Rooks Sitting on the Chessboard

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January 4, 2022

Abstract

Antique prime number problems are generalized as one family and transformed into quantitative domination. Introducing prime pair, the longest gap in Moiré Pattern, remainder sequence, primorial ring and etc. A makeover view of number as a closed ring.

Prime-Factorizing Arrays

Definition 1. A *number pair* is two integers, whose sum is equal to $2n$.

For example, there is a number-pair-array in $[0, 2n]$:

$$\begin{array}{ccccccc} 0 & 1 & 2 & \dots & n-2 & n-1 & n \\ 2n & 2n-1 & 2n-2 & \dots & n+2 & n+1 & n \end{array}$$

Obviously, there are total of $n+1$ number pairs in $[0, 2n]$.

Definition 2. A *prime pair* is a number pair, for which both members are prime.

For example, suppose $n = 10$, so there is 2 prime pairs in number-pair-array:

$$\begin{array}{cc} 3 & 7 \\ 17 & 13 \end{array}$$

Safe Zone $[0, p_{\dot{z}}^2]$

Any integer greater than 1 can be considered a composition of one or more prime factors. The number of same prime factors is irrelevant. All prime factors are cyclic with constantly equidistant spacing and are distributed in an integral array. There is only a need for every prime factor less than or equal to $\sqrt{2n}$ to sieve in $[0, 2n]$.

Moiré Pattern

Sieving number or number pairs with prime factor p will generate a Moiré pattern which repeats in p length period. Sieving with multiple prime factors can be considered as a certain combination of multiple layers of Moiré pattern.

Remainder Sequence

Definition 3. R_n is a remainder sequence generated by a prime sequence (usually $\{2, 3, 5, 7, ...\}$) dividing n.

$$R_n \quad = \quad \{r_i | r_i \equiv n \pmod{p_i}\}$$

Primorial Ring $\mathring{Z}$

There are not enough number elements in $[0, p_{\dot{z}}^2]$ to complete the full super-pattern. So, we extend the sieved number-array or number-pair-array.

Definition 4. $\mathring{Z} = 2 \cdot 3 \cdot 5 \dots p_{\dot{z}} = p_{\dot{z}}\#$ as a primorial ring which has every possible remainder sequence to complete the full super-pattern,.

Let M be a matrix that:

$$\begin{aligned} n &\equiv R_n M \pmod{\mathring{Z}} \\ -1 &\equiv \{-1,-1,...\}M \\ 0 &\equiv \{0,0,0,...\}M \\ 1 &\equiv \{1,1,1,...\}M \\ 2 &\equiv \{0,2,2,...\}M \\ 3 &\equiv \{1,0,3,...\}M \\ \dots \end{aligned}$$

Let / be integer division operator which takes only quotient. If $\mathring{Z}$ is even, then

$$\begin{aligned} \mathring{Z}/2 &\equiv \{1,0,0,...\}M \\ \mathring{Z}/2 &\equiv -\mathring{Z}/2 \end{aligned}$$

But if $\mathring{Z}$ is odd, then

$$\begin{aligned} \mathring{Z}/2 &\equiv \{5/2,7/2,...\}M \\ \mathring{Z}/2 &\equiv -\mathring{Z}/2-1 \\ (5\cdot 7)/2 &\equiv \{1,2,2,3\}M \\ (5\cdot 7\cdot 11)/2 &\equiv \{0,0,2,3,5\}M \end{aligned}$$

Chessboard Model

Analogically sieving away residue classes of prime factors in primorial ring operates like positioning one or more rooks on primorial hyperspace chessboard which causes all enemy pieces in same column or row with the rook be eliminated (in sight of a rook).

Short Terms

Define following short terms. *Universe* stands for the chessboard and rooks, namely a game setup. *Configuration* is short for rook configuration. *Picture* is short for the whole picture of the entire universe, the complete full pattern. *Residue* stands for the remaining element. *Remaining* and *unmasked* are synonyms. *Sieved*, *eliminating* and *masked* are synonyms. They are used interchangeably.

Longest Gap in Pattern

Definition 5. A *gap* is a distance difference between 2 neighbor residues, in between which there is no other residue.

Definition 6. The *longest-gap* is denoted as

$$G^+(\{\frac{r}{p}|0<r<p\}),$$

$G^+_{\dot r}(\{p\})$, $G^+_{\dot r}$ or simply G^+ in which $\dot r$ gives the count of rooks and + indicates that it will retrieves the maximum from all possible parallel universes.

Boring Universes

Empty and full universes are out of this game. A interesting and challenging game must have 1 or more rooks covering as large as possible area and there must be at least 1 residue can never be covered.

Gap's Limits

$$1 \le G \le \mathring{Z}$$

By definition, the length of gap can not be shorter than 1 or longer than its universe. Otherwise a gap can always be lengthened by masking its edges. $\forall p_{\dot z} > 2$, there is

$$G^+_1 < G^+_2$$

$$\begin{aligned} G^+_2(\{5\})-1 &= 2 \\ G^+_2(\{5,7\})-1 &= 2\cdot 2 \end{aligned}$$

The length of the longest gap can not be shorter than the count of layers or longer than the total of eliminated plus 1. For example, in case of 2-rook,

$$\begin{aligned} & 2\cdot(\dot z-2) \\ < \ & G^+_2(\{5,7,\dots,p_{\dot z}\}) \\ < \ & \mathring{Z}\cdot(1-\frac{3}{5}\cdot\frac{5}{7}\cdot\dots\cdot\frac{p_{\dot z}-2}{p_{\dot z}})+1. \end{aligned}$$

A universe has to exhaust its entropy or available possible configurations to lengthening a gap, until they are depleted.

In 2-rook situation, if it sieves away $p_{\dot z}-1$ residue classes instead of 2 residue classes of $p_{\dot z}$, then $\forall p_{\dot z} > 5$ there is

$$\begin{aligned} G^+_2 &< G^+(\{\frac{2}{5},\frac{2}{7},\dots,\frac{2}{p_{\dot z-1}},\frac{p_{\dot z}-1}{p_{\dot z}}\})\cdot 6 \\ &= p_{\dot z}\cdot G^+_2(\{\dots,p_{\dot z-1}\}). \end{aligned}$$

And if we substitute the largest prime factor $p_{\dot z}$ with 3, then there is

$$\begin{aligned} G_2^+ &= G_2^+(\{5,7,\dots,p_{\dot z-1},p_{\dot z}\})\cdot 6 \\ &< G_2^+(\{5,7,\dots,p_{\dot z-1},3\})\cdot 6 \\ &= 3\cdot G_2^+(\{5,7,\dots,p_{\dot z-1}\})\cdot 6 \\ &= 3\cdot G_2^+(\{\dots,p_{\dot z-1}\}). \end{aligned}$$

In general, with same count of layers, denser patterns get longer or equal result. It is false in critical cases which are rare. For example, in case of using 8 layers, there is a bump

$$\begin{aligned} 25 &= G^+(\{\frac{4}{5},\frac{4}{7}\}), \\ 20 &= G^+(\{\frac{2}{5},\frac{2}{7},\frac{4}{11}\}), \\ 25 &= G^+(\{\frac{2}{5},\frac{2}{7},\frac{2}{11},\frac{2}{13}\}). \end{aligned}$$

Determinating critical cases in comparing longest gap functions is empirical and manual.

1-Rook

There is a function estimating residue count in 1-rook situation:

$$f_1(l) = l\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{p_{\dot z}-1}{p_{\dot z}} \tag{1}$$

where l is the length of target interval on chessboard.

Exhaustive searching, we observe that

$$G_1^+ = 2\cdot p_{\dot z-1} \qquad (p_{\dot z}\geq 5). \tag{2}$$

There are always two conjugated solution gaps. Taking a closer look. Let a be middle element of either gap. R_a is

$$a \equiv \{...0,0,0,\pm 1,\mp 1\}M,$$

where $r_{\dot z-1}\equiv\pm 1 \pmod{p_{\dot z-1}}$ and $r_{\dot z}\equiv\mp 1 \pmod{p_{\dot z}}$ and all others are 0s. Their inner-structure is self-explanatory. Following is the analysis:

Rotate the picture, let the middle element of either solution gap be relocated from a to 0. Therefore the rook will be relocated from 0 to $\pm a$, at where the rook can not see two blind spots $\pm p_{\dot z-1}$. Therefore each solution gap is $p_{\dot z-1}-(-p_{\dot z-1}) = 2\cdot p_{\dot z-1}$ long. For example, in case of $p_{\dot z}=7$,

$$\begin{aligned} +p_{\dot z-1} = +5 &= \{1,2,0,5\}M \\ -p_{\dot z-1} = -5 &= \{-1,-2,0,-5\}M \end{aligned}$$

2 is included as one of prime factors and even length gap has one middle element. Therefore all gaps are even length gaps and each has its own middle element. It is legitimate and convenient to shift the middle element of a challenger gap to position 0 and to measure its length.

Configuration $\{0,0,0\dots\}$ leaves only 2 residues at position +1 and -1. If we ignore all even numbers, last two prime factors $p_{\dot z-1}$ and $p_{\dot z}$ are too long to mask more than 2 positions in $(-p_{\dot z-1},+p_{\dot z-1})$.

Axiom 1. In 1-rook situation, without introducing last two prime factors $p_{\dot z-1}$ and $p_{\dot z}$, no other but only the natural configuration $\{0,0,0\dots\}$ can leave less than 3 residues in $(-p_{\dot z-1},+p_{\dot z-1})$. So configuration $\{...0,0,0,\pm 1,\mp 1\}$ is the only solution for G_1^+.

Prime Gap as G_1^+

In natural configuration situation, position 1 is always the left edge of a gap, $\{2,\dots,p_{\dot z}-1\}$ are sieved away by prime factors $\{2,3,5,\dots,p_{\dot z-1}\}$ as the valley and $p_{\dot z}$ is always the right edge. Therefore, according to Formula (2), there is

$$\begin{aligned} p_{\dot z}-1 &\leq G_1^+(\{2,3,5,\dots,p_{\dot z-1}\}) \\ p_{\dot z}-1 &\leq 2\cdot p_{\dot z-2} \\ p_{\dot z} &\leq 2\cdot p_{\dot z-2}+1 \\ p_{\dot z}-p_{\dot z-1} &\leq 2\cdot p_{\dot z-2}-p_{\dot z-1}+1 \end{aligned}$$

Because $a=\{0,...,0,\pm 1,\mp 1\}M = k(2\cdot 3\cdot 5\cdot\cdots\cdot p_{\dot z-2})\neq 0$ which is growing by multiplying follow-up prime factors. Therefore, when $p_{\dot z}>11$, $p_{\dot z-1}+1$ can never be a. Therefore the middle element of the gap $[1,p_{\dot z}]$ is not to the middle element of the longest gap. Therefore the gap $[1,p_{\dot z}]$ is not the longest gap.

Therefore $\forall p_{\dot{z}} > 11$, $p_{\dot{z}} < 2 \cdot p_{\dot{z}-2} + 1$. For example:

$$\begin{aligned} 5 &= 2 \cdot 2 + 1 \\ 7 &= 2 \cdot 3 + 1 \\ 11 &= 2 \cdot 5 + 1 \\ 13 &< 15 = 2 \cdot 7 + 1 \\ 17 &< 23 = 2 \cdot 11 + 1 \\ \dots \end{aligned}$$

Legendre's Conjecture as G_1^+

Proof. Let $(p_{\dot{z}} + k + 1) \leq p_{\dot{z}+1}$ and $k \geq 0$. There is

$$(p_{\dot{z}} + k + 1)^2 - (p_{\dot{z}} + k)^2 = 2(p_{\dot{z}} + k) + 1 > G_1^+.$$

□

2-Rook

In brief, $[0, p_{\dot{z}}^2]$ is considered as a safe zone where any residue is a guaranteed prime pair. A gap must have 2 residues as its start and end element. If $\forall p_{\dot{z}}$ $G_2^+ < p_{\dot{z}}^2$, then all 2-rook problems always have at least one solution.

There is a function estimating residue count in 2-rook situation:

$$f_2(l) = l \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{p_{\dot{z}} - 2}{p_{\dot{z}}} \tag{3}$$

where l is the length of arbitrary interval on chessboard. Exhaustive searching, we got that $\forall p_{\dot{z}} \leq 23$,

$$G_2^+ < p_{\dot{z}}(p_{\dot{z}} - 1). \tag{4}$$

Polignac's Conjecture as G_2^+

If $\forall p_{\dot{z}}$ Formula (4) holds, then arbitrary even number $2n$ can be difference of two primes, $2n = p_a - p_b$. In order to set up each number pair as $\{k, k + 2n\}$, the first rook is always at natural position 0, the second rook is placed at $-2n$. The safe zone is $[3, p_{\dot{z}+1}^2 - 2n - 1]$.

$\dot{z}$	$p_{\dot{z}}$	G_1^+	G_2^+	$p_{\dot{z}}^2 - p_{\dot{z}}$
1	2	$\mathring{Z}$	$\mathring{Z}$	2
2	3	4	$\mathring{Z}$	6
3	5	6	18	20
4	7	10	30	42
5	11	14	66	110
6	13	22	150	156
7	17	26	192	272
8	19	34	258	342
9	23	38	366	506

Table 1: short table of $G^+[1]$

Goldbach's Conjecture as G_2^+

If $\forall p_{\dot{z}}$ Formula (4) holds, then arbitrary even number $2n$ can be sum of two primes, $2n = p_a + p_b$. In order to set up each number pair as $\{k, 2n - k\}$, the second rook is placed at $2n$. The safe zone is $[3, 2n - 3]$.

N-Rook Generalization

$$\lim_{p_{\dot{z}} \to \infty} G_{\dot{r}}^+ \leq k p_{\dot{z}} < p_{\dot{z}}(p_{\dot{z}} - 1) \tag{5}$$

where k and $\dot{r}$ are finite and

$$k p_{\dot{z}} \cdot \prod_{p=2, p \in \mathbb{P}}^{p_{\dot{z}}} \frac{\max\{1, p - \dot{r}\}}{p} \geq 2.$$

For example, in case of 2-rook, $k = 4$

$$f_2(4 p_{\dot{z}}) = 4 p_{\dot{z}} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{p_{\dot{z}} - 2}{p_{\dot{z}}} \geq 2.$$

If it holds, the count of residues in any $[k+1, k+4p_{\dot{z}}]$ is greater than 2 on average.

Discussion and Conclusion

The *quantitative domination* in sieving for residues in safe zone and further *ultimate equilibrium* that the longest gap converges to the average density of residues are obvious. According set theory, $|\mathbb{Q}| = |\mathbb{N}| = |k p_{\dot{z}}|$, therefore sieving will have a same result.

Conveniently, we consider infinity as an arbitrary large constant and anything goes towards it is same, since it will get there sooner or later. The nuance is running farther away from zero can not escape but expand finite domain. It implies a self-similar positive-feeding fractal structure. In Zeno's paradox, the arrow will eventually hit the wall at some point, exhaust every element in current set, and eternally stop in next cardinal steps or time scale. Looking for this wall is a reciprocal variant of Zeno's paradox. When the arrow hits something, within twice that far there will be next prime. Also Thomson's lamp. We can not assume a stationary wall.

In set theory, $\infty + 1 = \infty$, $\sup \mathbb{N} = \omega$. An analogical story, in the animal track game, all animals eventually meet at $n + k\omega = \omega$ due to bijection. Constants are fossil footprints. Variables (time varying function, eternal, temporal or even ephemeral) are dancing and moving animals which are out of its problem, in other words, which are invisible to set theory. It considers two sets are same sized if there is a one-to-one matching on foot marks. $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$. It is believed that cardinality is discrete. It handles everything discrete. Anything continuous is also out of its problem, or it handles as a union of discrete objects.

In ZF, everything is a set. All sets contain empty set. Zero is empty set as the first element of $\mathbb{N}$. Therefore all sets contain zero?! $\varnothing \subseteq \varnothing = 0 = \{\} < 1 = \{\{\}\} < 2 = \{\{\{\}\}\}$[2][3]. Empty set is subset of all sets but explicitly listing it out inside braces gives a different meaning. Axiom of regularity disallows a set containing itself. Anything recursive cyclic or self-referencing is out of its problem as nonexistence. Negatives are also out of its problem, it cannot represent a negative as a set or a set cannot have negative size.

In this paper, a black swan $\mathring{Z}$ closes itself as a complete ring in cyclic order. Each element is a unique configuration. If one is absent, the gap can not be filled by other one. It can not eliminate one debt by incurring another. So borrowing elements from infinity does not work. $p_{\mathring{z}}\# = 0 \in \mathring{Z}$. power towers ($\omega^{\omega}$) are fancy but indeterminate. Anything larger is unimaginable and weirder. Similar to $e^{2k\pi i} - 1 = 0$, zero as a constant can has many indistinguishable function equivalents. Zeros could be nothing or everything, the starting or finishing line.

In 1-rook situation, sieving by all prime factors, it should eliminate all but ± 1 . But by using more prime factors, correspondingly it also raises $\mathring{Z}$ bigger. Eventually residues will get even more by contrast, which is contrary to expectation. Almost all elements can not be fit in the one-dimensional static open flat space (number axis) in Euclidean geometry.

To satisfy both endless and bounded, we can assume a solution model whose manifold at global is a closed circle (ring) in spherical geometry. Time appears as intrinsic extra directional dimension in fractal space. Thinking of longitude and latitude, on a spherical surface, any two straight lines always meet. From the global perspective, every object stone or animal inside is shrinking or squeezed into zero by gravity as time goes (or its local attractor in a different model and setup). Inward gravity is the nature of positive curvature. All animals are both endlessly falling into a same rabbit-hole. From any object's local perspective, the space is expanding into infinity. Any event beyond finite limits is unmeasurable and unobservable directly.

Models in classical theories have bugs and loopholes when facing infinity, continuum (counting electrons), fractal or cyclic pattern, and mirrors (counting image copies in a kaleidoscope or mirror maze). The universe presents and maintains its consistency as pattern. It is quite rich, dynamic, even explosive and boisterous.

Is space expanding into infinity an illusion if it already always exists? In our physical universe, why gravity is squared (as if it is a spherical surface) rather than cubed? It leaves more questions.

References

[1] `https://github.com/n43e120/PrimeGap`

[2] `https://en.wikipedia.org/wiki/Set_(mathematics)`

[3] `https://en.wikipedia.org/wiki/Empty_set`

Acknowledgments

Thank God and Internet.