Critical line of nontrivial zeros of Riemann zeta function $\zeta(s)$

José Alcauza*

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Abstract

In this paper, we find a curious and simple possible solution to the critical line of nontrivial zeros in the strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$ of Riemann zeta function $\zeta(s)$. We show that exists $s_\sigma \in \mathbb{C}$ such that if $\{s_\sigma = \sigma + it : (\sigma \in \mathbb{R}, 0 < \sigma < 1); (\forall t \in \mathbb{R})\}$ with $i$ as the imaginary unit, then exactly satisfy:

$$\lim_{s \to s_\sigma} \zeta(s) = \zeta(s_\sigma) = 0 \quad \Rightarrow \quad s_\sigma = \frac{1}{2} + it$$

Therefore, all the nontrivial zeros lie on the critical line $\{s \in \mathbb{C} : \Re(s) = \frac{1}{2}\}$ consisting of the set complex numbers $\frac{1}{2} + it$, thus confirming Riemann’s hypothesis.

1 Introduction.

There is a large and extensive bibliography on the Riemann zeta function and its zeros, so we will not go into further details of it. Basically, Riemann zeta function is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$ by the absolutely convergent infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

Leonhard Euler already considered this series for real values of $s$. He also proved that it equals the Euler product:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

where the infinite product extends over all prime numbers $p$. However, we can also define the Riemann zeta function Eq.(1) as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \quad \Rightarrow \quad \zeta(s) = \frac{1}{2^s} \left( \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^s} \right)$$

Which can also be expressed as:

$$\zeta(s) = \frac{1}{2^s} \left[ \zeta(s) + B(s) \right] \iff B(s) = \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^s} \quad (2)$$

Thus, by Eq.(2) we can definitely express the Riemann zeta function as:

$$\zeta(s) = (2^s - 1)^{-1} B(s) \quad (3)$$

*alcauza.jose@gmail.com
As is well known, the Riemann zeta function $\zeta(s)$ and the Dirichlet eta function $\eta(s)$ satisfy the relation:

$$\eta(s) = (1 - 2^{1-s}) \zeta(s)$$

Thus, by Eq.(3) we can now express the Dirichlet eta function as:

$$\eta(s) = \left(\frac{1 - 2^{1-s}}{2^s - 1}\right) B(s)$$

### 2 Proof.

By Eq.(2), Eq.(4) and Eq.(5) we can obtain:

$$2^{1-s} = 1 - \frac{\eta(s)}{\zeta(s)} = 2^s \cdot \frac{\zeta(s) - \eta(s)}{\zeta(s) + B(s)} \quad \Rightarrow \quad 2^{1-2s} = \frac{\zeta(s) + \left(\frac{2^{1-s}-1}{2^s-1}\right) B(s)}{\zeta(s) + B(s)}$$

However, exists $s_\sigma \in \mathbb{C}$ such that $\{ s_\sigma = \sigma + it : (\sigma, t) \in \mathbb{R} \}$ with $i$ as the imaginary unit, such that exactly satisfy:

$$\lim_{s \to s_\sigma} \zeta(s) = \zeta(s_\sigma) = 0$$

Therefore, calculating $(\lim_{s \to s_\sigma})$ in Eq.(6), we obtain:

$$\lim_{s \to s_\sigma} (2^{1-2s}) = \lim_{s \to s_\sigma} \frac{\zeta(s) + \left(\frac{2^{1-s}-1}{2^s-1}\right) B(s)}{\zeta(s) + B(s)} \quad \Rightarrow \quad 2^{1-2s_\sigma} = \frac{\zeta(s_\sigma) + \left(\frac{2^{1-s_\sigma}-1}{2^s-1}\right) B(s_\sigma)}{\zeta(s_\sigma) + B(s_\sigma)}$$

However, since $[B(s_\sigma) \to 0 \iff \zeta(s_\sigma) \to 0]$ by Eq.(3), we obtain an indeterminacy of the type $\frac{0}{0}$. Then by successive applications of the L’hôpital rule until any $n$th derivative $B^{(n)}(s_\sigma) \neq 0$, that is: $(\forall j < n : B^{(j)}(s_\sigma) = 0)$, we obtain:

$$2^{1-2s_\sigma} = \frac{\left(\frac{2^{1-s_\sigma}-1}{2^s-1}\right) B^{(n)}(s_\sigma)}{B^{(n)}(s_\sigma)} \quad \Rightarrow \quad 2^{1-2s_\sigma} = \frac{2^{1-s_\sigma}-1}{2^s-1}$$

As $s_\sigma = \sigma + it$ then obtaining common factor $2^{-it}$ in numerator and $2^it$ in denominator of the fraction, we can express:

$$2^{1-2s_\sigma} = 2^{-2it} \cdot \frac{2^{1-\sigma} - 2^it}{2^\sigma - 2^{-it}}$$

Now, defining $s_0 \in \mathbb{C}$ such that $s_0 = \frac{1}{2} + it$, we can express previous equation as:

$$2^{2(s_\sigma - s_0)} = \frac{2^\sigma - 2^{-it}}{2^\sigma - 2^it}$$

Since by definition $s_\sigma = \sigma + it$ and $s_0 = \frac{1}{2} + it$ then $2(s_\sigma - s_0) = 2\sigma - 1$. Thus, developing in trigonometric form $2^it = e^{it\ln2}$ and $2^{-it} = e^{-it\ln2}$, we obtain:

$$2^{(2\sigma-1)} = \frac{2^\sigma - \cos(t\ln2) + is\eta(t\ln2)}{2^{1-\sigma} - \cos(t\ln2) - is\eta(t\ln2)}$$

since obviously as we know $\cos(-x) = \cos(x)$. Thus, by simplifying we have:

$$2^{(2\sigma-1)} = \frac{\cos(t\ln2) - is\eta(t\ln2)}{\cos(t\ln2) + is\eta(t\ln2)}$$
which by application of modulus, that is:

\[ |2^{(2\sigma - 1)}| = \frac{|\cos(tln2) - isen(tln2)|}{|\cos(tln2) + isen(tln2)|} \Rightarrow |2^{(2\sigma - 1)}| = \frac{|\cos(tln2) - isen(tln2)|}{|\cos(tln2) + isen(tln2)|} \]

since for any \( \{ z \in \mathbb{C} : |z| = |\overline{z}| \} \) we definitely obtain:

\[ |2^{(2\sigma - 1)}| = 1 \Rightarrow 2\sigma - 1 = 0 \Rightarrow \sigma = \frac{1}{2} \]

Therefore, since by definition \( s_\sigma = \sigma + it \), we obtain that for:

\[ \zeta(s_\sigma) = 0 \Rightarrow s_\sigma = \frac{1}{2} + it \]

Exactly, by Eq.(7) and Eq.(8) for \( \sigma = \frac{1}{2} \) we can verify:

\[ |2^{2(s_\sigma - s_0)}| = \left| \frac{(2^{\frac{1}{2}} - \cos(tln2)) + isen(tln2)}{(2^{\frac{1}{2}} - \cos(tln2)) - isen(tln2)} \right| = 1 \]

Thus, since by definition \( s_0 = \frac{1}{2} + it \), we have:

\[ 2(s_\sigma - s_0) = 0 \Rightarrow s_\sigma = s_0 \Rightarrow s_\sigma = \frac{1}{2} + it \]

Thus, all the nontrivial zeros lie on the critical line \( \{ s \in \mathbb{C} : \Re(s) = \frac{1}{2} \} \) consisting of the set complex numbers \( \frac{1}{2} + it \), thus confirming Riemann’s hypothesis.