The \( n \)-th Mean

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Abstract

We define the \( n \)-th mean of \( a_1, a_2, \ldots, a_k \) as

\[
M_n(a_1, a_2, \ldots, a_k) = I_{n+1} \left( \frac{k}{i=1} H_i(a_i), k \right)
\]

where \( a_1, a_2, \ldots, a_k \in \mathbb{R}, \ H_i(a_k) = H_i(a_1, H_i(a_2, \ldots, H_i(a_{n-1}, a_n)) \ldots) \)[1]

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1. Introduction

In this paper, we provide the definition of \( n \)-th mean and some examples.

2. The inverse operation of Hyperoperation

Definition 2.1. We define \( I_n \), the inverse operation of \( n \)-th hyperoperation as

\[
I_n(H_n(a, b), b) = a
\]

where \( H_n(a, b) \) is the hyperoperation of \( a \) and \( b \).

The next results follow easily from the definition.

Lemma 2.2. It is \( I_1(x, y) = x - y \) where \( x, y \in \mathbb{R} \).

Proof. By the definition, it is

\[
I_1(H_1(a, b), b) = I_1(a + b, b) = a.
\]

Here, let \( a = k - b \) (\( k \in \mathbb{R} \)), then

\[
I_1(a + b, b) = I_1(k, b) = k - b.
\]

Therefore, substituting \( k \) with \( x \), \( b \) with \( y \) yields the following formula.

\[
\therefore I_1(x, y) = x - y.
\]

Lemma 2.3. It is \( I_2(x, y) = x/y \) where \( x, y \in \mathbb{R} \), \( y \neq 0 \).

Proof. By the definition, it is

\[
I_2(H_2(a, b), b) = I_2(ab, b) = a.
\]

Here, let \( a = k/b \) (\( k \in \mathbb{R}, b \neq 0 \)), then

\[
I_2(ab, b) = I_2(k, b) = k/b.
\]

Therefore, substituting \( k \) with \( x \), \( b \) with \( y \) yields the following formula.

\[
\therefore I_2(x, y) = x/y.
\]
Lemma 2.4. It is $I_3(x, y) = \sqrt[3]{x}$ where $x, y \in \mathbb{R}$, $y \neq 0$.

Proof. By the definition, it is

$$I_3(H_3(a, b), b) = I_3(a^b, b) = a.$$ 

Here, let $a = \sqrt[3]{k}$ ($k \in \mathbb{R}, b \neq 0$), then

$$I_3(a^b, b) = I_3(k, b) = \sqrt[3]{k}.$$ 

Therefore, substituting $k$ with $x$, $b$ with $y$ yields the following formula.

$$\therefore I_3(x, y) = \sqrt[3]{x}.$$ 

Lemma 2.5. It is $I_4(x, y) = \sqrt[y]{x}$ where $x, y \in \mathbb{R}$, $y \neq 0$, and $^b a = H_4(a, b)$. Also, $\sqrt[y]{x}$ is super root which is the inverse operation of tetration. In other words, if $a = ^n b$, then $b = \sqrt[n]{a}$.\[2\]

Proof. By the definition, it is

$$I_4(H_4(a, b), b) = I_4(^b a, b) = a.$$ 

Here, let $a = \sqrt[4]{k}$ ($k \in \mathbb{R}, b \neq 0$), then

$$I_4(a^b, b) = I_4(k, b) = \sqrt[4]{k}.$$ 

Therefore, substituting $k$ with $x$, $b$ with $y$ yields the following formula.

$$\therefore I_4(x, y) = \sqrt[y]{x}.$$ 

3. The difference between arithmetic mean and geometric mean, and the $n$-th mean.

Definition 3.1. The arithmetic mean of $a_1, a_2, \ldots, a_n$ is

$$A(a_1, a_2, \ldots, a_n) = \frac{1}{n} \left( \sum_{k=1}^{n} (a_k) \right)$$ 

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.\[3\]

Definition 3.2. The geometric mean of $a_1, a_2, \ldots, a_n$ is

$$G(a_1, a_2, \ldots, a_n) = \left( \prod_{k=1}^{n} (a_k) \right)^{1/n}$$ 

3
where \( a_1, a_2, \ldots, a_n \in \mathbb{R} \).

The next results follow easily from the definitions and the lemmas.

**Lemma 3.3.** The arithmetic mean of \( a_1, a_2, \ldots, a_n \) can be expressed as

\[
I_2 \left( H_1^k(a_k), n \right)
\]

where \( a_1, a_2, \ldots, a_n \in \mathbb{R}, \ H_1^k(a_k) = H_i(a_1, H_i(a_2, \ldots, H_i(a_{n-1}, a_n) \ldots)) \).

**Proof.** Since \( H_1(a, b) = a + b \), \( I_2(a, b) = a/b \) so it is

\[
\left( \sum_{k=1}^{n} (a_k) \right) / n = \left( \sum_{k=1}^{n} (a_k) \right) / n = I_2 \left( H_1^k(a_k), n \right).
\]

**Lemma 3.4.** The geometric mean of \( a_1, a_2, \ldots, a_n \) can be expressed as

\[
I_3 \left( H_2^k(a_k), n \right)
\]

where \( a_1, a_2, \ldots, a_n \in \mathbb{R}, \ H_2^k(a_k) = H_i(a_1, H_i(a_2, \ldots, H_i(a_{n-1}, a_n) \ldots)) \).

**Proof.** Since \( H_2(a, b) = ab \), \( I_3(a, b) = \sqrt[n]{a} \) so it is

\[
\left( \prod_{k=1}^{n} (a_k) \right)^{1/n} = \left( \prod_{k=1}^{n} (a_k) \right)^{1/n} = I_3 \left( H_2^k(a_k), n \right).
\]

We can know that the arithmetic mean and the geometric mean of \( a_1, a_2, \ldots, a_k \) is the same as following formula at \( n = 1 \) and \( n = 2 \):

\[
I_{n+1} \left( H_i^k(a_i), k \right).
\]

**Definition 3.5.** We define the \( n \)-th mean of \( a_1, a_2, \ldots, a_k \) as

\[
M_n(a_1, a_2, \ldots, a_k) = I_{n+1} \left( H_i^k(a_i), k \right)
\]

where \( a_1, a_2, \ldots, a_n \in \mathbb{R}, \ H_i^k(a_k) = H_i(a_1, H_i(a_2, \ldots, H_i(a_{n-1}, a_n) \ldots)) \).
**Example 3.6.** Calculate the 1st mean of 1 and 2.

1st mean of 1 and 2 is
\[ M_1 (1, 2) = I_2 (H_1 (1, 2), 2) = I_2 (3, 2) = 3/2. \]

Therefore,
\[ \therefore M_1 (1, 2) = 1.5 \]

**Example 3.7.** Calculate the 3th mean of 2 and 3.

3th mean of 2 and 3 is
\[ M_3 (2, 3) = I_4 (H_3 (2, 3), 2) = I_4 (8, 2) = \sqrt[3]{8}. \]

Let
\[ x = M_3 (2, 3) = \sqrt[3]{8}, \]
then
\[ 2x = x^x = 8. \]

Here, take the logarithm on both sides.
\[ x \ln x = \ln x \cdot e^{\ln x} = \ln 8 \]

Then, take the Lambert W function of both sides.
\[ \ln x = W (\ln 8) \]
\[ \therefore x = e^{W(\ln 8)} \approx 2.38842348 \]

Therefore,
\[ \therefore M_4 (2, 3) = e^{W(\ln 8)} \approx 2.38842348. \]

**4. Conclusion**

In this paper, we provided the definition of inverse operation of hyperoperation, definition of n-th mean, and some examples. We defined the inverse operation of hyperoperation and found some formulas from the definition. And we found the difference between arithmetic mean and geometric mean. Finally, we defined the n-th mean of \( a_1, a_2, \ldots, a_k \) as
\[ M_n (a_1, a_2, \ldots, a_k) = I_{n+1} \left( H_n (a_i), k \right) \]
where \( a_1, a_2, \ldots, a_k \in \mathbb{R}, \ H_i (a_k) = H_i \left( a_1, H_i (a_2, \ldots, H_i (a_{n-1}, a_n) \ldots) \right). \]
Reference


