On the refractive index-curvature relation

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In a two-dimensional space, a refractive index-curvature relation is formulated using the second rank tensor of Ricci curvature. A scalar refractive index describes an isotropic linear optics. In a fibre bundle geometry, a scalar refractive index is related to an Abelian (a linear) curvature form. The Gauss-Bonnet-Chern theorem is formulated using a scalar refractive index. Because the Euler-Poincare characteristic is the topological invariant then a scalar refractive index is also a topological invariant.

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In the geometrical optics, the refractive index-curvature relation derived from the Fermat’s principle describes ray propagation in a steady (time-independent) state. The refractive index-curvature relation can be written as

$$\frac{1}{R} = \nabla \ln n(r)$$

where $R$ is a radius of curvature, $\nabla$ is an unit vector along the principal normal or has the same direction with $\nabla \ln n(r)$, $\nabla \ln n(r)$ means the gradient of a function ln n at a point r and n(r) is a space-dependent refractive index, a scalar function of the coordinates only (a smooth continuous function of the position). We see eq.(1) is a dot product of two vectors, so the result gives a scalar quantity, $\nabla \ln n(r) = \sum_{\dim = 1}^{12} N_i \nabla_i \ln n(r)$, $\dim$ is a number of dimension of space. Eq.(1) tells us that the rays are therefore bent in the direction of increasing refractive index.

In a 2-dimensional space, we write eq.(1) as

$$R_{\mu\nu} = g_{\mu\nu} N(\mu \partial_{\nu}) \ln n$$

where $R_{\mu\nu}$ is the second rank tensor of Ricci curvature, $N(\mu \partial_{\nu})$ is a scalar density, $g_{\mu\nu}$ is the metric tensor, $g = |(\det g_{\mu\nu})|$ is a real number, and $\mu, \nu$ run from 1 to 2. We write $N(\mu \partial_{\nu})$ in eq.(2) to accomodate the symmetry property of the second rank tensor of Ricci curvature, $R_{\mu\nu} \equiv R_{\nu\mu}$, where $N(\mu \partial_{\nu}) = \frac{1}{2}(N_{\mu \partial_{\nu}} + N_{\nu \partial_{\mu}})$.

The zeroth rank tensor (a scalar, a real number) of the refractive index (1), (2) describes an isotropic linear optics. But, the refractive index can be not simply a scalar. The refractive index can also be a second rank tensor which describes that the electric field component along one axis may be affected by the electric field component along another axis. The second rank tensor of the refractive index describes an anisotropic linear optics.

The geometrical optics can be derived from the Maxwell’s theory, an Abelian U(1) local gauge theory. That is why, in this article we also treat the geometrical optics as an Abelian U(1) local gauge theory. We will formulate a curvature in a fibre bundle. Is there a relationship between a fibre bundle and a gauge theory? Originally, a fibre bundle and a gauge theory are developed independently. Until it was realized that the curvature (in a fibre bundle) and the field strength (in Yang-Mills theory) are identical.

Why do we need to formulate the curvature in a fibre bundle instead of the Riemann-Christoffel curvature tensor? As a consequence of the geometrical optics is treated as an Abelian U(1) local gauge theory, so we need to formulate the curvature in a fibre bundle as what we call an Abelian (a linear) curvature form. A curvature form in a fibre bundle can be an Abelian or a non-Abelian (a non-linear). It differs with the Riemann-Christoffel curvature tensor which has the non-linear form only.

The curvature form, $\Omega_{\alpha\mu}$, in a fibre bundle can be written as

$$\Omega_{\alpha\mu} = \sum R_{\alpha\mu\beta\nu} \, du^\beta \wedge du^\nu$$

where $R_{\alpha\mu\beta\nu}$ is the fourth rank tensor of Riemann-Christoffel curvature (which has the algebraic properties as symmetry, anti-symmetry and cyclicity), $u^\alpha, u^\beta$ are local coordinates and $\wedge$ is a notation of the exterior (wedge) product (it satisfies the distributive, anti-commutative and associative laws). $\Omega_{\alpha\mu}$ is an anti-symmetric matrix of 2-forms.

If we reformulate eq.(3) using eq.(2) and the Ricci-Riemann relation in a 2-dimensional space, $R_{\alpha\mu\beta\nu} = (g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\mu\beta}) \frac{\partial \omega}{\partial x^\alpha}$, then we obtain

$$\sum (g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\mu\beta}) \, N(\mu \partial_{\nu}) \ln n \, du^\beta \wedge du^\nu = \Omega_{\alpha\mu}$$

Eq.(4) shows the relationship between the scalar refractive index and the curvature form in a 2-dimensional space. We see that the scalar refractive index "lives" in a 2-dimensional space.

Let us introduce a general form of the curvature matrix, $\Omega$, which is a matrix of exterior two-forms below

$$\Omega = dw - \omega \wedge \omega$$

where \( \omega \) is the connection matrix, one-form\(^{23,24} \). We see that \( \text{eq.}(5) \) is a non-linear equation due to the second term of the right hand side of \( \text{eq.}(5) \).

Can the curvature matrix equation \( (5) \) be an Abelian, a linear equation? A gauge potential, \( A \), can be regarded as a local expression for a connection in a principal bundle\(^{23} \) as written below

\[
A = \sigma^* \omega \tag{6}
\]

where \( \sigma \) is a local section defined on a chart \( U \) of manifold, base space, \( M \). The local form of the curvature is defined by\(^{23} \)

\[
\mathcal{F} \equiv \sigma^* \Omega \tag{7}
\]

where \( \mathcal{F} \) is identified with the field strength. In a general case, from Cartan’s structure equation, we find\(^{25} \)

\[
\mathcal{F} = \sigma^* (d \sigma^* \omega + \sigma^* \omega \wedge \sigma^* \omega) = dA + A \wedge A \tag{8}
\]

where \( d \) is the exterior derivative on \( M \). We see from eqs.(7), (8) that

\[
\Omega = d \sigma^* \omega + \sigma^* \omega \wedge \omega \tag{9}
\]

and

\[
\sigma^* d \sigma^* \omega = d \sigma^* \omega \tag{10}
\]

In a special case, for an Abelian \( U(1) \) local gauge theory, using eq.(6) and the fact that the exterior derivative obeys the Leibniz rule\(^{25} \), \( \mathcal{F} \) can be expressed in terms of the gauge potential \( A \)\(^{23} \) as below

\[
\mathcal{F} = dA
\]

\[
\sigma^* d \sigma^* \omega = d (\sigma^* \omega) = d \sigma^* \omega + \sigma^* d \omega \tag{11}
\]

Eq.(11) implies

\[
\Omega = d \sigma^* \omega \tag{12}
\]

Notation \( d_p \) means the covariant derivative of a vector valued one-form on a principal bundle, \( P(M, G) \), \( G \) is structure group\(^{23} \). We see that eq.(12) is an Abelian, a linear equation.

Let us consider \( dA \) in eqs.(8), (11). \( dA \) in such both equations should be the same or in other words as a consequence of eq.(10), \( d \omega \) in eq.(11) should be zero

\[
d \omega = 0 \tag{13}
\]

It means that the connection matrix, one-form, \( \omega \), is closed if \( d \omega = 0 \).\(^{23,26,27} \)

\[\text{Can we see something interesting in eq.(10)?} \]

We see that \( \text{eq.}(10) \) is analog with the Stokes theorem which can be written roughly\(^{17} \) as

\[
\int_D d \omega = \int_{\partial D} \omega \tag{14}
\]

So, we could say that \( d \omega = 0 \) is a consequence of the Stokes theorem. Using the Stokes theorem (14), we see that \( d \omega = 0 \) has the same meaning with \( \omega \) is closed, i.e. \( \partial D = 0 \). What does \( d \omega = 0 \) imply in physics? Can \( d \omega = 0 \) be related to a conserved quantity in physics?

Is there a relationship between the curvature matrix, \( \Omega \) (5), and the curvature form, \( \Omega_{\alpha \mu} \) (3)? Yes (there is)\(^{28} \). If \( \Omega_{\alpha \mu} \) and \( \omega_{\alpha \mu} \) denote the components of curvature and connection matrices, \( \Omega \) and \( \omega \), respectively then we can write\(^{16} \)

\[
\Omega = (\Omega_{\alpha \mu}), \quad \omega = (\omega_{\alpha \mu}) \tag{15}
\]

So, the curvature matrices in eqs.(9), (12) can be written using the curvature form \(^{17} \) respectively as below

\[
\Omega_{\alpha \mu} = d \omega_{\alpha \mu} - \omega_{\alpha \tau} \wedge \omega_{\mu \tau} \tag{16}
\]

and

\[
\omega_{\alpha \mu} = d \omega_{\alpha \mu} \tag{17}
\]

We call eq.(17) as an Abelian (a linear) curvature form equation.

As we mentioned that we treat the geometrical optics as an Abelian \( U(1) \) local gauge theory, so we choose the curvature form (17) to describe the geometrical optics. By substituting eq.(4) into eq.(17), we obtain

\[
\sum (g_{\alpha \beta} g_{\mu \nu} - g_{\alpha \nu} g_{\mu \beta}) N(\mu, \partial \nu) \ln n \, du^\beta \wedge du^\nu = d \omega_{\alpha \mu} \tag{18}
\]

We call eq.(18) as an Abelian curvature form-scalar refractive index relation.

Let us define the pfaffian\(^{29} \) of the curvature matrix, pf \( \Omega \), as below\(^{16,30} \)

\[
\text{pf} \, \Omega = \sum \epsilon_{\alpha_1 \mu_1 \ldots \alpha_q \mu_2} \Omega_{\alpha_1 \mu_1} \wedge \ldots \wedge \Omega_{\alpha_q \mu_2} \tag{19}
\]

where the curvature matrix, \( \Omega \), is any even-size complex \( 2q \times 2q \) anti-symmetric matrix (if \( \Omega \) is an odd-size complex anti-symmetric matrix then the corresponding pfaffian is defined to be zero), \( \epsilon_{\alpha_1 \mu_1 \ldots \alpha_q \mu_2 q} \) is the \( 2q \)-th rank Levi-Civita tensor which has value +1 or -1 according as its indices form an even or odd permutation of 1, ..., \( q \) and its otherwise zero, and the sum is extended over all indices from 1 to \( 2q \), \( q \) is a natural number. Here, \( \alpha_1 < \mu_1 \), ..., \( \alpha_q < \mu_2 q \) and \( \alpha_1 < \alpha_2 < \ldots < \alpha_q \).\(^{16,30} \)

Shortly, the pfaffian of \( \Omega \) (19) can be rewritten as

\[
\text{pf} \, \Omega = \sum \epsilon_{\alpha \mu} \Omega_{\alpha \mu} \tag{20}
\]

By substituting eqs.(17), (18) into eq.(20) we obtain

\[
\sum \epsilon_{\alpha \mu} \sum (g_{\alpha \beta} g_{\mu \nu} - g_{\alpha \nu} g_{\mu \beta}) N(\mu, \partial \nu) \ln n \, du^\beta \wedge du^\nu = \text{pf} \, \Omega \tag{21}
\]

Using the pfaffian of \( \Omega \), the Gauss-Bonnet-Chern theorem\(^{31–33} \) says that\(^{16,32} \)

\[
(-1)^q \frac{1}{2^{2q} \pi^q q!} \int_{M^{2q}} \text{pf} \, \Omega = \chi(M^{2q}) \tag{22}
\]
where $\chi(M^{2q})$ is the Euler-Poincare characteristic\textsuperscript{34,35} (a topological invariant\textsuperscript{16}, a global invariant\textsuperscript{21}) of the even dimensional oriented compact Riemannian manifold, $M^{2q}$. We consider $q$ in $M^{2q}$ is the same as $q$ in the description of pf $\Omega$. We interpret that the size (ordo) of curvature matrix of the corresponding pfaffian is related to the number of a dimension of space (manifold). The size of curvature matrix is the same as the number of a dimension of space.

By substituting eq.(21) into eq.(22), we obtain the Gauss-Bonnet-Chern theorem related to the scalar refractive index as below

\[
(-1)^q \frac{1}{2^{2q} \pi^q q!} \int_{M^{2q}} \sum \epsilon_{\alpha\mu} \\
\sum (g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\mu\beta}) N(\mu, \partial_{\nu}) \ln n \ du^\beta \wedge du^\nu
= \chi(M^{2q})
\]

(23)

In case of a 2-dimensional space, i.e. for $q = 1$, eq.(23) becomes

\[
\frac{1}{4\pi} \int_{M^2} \sum \epsilon_{\alpha\mu} \\
\sum (g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\mu\beta}) N(\mu, \partial_{\nu}) \ln n \ du^\beta \wedge du^\nu
= \chi(M^2)
\]

(24)

We see from eqs.(23), (24), the scalar refractive index is related to the Euler-Poincare characteristic. Because the Euler-Poincare characteristic is the topological invariant\textsuperscript{20,37} (the global invariant\textsuperscript{21}) we consider that the scalar refractive index is also the topological invariant (the local invariant). Eqs.(22), (23), (24) show that the integral of a local topological invariant gives result a global topological invariant.

The pfaffian of the curvature matrix (20) is defined to be zero or non-zero if the curvature matrix is an odd-size or an even-size complex anti-symmetric matrix respectively. In turn, the zero or non-zero curvature form (3) has a consequence that the Riemann-Christoffel curvature tensor is vanish or not vanish respectively. The vanishing Riemann-Christoffel curvature tensor means vacuum space. In other words, the Riemann-Christoffel curvature tensor must vanish in vacuum space\textsuperscript{39}. So, does it mean that the zero or non-zero curvature form is related to vacuum or non-vacuum space (in turn a vanishing or a non-vanishing field strength)?

The zero or non-zero Euler-Poincare characteristic (22) is a consequence of the zero or non-zero pfaffian of the curvature matrix respectively. Does it mean that the zero or non-zero Euler-Poincare characteristic is related to vacuum or non-vacuum space? What is the existence of a topological invariant of the zero Euler-Poincare characteristic or vacuum space?

We see from eq.(13) that the connection matrix, one-form, $\omega$, is closed. What is the meaning of a closed one-form physically? Could we interpret $d\omega = 0$ related to a conserved quantity (conservation law) in physics, especially in the geometrical optics? What is such conserved quantity in the geometrical optics?

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2In one dimension, the curvature tensor $R_{1111}$ always vanishes. In other words, a curved line should have zero curvature. The Riemann-Christoffel curvature tensor reflects only the inner properties of the space, not how it is embedded in a higher dimensional space (Steven Weinberg, Gravitation and Cosmology, John Wiley & Sons, 1972.)


7The dimension of the curvature in eq.(1) can be extended to any arbitrary number of dimensions (see Moshe Carmeli, Classical Fields: General Relativity and Gauge Theory, John Wiley and Sons, Inc., 1982.)


11Roniyus Marjunas, Private communication.


15The Christoffel symbol does not transform as a tensor, but rather as an object in the jet bundle (Wikipedia, Christoffel symbols). If the non-linear term (non-Abelian term) of the Christoffel symbol happens to be zero in one coordinate system, it will in general not be zero in another coordinate system\textsuperscript{16}.


18Anticommutativity is a specific property of some non-commutative operations. In mathematical physics, where symmetry is of central importance, these operations are mostly called antisymmetric operations (Wikipedia, Anticommutative property).

19We consider an anti-commutative (anti-symmetric) property of the wedge product as $du^\alpha \wedge du^\nu \neq du^\nu \wedge du^\alpha$. 

10
An antisymmetric matrix is a square matrix that satisfies the identity $A = -A^T$ where $A^T$ is the transpose matrix. All $n \times n$ antisymmetric matrices of odd size (i.e., if $n$ is odd) are singular (determinant of matrix is equal to zero). Antisymmetric matrices are commonly called "skew symmetric matrices" by mathematicians.

In Nakahara\textsuperscript{23}, the curvature matrix is written as $\Omega = d\omega + \omega \wedge \omega$.


For any smooth 1-form, $\omega$, and smooth vector fields, $X$ and $Y$, on a manifold, the exterior derivative of a 1-form is defined as $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$ (\url{https://idv.sinica.edu.tw/ftliang/diff_geom/*diff_geometry%28II%29/3.11/exterior_derivative_2.pdf}).

The exterior derivative of a connection can be written as $d\omega$. If $d\omega = 0$, it means that $[X,Y]$ is commute, $XY = YX$, as in gauge bosons. In case of QED, i.e., an Abelian U(1) local gauge theory, it has one gauge field, the electromagnetic four potential, with the photon being the gauge boson (Wikipedia, *Gauge theory*).

Shing-Tung Yau, Private communication.