ON THE NUMBER OF INTEGRAL POINTS BETWEEN A K DIMENSIONAL SPHERE AND A GRID

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ABSTRACT. Using the method of compression we show that the number of integral points in the region bounded by the $2r \times 2r \times \cdots \times 2r$ ($k$ times) grid containing the sphere of radius $r$ and a sphere of radius $r$ satisfies the lower bound

$$N_{r,k} \gg r^{k-\delta} \times \frac{1}{\sqrt{k}}$$

for some small $\delta > 0$.

1. Introduction

The Gauss circle problem is a problem that seeks to counts the number of integral points in a circle centered at the origin and of radius $r$. It is fairly easy to see that the area of a circle of radius $r > 0$ gives a fairly good approximation for the number of such integral points in the circle, since on average each unit square in the circle contains at least an integral point. In particular, by denoting $N(r)$ to be the number of integral points in a circle of radius $r$, then the following elementary estimate is well-known

$$N(r) = \pi r^2 + |E(r)|$$

where $|E(r)|$ is the error term. The real and the main problem in this area is to obtain a reasonably good estimate for the error term. In fact, it is conjectured that

$$|E(r)| \ll \frac{1}{r^{\frac{1}{2}} + \epsilon}$$

for $\epsilon > 0$. The first fundamental progress was made by Gauss [3], where it is shown that

$$|E(r)| \leq 2\pi r \sqrt{2}.$$

G.H Hardy and Edmund Landau almost independently obtained a lower bound [1] by showing that

$$|E(r)| \neq o(r^{\frac{1}{2}} (\log r)^{\frac{1}{4}}).$$

The current best upper bound (see [2]) is given by

$$|E(r)| \ll \frac{1}{r^{\frac{12}{25}}}.$$

In this paper we study a variant of this problem in the region between a general $k$ dimensional grid $2r \times 2r \cdots \times 2r$ ($k$ times) and the largest sphere contained in the
grid. In particular, we obtain the following lower bound for the number of integral points in this region

**Theorem 1.1.** Let \( N_{r,k} \) denotes the number of integral points in the region bounded by the \( 2r \times 2r \times \cdots \times 2r \) \((k \text{ times})\) grid covering a sphere of radius \( r \) and a sphere of radius \( r \). Then \( N_{r,k} \) satisfies the lower bound

\[
N_{r,k} \gg r^{k-\delta} \times \frac{1}{\sqrt{k}}
\]

for some small \( \delta > 0 \).

2. Preliminaries and background

**Definition 2.1.** By the compression of scale \( m > 0 \) \((m \in \mathbb{R})\) fixed on \( \mathbb{R}^n \) we mean the map \( V : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
V_m[(x_1, x_2, \ldots, x_n)] = \left( \frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n} \right)
\]

for \( n \geq 2 \) and with \( x_i \neq x_j \) for \( i \neq j \) and \( x_i \neq 0 \) for all \( i = 1, \ldots, n \).

**Remark 2.2.** The notion of compression is in some way the process of re-scaling points in \( \mathbb{R}^n \) for \( n \geq 2 \). Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

**Proposition 2.1.** A compression of scale \( 1 \geq m > 0 \) with \( V_m : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a bijective map.

**Proof.** Suppose \( V_m[(x_1, x_2, \ldots, x_n)] = V_m[(y_1, y_2, \ldots, y_n)], \) then it follows that

\[
\left( \frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n} \right) = \left( \frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n} \right).
\]

It follows that \( x_i = y_i \) for each \( i = 1, 2, \ldots, n \). Surjectivity follows by definition of the map. Thus the map is bijective. \( \square \)

2.1. The mass of compression. In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

**Definition 2.3.** By the mass of a compression of scale \( m > 0 \) \((m \in \mathbb{R})\) fixed, we mean the map \( M : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
M(V_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.
\]

It is important to notice that the condition \( x_i \neq x_j \) for \( (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take \( x_1 = x_2 = \cdots = x_n \), then it will follows that \( \text{Inf}(x_j) = \text{Sup}(x_j) \), in which case the mass of compression of scale \( m \) satisfies

\[
m \sum_{k=0}^{n-1} \frac{1}{\text{Inf}(x_j) - k} \leq M(V_m[(x_1, x_2, \ldots, x_n)]) \leq m \sum_{k=0}^{n-1} \frac{1}{\text{Inf}(x_j) + k}
\]

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and
the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) must satisfy \(x_i \neq x_j\) for all \(1 \leq i, j \leq n\). Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) is such that \(x_i \leq x_j\) for \(1 \leq i, j \leq n\).

**Lemma 2.4.** The estimate remain valid

\[
\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)
\]

where \(\gamma = 0.5772\cdots\).

**Remark 2.5.** Next we prove upper and lower bounding the mass of the compression of scale \(m > 0\).

**Proposition 2.2.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) with \(x_i \neq 0\) for each \(1 \leq i \leq n\) and \(x_i \neq x_j\) for \(i \neq j\), then the estimates holds

\[
m \log \left(1 - \frac{n - 1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathcal{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log \left(1 + \frac{n - 1}{\inf(x_j)}\right)
\]

for \(n \geq 2\).

**Proof.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) for \(n \geq 2\) with \(x_j \neq 0\). Then it follows that

\[
\mathcal{M}(\mathcal{V}_m[(x_1, x_2, \ldots, x_n)]) = m \sum_{j=1}^{n} \frac{1}{x_j} \leq m \sum_{k=0}^{n-1} \frac{1}{\inf(x_j) + k}
\]

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

\[
\mathcal{M}(\mathcal{V}_m[(x_1, x_2, \ldots, x_n)]) = m \sum_{j=1}^{n} \frac{1}{x_j} \geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.
\]

\(\square\)

**Definition 2.6.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) with \(x_i \neq 0\) for all \(i = 1, 2, \ldots, n\). Then by the gap of compression of scale \(m > 0\), denoted \(\mathcal{G} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)]\), we mean the expression

\[
\mathcal{G} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)] = \left\| \left( x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n} \right) \right\|
\]

**Definition 2.7.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) with \(x_i \neq 0\) for all \(1 \leq i \leq n\). Then by the ball induced by \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) under compression of scale \(m > 0\),
denoted $B_{\frac{1}{2}G \circ V_m}(x_1, x_2, \ldots, x_n)$] we mean the inequality
\[
\|\bar{y} - \frac{1}{2} \left( x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n} \right)\| < \frac{1}{2} G \circ V_m((x_1, x_2, \ldots, x_n)).
\]
A point $\bar{z} = (z_1, z_2, \ldots, z_n) \in B_{\frac{1}{2}G \circ V_m}(x_1, x_2, \ldots, x_n)$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n = 2$.

**Remark 2.8.** In the geometry of balls under compression of scale $m > 0$, we will assume implicitly that $1 \geq m > 0$. The circle induced by points under compression is the ball induced on points when we take $n = 2$.

**Proposition 2.3.** Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$, then we have
\[
G \circ V_m((x_1, x_2, \ldots, x_n))^2 = M \circ V_1 \left[ \left( \frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2} \right) \right] + m^2 M \circ V_1((x_1^2, \ldots, x_n^2)) - 2mn.
\]
In particular, we have the estimate
\[
G \circ V_m((x_1, x_2, \ldots, x_n))^2 = M \circ V_1 \left[ \left( \frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2} \right) \right] - 2mn + O \left( m^2 M \circ V_1((x_1^2, \ldots, x_n^2)) \right)
\]
for $\bar{x} \in \mathbb{R}^n$, where $m^2 M \circ V_1((x_1^2, \ldots, x_n^2))$ is the error term in this case.

**Lemma 2.9** (Compression estimate). Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ and $x_i \neq x_j$ for $i \neq j$, then we have
\[
G \circ V_m((x_1, x_2, \ldots, x_n))^2 < n \sup(x_i^2) + m^2 \log \left( 1 + \frac{n-1}{\inf(x_j^2)} \right) - 2mn
\]
and
\[
G \circ V_m((x_1, x_2, \ldots, x_n))^2 > n \inf(x_j^2) + m^2 \log \left( 1 - \frac{n-1}{\sup(x_j^2)} \right) - 2mn.
\]

**Theorem 2.10.** Let $\bar{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$. Then $\bar{z} \in B_{\frac{1}{2}G \circ V_m(\bar{y})} \bar{y}$ if and only if
\[
G \circ V_m[\bar{z}] < G \circ V_m[\bar{y}].
\]

Proof. Let $\bar{z} \in B_{\frac{1}{2}G \circ V_m(\bar{y})} \bar{y}$ for $\bar{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$, then it follows that $||\bar{y}|| > ||\bar{z}||$. Suppose on the contrary that
\[
G \circ V_m[\bar{z}] \geq G \circ V_m[\bar{y}],
\]
than it follows that $||\bar{y}|| \leq ||\bar{z}||$, which is absurd. Conversely, suppose
\[
G \circ V_m[\bar{z}] < G \circ V_m[\bar{y}]
\]
then it follows from Proposition 2.3 that $||\bar{z}|| < ||\bar{y}||$. It follows that
\[
\left\| \bar{z} - \frac{1}{2} \left( \frac{y_1 + m}{y_1}, \ldots, \frac{y_n + m}{y_n} \right) \right\| < \left\| \bar{y} - \frac{1}{2} \left( \frac{y_1 + m}{y_1}, \ldots, \frac{y_n + m}{y_n} \right) \right\| = \frac{1}{2} G \circ V_m[\bar{y}].
\]
This certainly implies $\bar{z} \in B_{\frac{1}{2}G \circ V_m(\bar{y})} \bar{y}$ and the proof of the theorem is complete. \qed
2.2. Admissible points of balls induced under compression. We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.11. Let \( \bar{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) with \( y_i \neq y_j \) for all \( 1 \leq i < j \leq n \). Then \( \bar{y} \) is said to be an admissible point of the ball \( B_{\frac{1}{2}G \circ V_m}([\bar{x}]) \) if

\[
\left\| \bar{y} - \frac{1}{2} \left( y_1 + \frac{m}{x_1}, \ldots, y_n + \frac{m}{x_n} \right) \right\| = \frac{1}{2}G \circ V_m([\bar{x}]).
\]

Remark 2.12. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball.

Theorem 2.13. The point \( \bar{y} \in B_{\frac{1}{2}G \circ V_m}([\bar{x}]) \) is admissible if and only if

\[
B_{\frac{1}{2}G \circ V_m}([\bar{y}]) = B_{\frac{1}{2}G \circ V_m}([\bar{x}]).
\]

and \( G \circ V_m([\bar{y}]) = \frac{1}{2}G \circ V_m([\bar{x}] \).

Proof. First let \( \bar{y} \in B_{\frac{1}{2}G \circ V_m}([\bar{x}]) \) be admissible and suppose on the contrary that

\[
B_{\frac{1}{2}G \circ V_m}([\bar{y}]) \neq B_{\frac{1}{2}G \circ V_m}([\bar{x}]).
\]

Then there exist some \( \bar{y} \in B_{\frac{1}{2}G \circ V_m}([\bar{x}]) \) such that

\[
\bar{y} \notin B_{\frac{1}{2}G \circ V_m}([\bar{y}]).
\]

Applying Theorem 2.10, we obtain the inequality

\[
G \circ V_m([\bar{y}]) \leq G \circ V_m([\bar{x}]) < G \circ V_m([\bar{x}]).
\]

It follows from Proposition 2.3 that \( ||[\bar{x}]|| < ||[\bar{y}]|| \) or \( ||[\bar{y}]|| < ||[\bar{x}]|| \). By joining this points to the origin by a straight line, this contradicts the fact that the point \( \bar{y} \) is an admissible point of the ball \( B_{\frac{1}{2}G \circ V_m}([\bar{x}]) \). The latter equality follows from assertion that two balls are indistinguishable. Conversely, suppose

\[
B_{\frac{1}{2}G \circ V_m}([\bar{y}]) = B_{\frac{1}{2}G \circ V_m}([\bar{x}] \)
\]

and \( G \circ V_m([\bar{y}]) = G \circ V_m([\bar{x}] \). Then it follows that the point \( \bar{y} \) lives on the outer of the indistinguishable balls and must satisfy the inequality

\[
\left\| \bar{y} - \frac{1}{2} \left( y_1 + \frac{m}{x_1}, \ldots, y_n + \frac{m}{x_n} \right) \right\| = \left\| \bar{y} - \frac{1}{2} \left( x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n} \right) \right\| = \frac{1}{2}G \circ V_m([\bar{x}]).
\]

It follows that

\[
\frac{1}{2}G \circ V_m([\bar{x}] = \left\| \bar{y} - \frac{1}{2} \left( x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n} \right) \right\|
\]

and \( \bar{y} \) is indeed admissible, thereby ending the proof.

Remark 2.14. We note that we can replace the set \( \mathbb{N}^n \) used in our construction with \( \mathbb{R}^n \) at the compromise of imposing the restrictions \( \bar{y} = (x_1, \ldots, x_n) \in \mathbb{R}^n \) such that \( x_i > 1 \) for all \( 1 \leq i \leq n \) and \( x_i \neq x_j \) for \( i \neq j \). The following construction in our next result in the sequel employs this flexibility.
3. The lower bound

**Theorem 3.1.** Let $N_{r,k}$ denotes the number of integral points in the region bounded by the $2r \times 2r \times \cdots 2r$ ($k$ times) grid covering a sphere of radius $r$ and a sphere of radius $r$. Then $N_{r,k}$ satisfies the lower bound

$$N_{r,k} \gg r^{k-\delta} \times \frac{1}{\sqrt{k}}$$

for some small $\delta > 0$.

**Proof.** Pick arbitrarily a point $(x_1, x_2, \ldots, x_k) = \vec{x} \in \mathbb{R}^k$ with $x_i > 1$ for $1 \leq i \leq k$ and $x_i \neq x_j$ for $i \neq j$ such that $\mathcal{G} \circ \mathcal{V}_m[\vec{x}] = 2r$. This ensures the ball induced under compression is of radius $r$. Next we apply the compression of fixed scale $m \leq 1$, given by $\mathcal{V}_m[\vec{x}]$ and construct the ball induced by the compression given by

$$B_{\frac{1}{2}\mathcal{G} \circ \mathcal{V}_m}[\vec{x}]$$

with radius $\frac{(\mathcal{G} \circ \mathcal{V}_m[\vec{x}])}{2} = r$. By appealing to Theorem 2.13 admissible points $\vec{x}_l \in \mathbb{R}^k$ ($\vec{x}_l \neq \vec{x}$) of the ball of compression induced must satisfy the condition $\mathcal{G} \circ \mathcal{V}_m[\vec{x}_l] = 2r$. Also by appealing to Theorem 2.10 points $\vec{x}_l \notin B_{\frac{1}{2}\mathcal{G} \circ \mathcal{V}_m}[\vec{x}]$ must satisfy the inequality

$$\mathcal{G} \circ \mathcal{V}_m[\vec{x}_l] \geq \mathcal{G} \circ \mathcal{V}_m[\vec{x}] = 2r.$$ 

In particular points in $\vec{x}_l \notin B_{\frac{1}{2}\mathcal{G} \circ \mathcal{V}_m}[\vec{x}]$ contained in the $2r \times 2r \times \cdots 2r$ ($k$ times) grid that covers this ball must satisfy for their coordinates

$$\max_{\vec{x}_l \in (2r)^k} \sup_{i=1}^k (x_{l_i})^k = (2r)^{1+\delta}$$

for some small $\delta > 0$ so that $\mathcal{G} \circ \mathcal{V}_m[\vec{x}_l] \geq 2r$. The number of integral points contained in the region between the $2r \times 2r \times \cdots 2r$ ($k$ times) grid covering the
ball and the ball is lower bounded by
\[
\mathcal{N}_{r,k} = \sum_{\vec{x}_i \in \|2r\|^k \subset \mathbb{N}^k} 1
\]
\[
\geq \sum_{\vec{x}_i \in \|2r\|^k \subset \mathbb{N}^k} \frac{2r}{G \circ V_m[\vec{x}_i]}
\]
\[
\gg \sum_{\vec{x}_i \in \|2r\|^k \subset \mathbb{N}^k} \frac{2r \sqrt{k}}{k \sup(x_{i_1})}
\]
\[
= 2r \sum_{\vec{x}_i \in \|2r\|^k \subset \mathbb{N}^k} \frac{1}{\sqrt{k} \sup(x_{i_1})}
\]
\[
\geq \frac{2r}{\sqrt{k}} \sum_{\vec{x}_i \in \|2r\|^k \subset \mathbb{N}^k} \frac{1}{\max_{\vec{x}_i \in \|2r\|^k} \sup(x_{i_1})}
\]
\[
= \frac{2r}{(2r)^{1+\delta} \sqrt{k}} \sum_{\vec{x}_i \in \|2r\|^k \subset \mathbb{N}^k} 1
\]
\[
\gg \frac{r}{r^{1+\delta} \sqrt{k}} \times r^k
\]
and the lower bound follows. \(\square\)

References