the correct value in \( \log(A) \).

Now about the sum \( \sum_{k=1}^{\infty} \frac{(1-\zeta(2k+1))q^{2k+2}}{(k+1)(2k+1)} \) (10)

Especially, there are no term in \( \log(\pi) \) in the final closed form.

When we have a sum where we find a term of \( \zeta(2k+a) \) where \( a \) is a odd positiv or negativ integer.

Just I see in the final closed form of the sum, we have no trigamma function and no Catalan’s constant.
5 Expressions of $G(1/8)$, $G(3/8)$, $G(5/8)$, $G(7/8)$

First I find $\log(G(7/8))$, it’s easy and we use the formula (5) if $z = -1/8$ we have:

$$\log \left( G \left( \frac{7}{8} \right) \right) = \frac{7}{128} - \frac{5 \log(2)}{32} - \frac{3 \log(\pi)}{16} - \int_0^{-1/8} \log(\Gamma(t+1)) \, dt - \frac{\log(1+\sqrt{2})}{16} + \frac{\log(\Gamma(\frac{1}{8}))}{8}$$

Now we search $\log(G(1/8))$ and I use for example:

$$\int_0^\pi x \cot(\pi x) \, dx = t \log(2\pi) + \log \left( \frac{G(t)}{G(1-t)} \right)$$

When $t = 1/8$ and with Maple I find the closed form of the integral and finally:

$$\log \left( G \left( \frac{1}{8} \right) \right) = \frac{7}{128} - \frac{5 \log(2)}{16} - \frac{3 \log(\pi)}{16} - \int_0^{-1/8} \log(\Gamma(t+1)) \, dt - \frac{K}{8\pi} \frac{7 \log(\Gamma(\frac{1}{8}))}{8}$$

Now we search $\log(G(3/8))$ and $\log(G(5/8))$

We know that $\prod_{k=1}^{\infty} \frac{(1-\zeta(2k+1))q^{2k+2}}{(2k+1)(2k+3)} = (\gamma - 2)q^2 - \frac{1}{6} + 2 \log(A) + \zeta(-1, 2-q) + \zeta(-1, 2+q)$

And using the relation $\zeta(-1, 1, t) = \frac{1}{12} - \log(A) - \log(G(t)) + (t-1)\log(\Gamma(t))$

I have $\sum_{k=1}^{\infty} \frac{1-\zeta(2k+1)}{(2k+1)(2k+3)} \left( \frac{q}{2} \right)^{2k+2} = \frac{\gamma}{64} - \frac{9}{64} + 2\int_0^{-1/8} \log(\Gamma(t+1)) \, dt + \frac{\log(\pi) + 2\Psi(1, 1) + (2\pi^2 + 16K)\sqrt{2} + 2\pi^2}{64\pi}$

I know that the partial closed form of $\int_0^{-1/8} \log(\Gamma(t+1)) \, dt$:

$$-\frac{\log(\pi)}{16} + \frac{-\sqrt{2}\Psi(1, 1) + (2\pi^2 + 16K)\sqrt{2} + 2\pi^2 - 8K}{128\pi}$$

Remember in the final closed form of the sum there are no trigamma function, no $\log(\pi)$ and no term in Catalan’s constant.
Using the relation in the paper Polygamma Function of Negativ Order, page 5:

\[\zeta\left(1, -1, \frac{a}{b}\right) - \zeta\left(1, -1, \frac{a}{b}\right) = -\frac{\sqrt{2}\Psi(1, \frac{1}{4}) + (2\pi^2 + 16K)\sqrt{2}\pi^2 + 8K}{64\pi}\]

Using the relation \(\zeta\left(1, -1, t\right) = \frac{1}{t} - \log\left(A\right) - \log\left(G\left(t\right)\right) + (t - 1)\log\left(\Gamma\left(t\right)\right)\) and the relation (5)

\[
\text{Equivalently } \int_0^{5/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt - \int_0^{3/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt = -\frac{\sqrt{2}\Psi(1, \frac{1}{4}) + (2\pi^2 + 16K)\sqrt{2}\pi^2 + 8K}{64\pi}
\]

In the same time \(\sum_{k=1}^{\infty} -\zeta\left(\frac{a}{b}, k+1\right) (\frac{a}{b})^2 k + 2 = \frac{9\gamma}{64} + \frac{43}{64} + 2 \log\left(A\right) + \zeta\left(1, -1, \frac{a}{b}\right)\)

\[
\text{Now we search } \int_0^{5/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt + \int_0^{3/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt\]

The closed form of \(\sum_{k=1}^{\infty} -\zeta\left(\frac{a}{b}, k+1\right) (\frac{a}{b})^2 k + 2\) look like as the closed form of \(\sum_{k=1}^{\infty} -\zeta\left(\frac{a}{b}, k+1\right) (\frac{a}{b})^2 k + 2\)

The beginning of \(\int_0^{5/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt + \int_0^{3/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt\) is:

\[-\frac{\sqrt{2}\Psi(1, \frac{1}{4}) + (2\pi^2 + 16K)\sqrt{2}\pi^2 + 8K}{64\pi} - 2 \int_0^{1/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt\]

Or

\[\frac{\sqrt{2}\Psi(1, \frac{1}{4}) + (2\pi^2 - 16K)\sqrt{2}\pi^2 + 8K}{64\pi} + 2 \int_0^{1/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt\]

We have two possibilities but I choose the first expression.

Now we calculate missing terms at the right.

But the problem that we calculate terms in \(\log\left(A\right)\) for a bound \(t\) between 0 and 1 and so the polynom is \(-6t^2 + 6t\), for terms in \(\log\left(\pi\right)\) the polynom is \(\frac{1}{4}\) and there are a constant who is \(-t\). And when the bound is \(a/b\) where \(b\) and \(a\) both positiv integer, a smaller than \(b\) and a prime there are a term who is \(\frac{a \log(a)}{b}\).

For \(\int_0^{5/8} \log\left(\Gamma\left(t + 1\right)\right) \, dt\) we have \(\frac{5 \log(\pi)}{16} + \frac{5 \log(5)}{8} - \frac{5}{8} + \frac{45 \log(A)}{32}\)
For \( \int_0^{3/8} \log (\Gamma (t + 1)) \, dt \) we have
\[
\frac{3 \log(\pi)}{16} + \frac{3 \log(3)}{8} - \frac{3}{8} + \frac{45 \log(A)}{32}
\]

For \( \int_0^{-1/8} \log (\Gamma (t + 1)) \, dt \) we have
\[
- \frac{\log(\pi)}{16} + \frac{21 \log(A)}{32}
\]

I make the calculation
\[
\frac{5 \log(\pi)}{16} + \frac{5 \log(5)}{8} - \frac{5}{8} + \frac{45 \log(A)}{32} + \frac{3 \log(\pi)}{16} + \frac{3 \log(3)}{8}
\]

So I find
\[
- \frac{3}{8} + \frac{45 \log(A)}{32} + 2 \left[ - \frac{\log(\pi)}{16} + \frac{21 \log(A)}{32} \right]
\]

Now using the command identify with Maple, I have the missing term in \( \log(2) \) and finally the closed form of
\[
\int_0^{3/8} \log (\Gamma (t + 1)) \, dt + \int_0^{-1/8} \log (\Gamma (t + 1)) \, dt
\]

is:
\[
-1 + \frac{33 \log(A)}{8} + \frac{5 \log(5)}{8} + \frac{3 \log(\pi)}{8} + \frac{3 \log(3)}{8}
\]

Now I have
\[
\int_0^{5/8} \log (\Gamma (t + 1)) \, dt + \int_0^{3/8} \log (\Gamma (t + 1)) \, dt \text{ and } \int_0^{5/8} \log (\Gamma (t + 1)) \, dt - \int_0^{3/8} \log (\Gamma (t + 1)) \, dt
\]

So I find respectively the two integrals and consequently \( \log(G\left(\frac{3}{8}\right)) \) and \( \log(G\left(\frac{5}{8}\right)) \)

### 6 List of conjectural formulas

\[
\log\left(G\left(\frac{1}{8}\right)\right) = \frac{7}{128} - \frac{\log(2)}{16} - \frac{\log(\pi)}{16} - \int_0^{-1/8} \log (\Gamma (t + 1)) \, dt - \frac{K}{8\pi} \frac{7 \log (\Gamma \left(\frac{1}{2}\right))}{8}
\]
\[
+ \frac{-\sqrt{2} \Psi \left(1, \frac{1}{8}\right) + (2 \pi^2 + 16 K) \sqrt{2} + 2 \pi^2}{64 \pi}
\]

\[
\log\left(G\left(\frac{3}{8}\right)\right) = \frac{15}{128} + \frac{11 \log(2)}{192} - \frac{\log(\pi)}{4} + \int_0^{-1/8} \log (\Gamma (t + 1)) \, dt - \frac{33 \log(A)}{16} + \frac{5 \log(1 + \sqrt{2})}{16}
\]
\[
+ \frac{K}{8\pi} - \frac{5 \log \left(\frac{\Gamma(1/8)}{\Gamma(1/4)}\right)}{8}
\]

\[
\log\left(G\left(\frac{5}{8}\right)\right) = \frac{15}{128} - \frac{43 \log(2)}{192} - \frac{\log(\pi)}{8} + \int_0^{-1/8} \log (\Gamma (t + 1)) \, dt - \frac{33 \log(A)}{16} + \frac{3 \log \left(\frac{\Gamma(1/8)}{\Gamma(1/4)}\right)}{8}
\]
\[
+ \frac{\sqrt{2} \Psi \left(1, \frac{1}{8}\right) + (-2 \pi^2 - 16 K) \sqrt{2} - 2 \pi^2}{64 \pi}
\]
\[
\log \left(G \left(\frac{7}{8}\right)\right) = \frac{7}{128} - \frac{5 \log (2)}{32} - \frac{3 \log (\pi)}{16} - \int_0^{1/8} \log (\Gamma (t + 1)) \, dt - \frac{\log (1 + \sqrt{2})}{16} + \frac{\log (\Gamma \left(\frac{1}{4}\right))}{8}
\]

Now we search \(G(1/12), G(5/12), G(7/12)\) and \(G(11/12)\): it's the same principle but just we work with the sum \(\sum_{k=1}^{\infty} \frac{1 - \zeta(2k+1)}{2k+1} \left(\frac{1}{12}\right)^{2k+2}\) and \(\sum_{k=1}^{\infty} \frac{1 - \zeta(2k+1)}{2k+1} \left(\frac{A}{12}\right)^{2k+2}\).

### 7 List of conjectural formulas

\[
\log \left(G \left(\frac{1}{12}\right)\right) = \frac{11}{288} + \frac{3 \log (2)}{16} - \frac{11 \log (3)}{32} + \frac{5 \log (\pi)}{12} - \int_0^{1/12} \log (\Gamma (t + 1)) \, dt - \frac{\log (1 + \sqrt{3})}{24} - \frac{K}{3 \pi}
\]

\[
\quad - \frac{11 \log (1 + \sqrt{3})}{24} - \frac{\log (\Gamma (1/3) \Gamma (1/4))}{12} + \frac{\left(-3 \Psi (1, 1/3) + 2 \pi^2\right) \sqrt{3}}{144 \pi}
\]

\[
\log \left(G \left(\frac{5}{12}\right)\right) = \frac{35}{288} - \frac{19 \log (2)}{48} + \frac{5 \log (3)}{72} - \frac{\log (\pi)}{4} + \int_0^{1/12} \log (\Gamma (t + 1)) \, dt + \frac{7 \log (1 + \sqrt{3})}{24} - \frac{23 \log (A)}{12} + \frac{7 \log \left(\frac{\Gamma (1/3)}{\Gamma (1/4)}\right)}{12} + \frac{\sqrt{3} \left(3 \Psi (1, 1/3) - 2 \pi^2\right)}{144 \pi}
\]

\[
\log \left(G \left(\frac{7}{12}\right)\right) = \frac{35}{288} - \frac{13 \log (2)}{48} - \frac{\log (3)}{18} - \frac{\log (\pi)}{6} + \int_0^{1/12} \log (\Gamma (t + 1)) \, dt + \frac{5 \log (1 + \sqrt{3})}{24} - \frac{23 \log (A)}{12} + \frac{K}{3 \pi} - \frac{5 \log \left(\frac{\Gamma (1/3)}{\Gamma (1/4)}\right)}{12}
\]

\[
\log \left(G \left(\frac{11}{12}\right)\right) = \frac{11}{288} - \frac{5 \log (2)}{48} + \frac{\log (3)}{32} - \frac{\log (\pi)}{6} - \int_0^{1/12} \log (\Gamma (t + 1)) \, dt - \frac{\log (1 + \sqrt{3})}{24} + \frac{\log (\Gamma \left(\frac{1}{4}\right) \Gamma \left(\frac{1}{4}\right))}{12}
\]

### 8 Applications with Wallis product (11)

First example

Consider and calculate the closed form

\[
\prod_{k=1}^{\infty} \left(\frac{(8k + 5)^2 (8k + 1) (8k + 7)}{(8k + 4)^2 (8k + 2) (8k + 8)}\right)^k
\]
So we have \( \frac{(\frac{d(\frac{1}{4})}{d(\frac{1}{3})})^2}{d(\frac{1}{2})^2} \)

We obtain

\[
\Gamma \left( \frac{1}{4} \right) \left( \Gamma \left( \frac{1}{8} \right) \right)^{-2} 2\pi^2 \left( 1 + \sqrt{2} \right) \frac{\pi}{\sqrt{2} \pi} e^{\frac{\sqrt{\pi \Psi(1, 1/8) + \left( -2 \pi^2 - 16 \pi \right) \sqrt{2} \pi^2 + 88 K}}{\sqrt{2} \pi^2 + 88 K}}
\]

**Second example**

Consider and calculate the closed form

\[
\prod_{k=1}^{\infty} \left( \frac{(8k + 1)^2 (8k + 3)^3 (8k + 4)}{(8k + 2)^5 (8k + 5)} \right)^k
\]

So we have \( \frac{(\frac{d(\frac{1}{4})}{d(\frac{1}{3})})^2}{d(\frac{1}{2})^2} \)

We obtain

\[
\left( \Gamma \left( \frac{1}{4} \right) \right)^{-3} \left( \Gamma \left( \frac{1}{8} \right) \right)^2 2\pi^2 \left( 1 + \sqrt{2} \right) \frac{\pi}{\sqrt{2} \pi} e^{\frac{\sqrt{3} \Psi(1, 1/3) + \left( -3 \pi^2 + 48 \pi \right) \sqrt{3} \pi^2 + 36 K}}{\sqrt{3} \pi^2 + 36 K}}
\]

**Third example**

Consider and calculate the closed form

\[
\prod_{k=1}^{\infty} \left( \frac{(12k + 1)(12k + 4)^2 (12k + 5)}{(12k + 2)(12k + 3)^2 (12k + 6)} \right)^k
\]

So we have \( \frac{(\frac{d(\frac{1}{4})}{d(\frac{1}{3})})^2}{d(\frac{1}{2})^2} \)

We obtain

\[
3\pi^2 \left( 1 + \sqrt{3} \right) \frac{\pi}{\sqrt{3} \pi} e^{\frac{2 \sqrt{3} \pi^2 - 3 \sqrt{3} \Psi(1, 1/3) + 36 K}}{\sqrt{3} \pi^2 + 36 K}}
\]

8
Fourth example

Consider and calculate the closed form

\[
\prod_{k=1}^{\infty} \left( \frac{(12k + 1)^2 (12k + 5)^4 (12k + 6)}{(12k + 2) (12k + 3)^4 (12k + 7)^2} \right)^k
\]

So we have

\[
\frac{(\Gamma(1/4))^2 (\Gamma(1/4))^4 (\Gamma(1/4))^4}{\Gamma(1/4)^4 (\Gamma(1/4))^4 (\Gamma(1/4))^4}
\]

We obtain

\[
\left( \Gamma \left( \frac{1}{4} \right) \right)^2 \left( \Gamma \left( \frac{1}{3} \right) \right)^{-3} 3^{-\frac{1}{2}} 2^{\frac{1}{4}} \pi^{\frac{1}{4}} \left( 1 + \sqrt{3} \right)^{-\frac{1}{6}} e^{-\frac{1}{2} \sqrt{3} \pi^2 \frac{1}{48} \zeta(3/4) - 12 k}
\]

Remark and conclusion: this is a first approach with these particular values of the Barnes function and any improvement for these eight expressions is welcome.

9 References

(1): Denis Gallet, Values of the Barnes Function (2021)


(3), (4), (5) and (6): https://dlmf.nist.gov/5.17


(9): https://en.wikipedia.org/wiki/Trigamma-function
