The Fundamental Conservation Law in the Theory of General Relativity: an unconventional approach is feasible and correct?

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Abstract
The author derives the fundamental and necessarily existing conservation law in the theory of General Relativity (GR). Thereby, the energy-momentum density tensor of the gravitational field is found. Additionally, the cosmological constant takes on a completely new meaning, which solves the cosmological constant problem. This new interpretation of the cosmological constant also explains the dark energy and the dark matter phenomenon.

Keywords
General Relativity, conservation law, energy-momentum density tensor of the gravitational field, cosmological constant, cosmological constant problem, dark energy, dark matter.

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1. Introduction

Regarding the vanishing covariant divergence of the energy-momentum density tensor of matter,

$$T^{k}_{i;k} = 0 \,, \tag{1}$$

it is justifiably stated in § 96 of Ref. [1]:

> In this form, however, this equation does not generally express any conservation law whatever. This is related to the fact that in a gravitational field the four-momentum of the matter alone must not be conserved, but rather the four-momentum of matter plus gravitational field; the latter is not included in the expression for T^{k}_{i}.

In order to remedy this, pseudotensors assume the role as energy-momentum density "tensors" of the gravitational field. However, pseudotensors are no true tensors. They are not form-invariant with respect to coordinate transformations [2]. Regarding pseudotensors, it is stated in § 20.4. of Ref. [3], that there is no unique formula for "local gravitational energy-momentum", but an infinite amount of quite distinct formulas. Moreover, pseudotensors vanish in a local reference frame, where the energy-momentum density tensor of the gravitational field must not vanish because of the following reason: The gravitational field strength is represented by the Christoffel symbols. While in a local reference frame the first derivatives of the metric tensor and hence also the Christoffel symbols vanish, the energy density of the gravitational field must not vanish, otherwise there would be no free fall.

The energy-momentum density "tensor" of the gravitational field $t_{\mu\kappa}$ is given by Eq. (7.6.4) in Ref. [4]. However, it turns out, that this quantity is not being about a true tensor, but rather it is a tensor fragment, wherefore its designation is written in quotation marks. Moreover, Eq. (7.6.3) in Ref. [4] has the form of a wave equation, wherein there are remaining terms on its left hand side, which definitely belong to the energy-momentum density of the gravitational field. The reason for this is, that the first order of the Ricci tensor and that of the Ricci scalar contain the fields $h_{\mu\nu}$, which are a part of the metric tensor and contain the expression of the Newtonian gravitational potential. These terms therefore belong to the gravitational field. Consequently, they are missing in a true tensor, which represents the energy-momentum density of the gravitational field.

In § 20.4. of Ref. [3] it is argued why one cannot define a localized energy-momentum for the gravitational field. Similarly, it is explained in Sec. 3.4 of Ref. [5], that ...

> ... a general conservation law for energy and momentum does *not* exist in GR. This has been disturbing to many people, but one simply has to get used to this fact. There is no "energy-momentum tensor for the gravitational field". Independently of any formal arguments, Einstein's equivalence principle tells us directly that there is no way to localize the energy of the gravitational field: The "gravitational field" (the connection $\Gamma^{\mu}{}_{\alpha\beta}$) can be locally transformed away. But if there is no field, there is locally no energy and no momentum.

2. Theory

By using the metric signature $(-,+,+,+)$, Einstein's field equations with the cosmological constant Λ read [5],

$$G_{\mu\nu} = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \ , \tag{2}$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant, G is the gravitational constant, and c is the speed of light. For the energy-momentum density tensor of matter $T_{\mu\nu}$, the energy-momentum density tensor of a perfect fluid is utilized,

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu} \ , \tag{3}$$

where ρ is the mass density, P is the pressure, and u^μ is the four-velocity of the fluid.

The Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \ ,$$

is formed by the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ and the Ricci tensor,

$$R_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu,\lambda} - \Gamma^{\lambda}{}_{\mu\lambda,\nu} + \Gamma^{\sigma}{}_{\mu\nu}\Gamma^{\lambda}{}_{\sigma\lambda} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\sigma\nu} \ ,$$

in which appear the Christoffel symbols of the second kind,

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{g^{\lambda\sigma}}{2} \left(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma} \right) \ .$$

Unless mentioned otherwise, in this article with Einstein's field equations are always meant those ones, wherein the cosmological term is included. Because of Lovelock's theorem [6], Eqs. (2) show Einstein's field equations in their most possible modified form.

Einstein's field equations (2) can be algebraically transformed into the mixed-tensor representation,

$$\kappa^{-1}\Lambda\delta^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} - \kappa^{-1} G^{\mu}{}_{\nu} \ . \tag{4}$$

Einstein's field equations in mixed-tensor representation (4) as well as Einstein's field equations (2) are valid in any reference frame.

By utilizing a reference frame, wherein the fluid velocity $u^i = 0$, the energy-momentum density tensor of matter (3) in mixed-tensor representation reads

$$T^{\mu}{}_{\nu} = \mathrm{diag}\left(-\rho c^2, P, P, P\right) \ ,$$

which does not depend on any metric coefficients. The term on the left hand side of Einstein's field equations in mixed-tensor representation (4) is proportional to the Kronecker tensor $\delta^{\mu}{}_{\nu}$. Consequently, in Eqs. (4) all metric tensors and its first and second derivatives therein appear in the Einstein tensor in mixed-tensor representation $G^{\mu}{}_{\nu}$, while in Eqs. (2) this is not in case.

This suitable separation of the metric tensors from other quantities in Einstein's field equations in mixed-tensor representation (4) is neatening them in the sense, that one is able to designate and to match the tensors to their physical meaning as is shown in the following.

3. Discussion

In order to overcome all these shortcomings and contradictions regarding the energy-momentum density tensor of the gravitational field and the denial of a necessarily existing conservation law in GR, which are demonstrated in the introduction of this article, one can introduce an initially unknown energy-momentum density tensor of the gravitational field $A^{\mu}{}_{\nu}$ besides the energy-momentum density tensor of matter $T^{\mu}{}_{\nu}$. The wanted and necessarily existing conservation law then reads

$$\nabla_\mu \left(T^{\mu}{}_{\nu} + A^{\mu}{}_{\nu}\right) = 0 \ , \tag{5}$$

which is the logical and reasonable extension of Eq. (1) in order to overcome its shortcoming of not being a conservation law.

In order to find the correct expression for the energy-momentum density tensor of the gravitational field, the following requirements have to be fulfilled:

- The energy-momentum density tensor of the gravitational field must be a true tensor and neither a pseudotensor nor a tensor fragment.
- The energy-momentum density tensor of the gravitational field must consist of all metric tensors and its derivatives, which appear in Einstein's field equations, cf. Eqs. (4), since the metric tensor contains the Newtonian gravitational potential, which is a quantity, that definitely belongs to the gravitational field and therefore to its energy-momentum density.

- The energy-momentum density tensor of the gravitational field must contain the Christoffel symbols squared. This is because in electrostatics, there appears the electric field strength squared in the expression for the energy density of the electrostatic field. Analogously, in Newtonian gravity there appears the gravitational acceleration a_g squared in the formula for the energy density of the gravitational field,

$$\varepsilon = -\frac{a_g^2}{8\pi G} \ . \tag{6}$$

  Hence, the gravitational field strength in GR, which is represented by the Christoffel symbols, is also expected to appear squared in the energy-momentum density tensor of the gravitational field.

- The energy-momentum density tensor of the gravitational field must not vanish in a local reference frame, because otherwise there would not be free fall. It therefore has to contain non-vanishing terms of second derivatives of the metric tensor or such with derivatives of the Christoffel symbols, respectively.

- The energy-momentum density tensor of the gravitational field must have the unit of measurement of an energy density.

- Because of Eqs. (1) and (5), the covariant divergence of the energy-momentum density tensor of the gravitational field must vanish,

$$A^{\mu}{}_{\nu;\mu} = 0 \ .$$

- Because of the conservation law (5), there must exist an initially unknown total energy-momentum density tensor,

$$L^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} + A^{\mu}{}_{\nu} \ , \tag{7}$$

  of which the covariant divergence vanishes,

$$L^{\mu}{}_{\nu;\mu} = 0 \ .$$

All these requirements can only be fulfilled with the following expression for the energy-momentum density tensor of the gravitational field,

$$A^{\mu}{}_{\nu} = -\kappa^{-1} G^{\mu}{}_{\nu} \ . \tag{8}$$

Because of Eqs. (4), (7), and (8), the total energy-momentum density tensor reads

$$L^{\mu}{}_{\nu} = \kappa^{-1} \Lambda \delta^{\mu}{}_{\nu} \ . \tag{9}$$

Its covariant divergence,

$$L^{\mu}{}_{\nu;\mu} = L^{\mu}{}_{\nu,\mu} + \Gamma^{\mu}{}_{\mu\lambda} L^{\lambda}{}_{\nu} - \Gamma^{\lambda}{}_{\mu\nu} L^{\mu}{}_{\lambda} \ ,$$

is equal to its partial divergence, because the terms, which contain the Christoffel symbols, drop out, so that the wanted and necessary conservation law,

$$L^{\mu}{}_{\nu;\mu} = L^{\mu}{}_{\nu,\mu} = 0 \ ,$$

is obtained.

One can even go a step further and take the covariant derivative of Eqs. (9),

$$L^{\mu}{}_{\nu;\sigma} = L^{\mu}{}_{\nu,\sigma} + \Gamma^{\mu}{}_{\sigma\lambda} L^{\lambda}{}_{\nu} - \Gamma^{\lambda}{}_{\sigma\nu} L^{\mu}{}_{\lambda} \ .$$

The terms, which contain the Christoffel symbols, drop out again. Therefore, the covariant derivative of the total energy-momentum density tensor is equal to its partial derivative, so that the necessary and fundamental conservation law in GR reads

$$L^{\mu}{}_{\nu;\sigma} = L^{\mu}{}_{\nu,\sigma} = 0 \ .$$

Thereby, the total energy and the total momentum within a certain flat three dimensional spatial volume are conserved,

$$\frac{\partial}{\partial t} \int \kappa^{-1} \Lambda \delta^{\mu}{}_{\nu} \, \mathrm{d}V = \frac{\partial}{\partial t} \int \left(T^{\mu}{}_{\nu} - \kappa^{-1} G^{\mu}{}_{\nu} \right) \mathrm{d}V = 0 \ . \tag{10}$$

Consequently, Einstein's field equations in their mixed-tensor representation (4) form the fundamental conservation law in gravity.

Because of this essential finding, Einstein's field equations without the cosmological constant,

$$G_{\mu\nu} = \kappa T_{\mu\nu} \ ,$$

are incomplete and thus violate the fundamental conservation law in GR. Nonetheless, they can be utilized for "short" distances, where the contribution of the cosmological constant does not play a significant role because of its tininess. Hence, in empty space-time, for example around a star $G_{\mu\nu} = 0$ is still a very good approximation for "short" distances. However, it demonstrates its incompleteness by having a vanishing Einstein tensor, that is not balanced with the cosmological term.

With these findings, the cosmological constant Λ, which is proportional to the total energy density, can never be a universal constant, but rather a constant parameter, that is constant only with respect to the metric under consideration, which means, that there are different cosmological constants with respect to different metrics. That this is indeed true is exemplified in the following:

In absence of any matter, $T_{\mu\nu} = 0$, GR becomes Special Relativity (SR) by considering the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$. Hence, there neither is a gravitational field nor a gravitational field energy-momentum density, $G_{\mu\nu} = 0$. Since Einstein's field equations must be fulfilled, the cosmological constant consequently has to vanish, $\Lambda = 0$.

From observations, we know, that our universe is accelerated expanding [7]. This means, that with respect to the

Friedmann-Lemaître-Robertson-Walker (FLRW) metric, there must exist a positive value for the cosmological constant, $\Lambda = 1.1056 \cdot 10^{-52}\,\mathrm{m}^{-2}$, see Ref. [8].

Up to now, the cosmological constant erroneously is related to the energy density of the vacuum. Thereby, the cosmological constant problem arises, see e.g. Ref. [9]. There appears a huge mismatch between the theoretical and the observed value of the vacuum energy density, which cannot be overcome by keeping this hypothesis. The findings in this article give a logical and reasonable explanation for the cosmological constant being proportional to the total energy density with respect to the metric under consideration. Consequently, by considering the FLRW metric, its cosmological constant is proportional to its total energy density. This fact solves the cosmological constant problem and explains the dark energy phenomenon.

Since $\Lambda > 0$ with respect to the FLRW metric of an expanding universe, consequently also the energy density of the gravitational field is positive, $-\kappa^{-1}G^0{}_0 > 0$, which is simply reasoned by using Einstein's field equations in mixed-tensor representation (4). By considering the empty spacetime around a star for example, the total energy density equals the energy density of the gravitational field,

$$\kappa^{-1}\Lambda = -\kappa^{-1}G^0{}_0 \, .$$

One knows owing to Eq. (6), that the Newtonian value for the energy density of the gravitational field around a star is always negative, $\varepsilon < 0$. Consequently, also the energy density of the gravitational field is negative around a star in GR, $-\kappa^{-1}G^0{}_0 < 0$. Hence, the total energy density and thereby the cosmological constant with respect to the metric of a star or of any other celestial object is negative, $\Lambda < 0$. This fact explains the dark matter phenomenon. With this finding, flat rotation curves of spiral galaxies are obtained in Ref. [10].

The two quantities κ and Λ, which appear in Einstein's field equations, differ in their nature. While the cosmological constant Λ is a different constant parameter with respect to different metrics, Einstein's gravitational constant κ is a universal constant and assumes the role of a coupling constant, which is the same constant in every reference frame.

4. History

In February 1917, Einstein introduced the cosmological term into his theory [11], because he wanted to have a static universe, wherefore it was necessary to implement it.

Levi-Civita correctly suggested already in April 1917, that the Einstein tensor is proportional to the energy-momentum density tensor of the gravitational field [12], a fact, that Einstein denied, whereof the famous controversy between Levi-Civita and Einstein arose in August 1917, see Refs. [13, 14].

In January 1918, Einstein stated the following [15]:

> A *logical* objection can, of course, not be raised against such wording. But I find that (37) does not allow us to draw these conclusions which we are used to drawing from the conservation theorems. This is connected to the fact that in (37) the components of the *total energy* vanish everywhere. The equations (37), for example, do not exclude the possibility (and this in contrast to the equations [35]) that a material system dissolves into just nothing without leaving any trace. Because the total energy in (37) — but not in (35) — is zero from the beginning: the conservation of this energy value does not demand the continued existence of the system in any form.

By including the cosmological term, Einstein's objection becomes redundant, see Eqs. (4) and (10). Interestingly, and surprisingly, neither Levi-Civita nor Einstein found such a simple explanation, although Einstein had already included the cosmological term in his theory.

5. Summary and outlook

Einstein's field equations in their mixed-tensor representation (4) form the necessary and fundamental conservation law in gravity, wherein

- $T^\mu{}_\nu$ is the energy-momentum density tensor of matter,
- $-\kappa^{-1}G^\mu{}_\nu$ is the energy-momentum density tensor of the gravitational field,
- $\kappa^{-1}\Lambda\delta^\mu{}_\nu$ is the total energy-momentum density tensor,

so that the total energy-momentum density tensor equals the energy-momentum density tensor of matter plus the energy-momentum density tensor of the gravitational field.

The cosmological constant Λ is a constant parameter, that is proportional to the total energy density with respect to the metric under consideration, which means, that there are different cosmological constants with respect to different metrics.

With this finding, the cosmological constant problem is solved, and the dark energy as well as the dark matter phenomenon can be explained. Thus, flat rotation curves of spiral galaxies are obtained in Ref. [10].

Appendix

Debate with the referee

Thanks to the referee, Prof. Dr. Lavenda, for the following debate, wherein **R:** means objection of the referee and **A:** clarification of the author:

R: Regarding the paper you sent me, my opinion is that it is nonsense. The author deludes himself into thinking that he has solved the dark energy problem by adding the negative trace of the Einstein tensor to the energy-momentum tensor, Eq. (6).

A: I didn't add something to Eq. (6). Eq. (6) is the Newtonian formula for the energy density of the gravitational field.

R: Introducing the Kronecker delta into the left -hand side of his Eq (4) necessitates the same on the right-hand side. So I do not understand his insistence on the "mixed-representation" being superior Thus, his statement

A: Referee 1 obviously does not understand how to obtain Einstein's field equations in mixed-tensor representation. I definitely did not include the Kronecker tensor, I obtain it by converting Einstein's field equations into mixed-tensor representation.

R: "Consequently, in Eqs (4) all metric tensors and its [sic[first and second derivatives therein appear in the Einstein tensor in the mixed-tensor representation, while in equation (2) this is not the case."

Why is the equation following his equation (4) in the mixed representation when it is diagonal?

A: This statements again show, that Referee 1 obviously does not understand how to obtain Einstein's field equations in mixed-tensor representation.

R: The last sentence before Sec 3 is not English. What does "neatening" mean?

A: to neaten means: make neat; arrange in an orderly, tidy way.

R: It's the vanishing of the four-divergence that represents a conservation law, not the vanishing of the time derivative, as in his Eq. (10). And if the Kronecker delta applies to the left hand side, it must also apply to the right hand side. So what is being conserved?

A: The vanishing covariant divergence is shown before that in my article. Then, after this has been done, one can even go a step further and even take the time derivative, that is zero and which shows, that the expression does not depend on time. This is essential for having energy and momentum conservation.

R: The trace of the Einstein tensor is not "the energy-momentum density tensor of the gravitational field," contrary to what he claims.

A: -kappa^{-1} G^mu_nu is the energy-momentum density tensor of the gravitational field. Its trace is of course no tensor, but can be called energy-momentum density of the gravitational field (without the designation "tensor").

R: There is nothing in GR that would allow him to conclude that the vanishing of the Einstein tensor is "still a very good approximation for 'short' distances."

A: Without the cosmological term, the dark matter phenomenon cannot be explained within the framework of GR. The cosmological term balances the Einstein tensor. Without it, it is just an approximation.

R: In the "absence of any matter, GR becomes SR" is nonsense. What is the Schwarzschild solution–SR then?

A: In the context in this paragraph of my article, I consider flat space-time, which is the Minkowski metric and not the Schwarzschild metric.

R: I see the author has published another paper along the same lines in your journal. The FLRW metric shows that the expressions deduced from the Einstein equations require either the density or pressure to be negative. There is no "missing" energy-momentum density in the Einstein equations. It simply cannot be put in.

A: This is nonsense because pressure cannot be negative. Pressure is always a positive quantity. The energy-momentum density tensor of the gravitational field is not contained within T^k_i, see Landau Lifshitz §96. In my article, I have shown that it is represented by -kappa^{-1} G^mu_nu.

R: 1. To go from the mixed representation to any other, just multiply through by the metric tensor and sum over repeated indices. There is nothing special about the mixed representation!

A: The Referee overlooks, that in the mixed-tensor representation, the tensors are in a "pure" energy-momentum density form. This can be directly seen by the energy-momentum density tensor of matter in mixed-tensor representation, T^mu_nu, which is without any metric coefficients. The same with the cosmological term, which also shows no metric coefficients any longer. So, there is indeed something special.

R: 2. To go from GR->SR just set the energy momentum tensor equal to zero. No! Einstein's condition of emptiness is the vanishing of the eigenvalues of the Ricci tensor. This was used by Schwarzschild to obtain his "vacuum" solution. It is certainly not SR!

A: Referee 1 still does not understand, that I CONSIDER FLAT space-time, i.e. the Minkowski metric in this paragraph and not the Schwarzschild metric.

R: 3. The author never heard of negative pressure?

A: I know this. But this is nonsense. Pressure is always positive.

R: What is inflation all about then?? Negative pressure is the driving force behind inflation.

Yes, I agree with the author on this that negative pressure cannot result in a stable configuration. But this is what the Einstein equations imply when the Robertson-Walker metric is used. And this is precisely what he uses in the other paper referred to.

A: In my articles I found a new interpretation for all this. The cosmological constant is related to the total energy density with respect to the metric under consideration. This has nothing to do with a negative pressure, which in my opinion is unrealistic.

R: 4. Attribution to G_mu^nu a physical meaning of being an energy-momentum "tensor" density goes against the grain of GR. G is geometry and T is energy-momentum. If the former is the latter then why do you need T?

A: Yes, but the Referee forgets to take into account the cosmological term. With it you'll have:

total energy-momentum density tensor = energy-momentum density tensor of matter plus energy-momentum density tensor of the gravitational field

And this is what makes sense. The Einstein tensor in mixed-tensor representation indeed is up to a constant (-1/kappa) the energy-momentum density tensor of the gravitational field. This already was suggested by Levi-Civita in 1917, and he was right!

R: Surely, Einstein would not have resorted to a pseudo-tensor–a bilinear product of Christoffel symbols if this were so! You get into trouble only when you consider gravitational energy. If gravitation is geometry, it cannot be included in T. Therefore the need of a pseudo tensor to be attached to T. This cannot be G_mu^nu!!

A: No. It definitely is G_mu^nu. Pseudotensors do not solve the problem! Please read again the introduction in my new article, then you see which problems occur by using pseudotensors.

R: 5. The author would have us believe that all static solutions to the Einstein equations satisfy energy and momentum conservation? You could have a mass density that depends on time which is conserved by equating it to the negative divergence of its flux! i.e., conservation of mass.

A: Yes, there is mass conservation by having nabla^mu T_{mu nu} = 0. The conservation law, Eq. (10), is much more than that. It is not mass conservation, it is the conservation of TOTAL energy and TOTAL momentum.

R: 6. Why should I have to understand Einstein's equations in the mixed representation as distinct from its covariant or contravariant expressions. There is absolutely no physics in lowering or raising indices!

A: It is an algebraic transformation but there is much more behind it as one may think at first sight. My explanations for this are demonstrated in the article.

R: The author claims that G_mu^nu is an energy-momentum "tensor" density. Let him give one concrete example of this! If I take the 4 divergence of G and set it equal to zero, and then take the 4-divergence of T and set it equal to zero, what then would be the difference if G_mu nu, G^mu nu or G_mu^nu contains information about energies and stresses in the system? The Ricci tensor contains the metric tensor g_mu nu and its derivatives so you cannot get around it not appearing in the mixed representation. And the metric tensor ALONE discriminates between GR and SR.: only if the metric tensor is space independent, as well as being time independent will GR reduce to SR. This discriminates between Schwarzschild and SR, and not as the author claims.

GR can very well do without the cosmological constant and still hold together (or fall) as a theory. The cosmological constant was an after thought of Einstein. who later repented saying it was "the biggest blunder of my life." He needn't have been so melodramatic about it.

I have read Levi-Civita's 1917 paper. It was meant as a criticism of Einstein's theory in that he points out that energy cannot be localized. To the best of recollection there is notreference to the mixed tensor being associated with an energy-momentum "tensor". If that was the case he could have patched up Einstein's theory without merely criticising it! See his last paragraph.

A: -kappa^{-1} G^mu_nu must be the energy-momentum density tensor of the gravitational field. The reasons for this are given as it fulfills all the necessary requirements (listed items) in the section "Discussion" of my article. I show here now the 00-component of Einstein's field equations in mixed-tensor representation:

Lambda/kappa = -rho*c^2 - 1/kappa*G^0_0.

It is absolutely clear, that T^0_0 = -rho*c^2 is the energy density of matter. The left hand side of this equation, Lambda/kappa, is a constant (constant parameter, which has a unit of measurement of an energy density). Consequently, it is time independent. It therefore must be the total energy density. Consequently, -1/kappa*G^0_0 must be the energy density of the gravitational field. This is a logical consequence.

This is a conservation law of energy! Because of that, one in principal mustn't neglect the cosmological term in Einstein's field equations. If one does this, one violates this conservation law. Nonetheless, it can be done because the contribution of the cosmological term is tiny. So, Einstein's field equations without the cosmological term are an approximation. The cosmological term explains dark energy. Within the framework of my article it is also able to explain the dark matter phenomenon. Also the cosmological constant problem is solved.

Referee 1 still does not understand, that if one a priori consideres a flat space-time, then there must be no masses, which curve space-time. Exactly this is written in my article: "In absence of any matter, . . . ". This also implies

T_{mu nu} = 0 and G_{mu nu} = 0 from which immediately follows Lambda = 0. No doubt.

R: No. The vanishing of the Einstein tensor, or what amounts to the same, the vanishing of the eigenvalues of the Ricci tensor, is Einstein's condition of emptiness! Schwarzschild used this to derive his outer solution in 1916. The central mass enters only when the arbitrary constant of integration is identified as a mass in the asymptotic limit where Newton's potential appears. This is not part of the solution. In Barrett O'Neill's words, the central mass is "not to be modelled." Yet, the Schwarzschild solution is not a flat metric! The author considers the presence of matter as the source for the non-Euclidean nature of the metric. According to Dirac, the gravitational field does not affect the emptiness of the universe, but all other fields do. This is a basic point which the author should retract for his own sake. Whether or not it is logical is another matter.

A: Einstein's field equations without the cosmological term are just an approximation. The Einstein tensor must be balanced with the cosmological term, because in empty spacetime, the total energy-momentum density tensor equals the energy-momentum density tensor of the gravitational field. If one does not this, one violates the conservation law. It can be done because the cosmological constant is tiny, but then it is only a good approximation.

R: The divergence of the stress-energy tensor is sufficient to guarantee the conservation of mass without any additional term, cf. Landau and Lifshitz, Fluid Mechanics, the chapter on general relativity. He can add whatever he likes to the energy-stress tensor, however neglecting gravitational energy, it accounts for all other forms of energy without introducing the mixed Einstein tensor.

A: nabla^mu T_{mu nu} = 0 means mass conservation. The conservation law which I found in my article means conservation of total energy and total momentum.

R: For the life of me, I still don't understand why he insists on the mixed representation when the metric tensor g_ij is contained in G_ij as well as its first and second derivatives. It's the latter which is missing in the pseudo tensor, which is a bilinear product of Christoffel symbols, and hence, not a real tensor. Thus, the author wants to replace the pseudo tensor by the mixed tensor G. This will not go over well with the relativists. Question: Does the pseudo-tensor appear in the Einstein equations, or only in the divergence expression? According to 't Hooft, the addition of the pseudo-tensor to the Einstein equations is "blatantly wrong." But, then, he is no authority on general relativity.

A: With pseudotensors occur the problems which are mentioned in the introduction of my article.

R: The author has still to answer where in the 1917 article by Levi-Civita does he mention the mixed tensor G as a panacea for accounting for gravitational energy in the Einstein equations. He can claim whatever he wants, but, in the end, he has to bring home the bacon! I haven't read the previous article, but I suspect it contains similar assertions.

A: I wrote in my article:

"Levi-Civita correctly suggested already in April 1917, that the Einstein tensor is proportional to the energy-momentum density tensor of the gravitational field ... "

You can read that the Einstein tensor in mixed-tensor representation (up to a constant) was suggested by Levi-Civita to be the energy-momentum density tensor of the gravitational field. Please have a look here:

https://einsteinpapers.press.princeton.edu/vol8-trans/392

Einstein understood, that Levi-Civita meant the Einstein tensor in mixed-tensor representation.

R: But, it is precisely ref. 13 that the author does not address, although he references Einstein's response. That response criticizes Levi-Civita's interpretation and the author's eqn 10 (without the cosmological constant).

Before entering into Einstein's argument, let me address the Schwarzschild solution. There T_mu^nu=0, which is Einstein's condition of emptiness. It is also static so that there is no acceleration in that frame, and yet, G_mu^nu would be non-zero, and, according to the author, must obey a conservation of energy equation. Conservation of energy of what I would ask?

Now to Einstein's letter. He specifically criticizes

T_4^4 + G_4^4 = 0

According to Einstein

"But with such an approach it is completely incomprehensible why such a thing as an energy law exists in spaces in which gravitation can be neglected."

He uses the example of a pendulum in motion in two reference frames K and K', where the first is static and the second is accelerating. In K, G_4^4 would be absent while in K' it would be present. In the latter, the "body would be able to cool down without emitting heat outwards." The energy equation allows T_4^4 to diminish only the comprenstation in G_4^4 to which Einstein argues "the absolute value of the quantity G_4^4 does not fall within physical observation. That is why I contend that what you call an energy has nothing to do with what is otherwise known as such a law in physics."

Einstein goes on to replace G_nu^mu with his pseudo-tensor t_nu^mu because the consequences that are drawn from the conservation of energy "are correct independently of whether one concedes that the t_nu^mu are 'real' energy components of the gravitational field. For my deduction it is only necessary that T_4^4 be the energy density of the mass, which neither of us doubt."

In essence, associating gravitational energy with a true tensor has problems that would raise havoc with general relativity. It could not disappear by a mere change in reference frames, and the field at infinity must be determined exactly as it is in Newton's theory, that is through the masses. In Einstein words, "it seems to me beyond doubt that (in the static case) the field at infinity must be fully determined by the energy of the mass and the gravitational field together. This fits with my interpretation of the t_nv^mu's" which vanish a spatial infinity.

Thus, associating the gravitational field, which can be nullified by a change in reference frame cannot be associated with a real tensor. Alternatively, the pseudo-tensor can be made to vanish by a mere change in the frame of reference, and, according to Einstein, the "energy components of the gravitational field should just be dependent on the first derivatives of the g_nu mu's because it is also valid for the forces exerted by the fields. There are no first order tensors (dependent on the g_nu mu and their first derivatives) however."

I can take exception of Einstein's referral to forces, because they don't exist in GR, except at spatial infinity where for weak fields Newtonian gravitation must apply.

This is the essence of the criticism to which the author does not address.

A: I wrote in my article: "Levi-Civita correctly suggested already in April 1917, that the Einstein tensor is proportional to the energy-momentum density tensor of the gravitational field [12], ...". In this statement, there is no mixed-tensor representation mentioned. It was understood by Einstein, that the Einstein tensor in mixed-tensor representation was meant, see Ref. [13]. I understand it in the same way.

It is shown in my article, that Levi-Civita's interpretation is indeed correct. Moreover, the cosmological term is necessary. Thereby, Einstein's objections become redundant.

In empty space-time, the energy-momentum density tensor of the gravitational field must be equal to the total energy-momentum density tensor, -kappa^{-1} G^mu_nu = kappa^{-1} Lambda delta^mu_nu.

-kappa^{-1} G^mu_nu being the energy-momentum density tensor of the gravitational field causes no problems in contrast to pseudotensors. These are shown in the introduction of my article. In case of free fall for example, the gravitational energy density must not vanish. But exactly this would be in case with pseudotensors. Without having gravitational energy density, there is no reason for the existence of free fall.

Newton's gravitational theory is just an approximation. It cannot be used as a reference theory for being correct at infinity, because this has not been verified. We know, that it works properly in our solar system, but at larger distances it definitely fails. This can immediately be seen by computing the rotation curves of spiral galaxies. In order to obtain the observed results, Newtonian gravitational theory has to be modified. Such a modification is done within the framework of MOND theory (Modified Newtonian Dynamics), which also is discussed controversally. By including the necessary cosmological term into Einstein's theory, a modification of the Newtonian Poisson equation is obtained, Laplace Phi = 4 pi G (rho - 2 rho_Lambda), wherein the cosmological constant occurs, Lambda = kappa*rho_Lambda*c^2. With the inclusion of the cosmological term, i.e. by using this modified Newtonian Poisson equation, one obtains FLAT rotation curves of spiral galaxies, see Ref. [10]. This reasonable modification also questions the controversally discussed MOND theory.

R: In my opinion the author has not made any changes in the text, and, as it stands, it is unacceptable.

A: In my opinion, the referee has a wrong view.

The author agrees to change the title of this article from "The Fundamental Conservation Law in the Theory of General Relativity" to "The Fundamental Conservation Law in the Theory of General Relativity: an unconventional approach is feasible and correct?". The author hopes with this change, that relativists think about the derivation and the results, which are obtained in this article. Also the implications are important, see Ref. [10]. It is definitely necessary, that relativists rethink and hopefully finally change their view.

The author thinks, that further changes in the text are not necessary since there occur no mistakes to his best knowledge.

R: His motto is "blame it on a constant." And that constant is the cosmological constant which can be either negative or positive. In the vacuum where it should be most easily observed, it plays the role of an energy density, but this energy is unacceptable since the pressure is negative. According to the Wikipedia article (which I don't put too much faith in, in general, "If the energy density is positive, the associated negative pressure will drive an accelerated expansion of the universe, as observed."

A: The referee still does not understand, that there are different cosmological constants with respect to different metrics. The author found out, that the cosmological constant is not related to the vacuum energy density or a negative pressure, respectively. These explanations are nonsense, unphysical. Moreover, the huge difference between theory and observation can never be overcome by keeping such an unrealistic hypothesis!

R: The author keeps on claiming that "In free fall the gravitational energy must not vanish, Without having gravitational energy density, there is no reason for free fall."

A: Exactly.

R: This contradicts–to say the least–Einstein's equivalence principle. An accelerometer does not measure any acceleration when in free fall. Free fall is really a local inertial reference frame! The author failed to discuss Einstein analogy with a pendulum in static and accelerating frames. The gravitational field can be made to vanish simply by a change the frame of reference, and this is why it can't be identified with the Einstein tensor. This is reflected in the annihilation of the pseudo- tensor by a coordinate transform–it is tantamount to the denial of the localization of energy!

A: It does not contradict Einstein's equivalence principle. Pseudotensors do not solve the problem, instead they make them. Please reread the introduction of this article.

The referee correctly mentions, that there is no gravitational field strength, i.e. vanishing Christoffel symbols, in the free-falling reference frame, where there is locally a Minkowski metric eta_{mu nu}, so that

G^(local)_{mu nu} = -Lambda eta_{mu nu}.

By using the Jacobean, one makes a coordinate transformation from the local reference frame (that is without a gravitational field, i.e. Christoffel symbols vanish) to that one of the earth (which possesses a gravitational field, i.e. Christoffel symbols do not vanish),

G_{mu nu} = -Lambda g_{mu nu},

where g_{mu nu} is the earth's metric. This immediately shows that there exists the same value for the gravitational field energy density in both reference frames as it should be. This makes absolutely sense. By using pseudotensors, there neither is a gravitational field nor gravitational energy density in the local reference frame, and therefore no reason for a free fall.

R: The author would have us believe that the Einstein tensor (in mixed representation as he would like to believe) is a real energy "density" like that of the electromagnetic field. It isn't because such a density would be negative, an aspect that turned Maxwell away from applying his theory, or an extension of it, to the gravitational field.

A: The referee is definitely wrong. The author already has shown, that T^0_0 = - rho c^2 is the energy density of matter. Also Lambda/kappa is an energy density, and because its time derivative vanishes it must be the total energy density. Consequently, -1/kappa G^0_0 must be the energy density of the gravitational field. This is so obvious. Is this so difficult to understand?

R: Consider a charged, verus neutral, ball of matter. In the former the electromagnetic energy is distributed throughout space according to Maxwell's equations. Where is the (gravitational) energy in the latter found, or doesn't it have an energy at all? The neutral ball is in a vacuum with no other masses.

A: In principle, one can also implement electromagnetic fields into Einstein's GR theory. Therefore, just put the electromagnetic energy-momentum density tensor inside the energy-momentum density tensor of matter. The latter one is defined as the tensor, wherein all contributions to energy and momentum appear except those ones, which belong to the gravitational field. The author used in this article a very simple form of the energy-momentum density tensor of matter namely that one of a perfect fluid. However, this simplification is completely adequate for the study. More complicate energy-momentum density tensors of matter, such which contain electromagnetic fields for example, are out of the scope of this work.

R: Levi-Civita argued for a strict analogy with d'Alembert's principle. The sum of the mass tensor T_ij and the inertial-gravitational tensor G_ij vanishes identically. According to him "This fact entails a total lack of stresses, of energy flow, and also of simple energy localization."

This would necessarily exclude a priori the physical significance of gravitational radiation and other purely gravitational phenomena (like free fall), In other words, general relativity does not allow any intrinsic, tensorial definition of gravitational energy.

This is to say that GR is not free of contradictions. While admitting the existence of a pseudo tensor (when when expressed in Kerr-Schild form in which all its components vanish globally), Einstein goes on to calculate the loss of energy due to gravitational waves using an electromagnetic formalism where the rate of energy loss is set equal to the time rate of change of the mass quadrupole tensor.

A: In this point Levi-Civita is wrong because the sum must be the total energy-momentum density tensor, which only vanishes in absence of any matter, i.e. total emptiness.

R: Even if I were to remove all reservations, the author has disregard for an entire history of the problem. It started with Lorentz (1916) and Levi-Civita (1917) but it was carried further by Tolman, Eddington, Bondi, and a whole host of other authors. Some have argued that energy can be localized only for spherically symmetrical sources. More recently, my deceased friend, Fred Cooperstock argued that energy exists only in regions of nonvanishing energy-momentum tensor. See his book, with Tau, "Einstein's Relativity" for a list of other references. His view would be in contradiction with Feynman's sticky bead argument that claims gravitational waves can generate heat.

A: Here, I agree with the referee, because I didn't show the whole history of this problem. The reason for this is, that I simply didn't want to do this, because this is not the main task of this article. The author only wants to show the reader "*Where Physics Went Wrong*".

R: Finally, the author has refused to correct his claim that in the absence of T_ij, GR reduces to SR. This is clearly wrong for it would have given Schwarzschild no means of calculating the metric tensor since all the eigenvalues of the Ricci tensor would be identically zero.

A: The referee still does not understand what I mean. There is a big difference between the designation "empty space-time", which is used by considering the empty space-time around a star for example, and the designation "in absence of any matter" whereby the author means total emptiness.

R: I would give him the benefit of the doubt. If he comes up with an example where his equation (5) is shown to work, then it would be extremely interesting. There is nothing holy about GR, and you can find fault with anything you add on to Einstein's equations to take gravitational energy into account. Rather, if he wants to question whether there is a conservation of energy, or a localization of energy, in GR, he should give the history of this and give his reasons why he thinks there is..

A: Eq. (5) works in any reference frame. This equation contains the energy-momentum density tensor of the gravitational field, that is shown in Eq. (8), wherein occurs the Einstein tensor, the covariant divergence of which vanishes. Consequently, nabla_mu A^mu_nu = 0. Eq. (5) also consists of the energy-momentum density tensor of matter. Its covariant divergence also vanishes.

Two simple examples, where Eq. (5) works, are: the FLRW metric and the inner de Sitter-Schwarzschild metric [10]. By considering the FLRW metric, nabla_mu T^mu_0 = 0 implies mass conservation, dot{rho} + 3 dot{a}/a (rho + P/c^2) = 0.

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