REPRESENTATIONS FOR β(10)

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Abstract. Here two novel expressions of the Dirichlet Beta function at 10 are provided.

The Dirichlet Beta function is defined by the sum

$$\beta(m)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^m}, \qquad \left.\begin{array}{c} n=1,2,3,\cdots \\ m=1,2,3,\cdots \end{array}\right\} \in \mathbb{N}. \tag{1}$$

When m are even positive integers, no one knows the defined formula above whether there are/exist closed-form solutions.

As a typical example of β(even), β(2), commonly appears in the form of infinite series and definite integrals, while other β(4), β(6), β(8) and β(10) etc. can rarely be seen.

Obviously, β(10) is the fifth one of the Dirichlet Beta function at even. Two novel expressions with infinite series for β(10) derived in last year are first shown below (the formulas are a little long, see Appendix for more details)

$$\beta(10)=\pi^9\left(\begin{array}{c}\dfrac{12497299839061}{43919715926016000}-\dfrac{25780953733\,\ln 2}{121999210905600}-\dfrac{1734634309\,\ln\pi}{17428458700800}+ \\ +\displaystyle\sum_{n=1}^{\infty}\frac{\left\{35458290+(2n+9)\left[\begin{array}{c}(11785005)\,2^{4n+13} \\ +\boxed{1}-(135)\,2^{18}\end{array}\right]\right\}|B_{2n}|\,\pi^{2n}}{(17729145)\,2^{2n+11}\,n\,(2n+9)!}\end{array}\right), \tag{2}$$

and

$$\beta(10)=\pi^9\left(\begin{array}{c}\dfrac{8683836877}{25819938816000}-\dfrac{601631911\,\ln 2}{903697858560}+\dfrac{243\,\ln 3}{2293760}+\dfrac{11659327\,\ln\pi}{215166156800} \\ +\displaystyle\sum_{n=1}^{\infty}\frac{\left\{(2n+9)\left[\begin{array}{c}(3943635)\,2^{4n+13} \\ -\boxed{2}+(15)\,2^{18}\end{array}\right]-(1313270)3^{2n+9}\right\}|B_{2n}|\,\pi^{2n}}{(656635)\,2^{2n+11}\,n\,(2n+9)!}\end{array}\right), \tag{3}$$

where, B_{2n} are Bernoulli numbers, and

$$\boxed{1}=(2n+6)(2n+7)(2n+8)\,2^{2n}[(10809433)2^{2n+4}-24337685],$$

$$\boxed{2}=(2n+6)(2n+7)(2n+8)\,2^{2n}[(50968720)2^{2n}+41016827].$$

Comparison of Calculation Results

№	Source (Come from)	Formulas	Value with first 10 terms	Value with first 20 terms
0	β(10) is approximately equal to (17 exact decimal digits)		β(10) = 0.999983164026197	β(10) = 0.999983164026197
1	Definition	(1)	≈0.99998316402615 4	≈0.99998316402619 69
2	in this paper	(2)	≈0.9999750889381567	≈0.99998279795 62427
3		(3)	≈1.0000451671698933	≈0.9999860566703379

Remark:

(a) The definition expression of $\beta(10)$ converges rapidly;

(b) Formulae (2) and (3) are for research purposes, not primarily for calculations. Their accuracy increases as the number of series terms increases.

(c) The red digits in the table above are the exact digits of the every computed value for $\beta(10)$ by comparing to its exact decimal digits.

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Appendix

Formula (2) for β(10)

$$\beta(10) = \pi^{9}\cdot\left[\frac{12497299839061}{43919715926016000} - \frac{1734634309\ln(\pi)}{17428458700800} - \frac{25780953733\ln(2)}{121999210905600} + \sum_{n=1}^{\infty}\frac{\left[(7\cdot 18761)\cdot 6\cdot 45 + \dfrac{\left[(26188945)\cdot 2^{4n+13} + 2^{2n}\cdot[(2n+6)\cdot(2n+7)\cdot(2n+8)]\cdot\left[(10809433)\cdot 2^{2n+4} - 243357685\right] - (135)\cdot 2^{18}\right]}{(2n+9)^{-1}}\right]\cdot\left|B_{n}\right|\cdot\pi^{2n}}{945\cdot(18761)\cdot 2^{2n+11}\cdot n\cdot(2n+9)!}\right]$$

Formula (3) for β(10)

$$\beta(10) = \pi^{9}\cdot\left[\frac{8683836877}{25819938816000} - \frac{601631911\ln(2)}{903697858560} + \frac{243\cdot\ln(3)}{2293760} + \frac{11659327\ln(\pi)}{215166156800} + \sum_{n=1}^{\infty}\frac{\left[\dfrac{\left[(15\cdot 262909)\cdot 2^{4n+13} - 2^{2n}\cdot(2n+6)\cdot(2n+7)\cdot(2n+8)\cdot\left[(50968720)\cdot 2^{2n} + 41016827\right] + 15\cdot 2^{18}\right]}{(2n+9)^{-1}} - 70\cdot(18761)\cdot 3^{2n+9}\right]\cdot\left|B_{n}\right|\cdot\pi^{2n}}{35\cdot(18761)\cdot 2^{2n+11}\cdot n\cdot(2n+9)!}\right]$$